J.evol.equ. 2 (2002) 00 – 00
0000–0000/02/010001 – 19 \$ 1.50 + 0.20/0

Journal of Evolution Equations

Asymptotic behaviour for the porous medium equation posed in the whole space

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Dédié à la mémoire de Philippe Bénilan, ami et maître

This paper is devoted to present a detailed account of the asymptotic behaviour as $t \to \infty$ of the solutions $u(x,t)$ of the equation

$$u_t = \Delta(u^m) \tag{0.1}$$

with exponent $m > 1$, a range in which it is known as the *porous medium equation*, written here PME for short. The study extends the well-known theory of the *classical heat equation* (HE, the case $m = 1$) into a nonlinear situation, which needs a whole set of new tools. The space dimension can be any integer $n \geq 1$. We will also present the extension of the results to exponents $m < 1$ (*fast-diffusion equation*, FDE). For definiteness we consider the Cauchy Problem posed in $Q = \mathbb{R}^n \times \mathbb{R}^+$ with initial data

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n \tag{0.2}$$

chosen in a suitable class of functions. In most of the paper we concentrate on the class $\mathcal{X}_0$ of integrable and nonnegative data,

$$u_0 \in L^1(\mathbb{R}^n), \quad u_0 \geq 0, \tag{0.3}$$

which is natural on physical grounds as the density or concentration of a diffusion process, the height of a ground-water mound, or the temperature of a hot medium (see a comment on the applications at the end). Consequently, we will deal mostly with nonnegative solutions $u(x,t) \geq 0$ defined in Q. An existence and uniqueness theory exists for this problem so that for every data u_0 we can produce an orbit $\{u(\cdot,t) : t > 0\}$ which lives in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and describes the evolution of the process. The solution is not classical for $m > 1$, but it is proved that there exists a unique weak solution for all $m > 0$. It is also unique in the sense of *mild solution*, obtained as limit of the Implicit Time-Discretization Scheme, a method that was carefully studied by Bénilan and Crandall [BC1, BC2]. They also proved that the mass, $M = \int u(x,t)\,dx$, is an invariant of the motion for all $t > 0$ as long as $m \geq (n-2)/n$, a number that will appear again below. The solution of the Cauchy Problem, (CP) for short, has a number of useful additional properties, in particular it is continuous in Q and bounded for $t \geq \tau > 0$.

The investigation of the existence of generalized solutions of the initial and boundary-value problems for the equations of nonlinear diffusion of this type and more general forms, like

$$u_t = \Delta\Phi(u) + f \tag{0.4}$$

(with Φ a monotone nondecreasing function and $f \in L^1(Q)$), was a main chapter of Ph. Bénilan's Thesis in 1972, [Be], and he continued during his life to enlarge the scope of the investigation to the whole range of second-order nonlinear equations of parabolic or hyperbolic-parabolic type. This is reflected in a large number of papers and in the book [Bk], unfortunately still not available to the wide community of researchers and students. In the best tradition of Applied Functional Analysis, Bénilan's way of attacking these equations was to provide an abstract framework, where the typical problems are well-posed and generate semigroups or flows in Banach spaces, mainly $L^1(\mathbb{R}^n)$. Concepts like integral solutions, mild solutions, renormalized solutions, entropy solutions appear in the work of Ph. Bénilan and his collaborators, many of the ideas were inspired by him and have become standard today in the study of nonlinear parabolic and hyperbolic phenomena. His work and teaching has illuminated the life of many friends and followers that contribute to this wide subject at the intersection of PDE's, Mechanics and Functional Analysis. Applications to problems in the oil industry and biology played a role in formulating new problems and questions in Bénilan's research and that of his collaborators, but I will not pursue this interesting direction here. Summing up this paragraph, I contend that the close combination of very concrete and important equations and very abstract and powerful methods is a most appealing way of doing PDEs, and we hope it will last in the work of younger generations.

The main part of the following material has been taught to graduate students at the Univ. Autónoma de Madrid in 1997 [V5]. Further progress is reported here, and a number of unpublished results are added at convenient places to close some arguments and solve some open problems. Let us mention that the general picture for (CP) is now quite well known, but some interesting problems still remain open!

1. Long-time asymptotics

Our main concern in this paper will be the study of the asymptotic behaviour of the solutions of equation (0.1). This is still a quite large subject, hence we will concentrate in the sequel on the initial-value problem (CP) posed in the whole space with integrable data. The behaviour is different for other classes of data or for boundary-value problems. A survey of results for the Dirichlet problem posed in a bounded domain is [V5], for the Neumann problem see [AR], for the exterior Dirichlet problem [QV]. On a general level, it has been pointed out in many papers and corroborated by numerical experiments that similarity solutions furnish the asymptotic representation for solutions of a wide range of problems in mathematical physics. The reader is referred to the book of G.I. Barenblatt

[B2] for a detailed discussion of this subject. Self-similar solutions and the forthcoming scaling techniques will play a prominent role in our asymptotic study.

The model. The asymptotic behaviour is a classical result for the linear heat equation, $m = 1$, and is now well known for the PME, $m > 1$: this is the theory that we want to present in a systematic way. In the first case, the classical result says that there is convergence of any solution of the Cauchy problem under conditions (0.3) towards the Gaussian kernel

$$u(x,t) \sim \frac{M}{(4\pi t)^{n/2}} \exp(-x^2/4t), \tag{1.1}$$

where $M = \int u_0(x)\,dx$ is the mass of the solution (space integration is performed by default in $\mathbb{R}^n$). In the case $m > 1$ the behaviour of our class of solutions can be described for large t by a one-parameter family of special solutions of (0.1)

$$\mathcal{U}(x,t;C) = t^{-\alpha} F(xt^{-\beta};C), \tag{1.2}$$

with parameter $C > 0$. The functions $\mathcal{U}(x,t;C)$ are variously called *source-type solutions*, *fundamental solutions*, *Barenblatt solutions*, or *BZKP solutions*, cf. the original papers [ZK], [B1], [Pa]. They are given by the explicit formula

$$F(\eta) = (C - k\,\eta^2)_+^{\frac{1}{m-1}}, \quad \alpha = \frac{n}{n(m-1)+2}, \ \beta = \frac{1}{n(m-1)+2}. \tag{1.3}$$

F is called the profile, α and β are the similarity exponents (that we call Barenblatt exponents). $C > 0$ is a free constant and k is fixed, $k = (m-1)\beta/2m$. We have $\mathcal{U}^{m-1} = (Ct^{2\beta} - k\,x^2)_+/t$. The fact that the fundamental solutions are self-similar is important in what follows, the fact that they are explicit is not.

Main result. We will prove that $u(x,t) \sim \mathcal{U}(x,t;C)$ for large t. For a given u there is a correct choice of the constant $C = C(u_0)$ in this asymptotic result which agrees with the rule of mass equality:

$$\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} \mathcal{U}(x,t;C)\,dx. \tag{1.4}$$

It follows that $C = c(m,n)\,M^{2(m-1)\beta}$. We also write $\mathcal{U}_M$ for the solution with mass M and F_M for its profile. This is the precise statement of the asymptotic convergence result:

THEOREM 1.1. *Let $u(x,t)$ be the unique weak solution of problem* (CP) *with initial data $u_0 \in L^1(\mathbb{R}^n)$, $u_0 \ge 0$. Let $\mathcal{U}_M$ be the Barenblatt solution with the same mass as u_0. Then as $t \to \infty$ we have*

$$\lim_{t\to\infty} \|u(t) - \mathcal{U}_M(t)\|_1 = 0. \tag{1.5}$$

Convergence holds also in L^∞-norm in the proper scale:

$$\lim_{t\to\infty} t^\alpha \|u(t) - \mathcal{U}_M(t)\|_\infty = 0 \tag{1.6}$$

with $\alpha = n/(n(m-1)+2)$. *Moreover, for every* $p \in (1,\infty)$ *we have*

$$\lim_{t\to\infty} t^{\alpha(p)} \|u(t) - \mathcal{U}_M(t)\|_{L^p(\mathbb{R}^n)} = 0, \tag{1.7}$$

with $\alpha(p) = \alpha(p-1)/p$.

The last result follows from (1.5) and (1.6) by simple interpolation, but (1.6) and (1.5) are (to an extent) independent. The main body of the paper is devoted to giving a detailed proof of this theorem, and to report on its many extensions. We will follow the "four-step method", a general plan to prove asymptotic convergence devised by Kamin and Vazquez in 1988, [KV1], who settled in this way the asymptotic behaviour both for the p-Laplacian equation, $u_t = \nabla\cdot(|\nabla u|^{p-2}\nabla u)$, and for the PME. But the first proof of convergence for the PME in several dimensions appeared in a celebrated paper by Friedman and Kamin in 1980 [FK]: it uses the method of lower barriers that will be presented in Section 14 and gets uniform convergence on compact expanding sets of the form $\{|x| \le C\, t^\beta\}$, a weaker form of (1.6). The initial data belong to $L^1 \cap L^2$, but this is not a real restriction after the regularizing effect of Bénilan [Be2] and Véron [Ve]. A previous proof in one space dimension is due to S. Kamin, 1973 [K1].

The proof of the full result with uniform convergence in the whole space, formula (1.6), has first appeared in the notes [V5]. There, six different proofs of the main result are presented, all of them sharing a common 4-step basis, but concluding through quite different ideas, thus summing up the state of asymptotic convergence methods for the PME at the time. The interest in displaying all of these approaches lies in the fact that they call on many of the asymptotic methods available in the study of nonlinear parabolic problems. In the study of similar problems for other equations and/or classes of data some of these techniques will be applicable and have in fact been applied, but the best technique has to be chosen depending on the case.

Discussion. Immediate issues after Theorem 1.1. are discussing whether the class of data may be extended with the same type of convergence, or the rate of convergence can be improved. Both questions have been explored, but let us begin with two negative results.

For the first question, the restriction $u_0 \in L^1(\mathbb{R}^n)$ cannot be ignored if we want to keep the asymptotic size, since non-integrable data have a different behaviour.

COROLLARY 1.2. *If u is a nonnegative global solution of the PME with* $u_0 \in L^1_{loc}(\mathbb{R}^n)$ *and* $\int u_0\,dx = \infty$ *then*

$$\lim_{t\to\infty} t^\alpha u(y\, t^\beta, t) = \infty \tag{1.8}$$

uniformly on compact sets $\{y \in K\} \subset \mathbb{R}^n$.

The proof of this result is very easy after approximating u_0 by an increasing sequence of integrable data u_{0n}, applying Theorem 1.1 and passing to the limit. Consequently, non-integrable solutions have other asymptotic size and shape, which has been partially investigated in [AR2] and other papers. We recall that a theory of global solutions of the PME with non-integrable data was constructed by Bénilan, Crandall and Pierre in [BCP] in 1984. Let us also refer to asymptotic results with a completely different flavor from Theorem 1.1: a general study of the asymptotic behaviour of the PME and other evolution equations in the framework of $\mathcal{X} = L^\infty(\mathbb{R}^n)$ has been performed recently in a paper with E. Zuazua [VZ]; the situation is quite different, there is no simple model of asymptotic attractor, the appropriate word is rather *complexity*, i.e., the possibility of chaos.

The second direction is improving the rates, and there we again find a limitation.

THEOREM 1.3. *The rates of convergence of Theorem* 1.1 *are optimal in the class of initial data* $\mathcal{X}_0 = \{u_0 \in L^1(\mathbb{R}^n),\ u_0 \geq 0\}$.

The main idea in constructing a counterexample consists of placing small bits of mass far enough at $t = 0$. The detailed proof is presented in Section 11. Better rates can be found if we restrict the class of data. This subject, which has been the object of an active interest in recent years, will discussed in Sections 16 to 19. A first natural idea is to consider initial data with compact support and to control the expansion of the support for large times. Then it is proved that the support of the solution $u(\cdot, t)$ approaches the size of the support of the source-type solution, a ball of radius $\mathcal{R}(t) = C_0\, t^\beta$ with $C_0 = c(m, N)\, M^{(m-1)\beta}$, cf. Section 9, Theorem 9.1. A larger class of solutions is considered in Section 19 using so-called entropy methods.

Another idea is to extend the convergence to the derivatives, i.e., convergence in C^k. The presence of a free boundary for compactly supported solutions makes the question tricky because the variable domain of positivity: though it converges to the ball of the source-type solution, it does not coincide with it. The question has been recently studied by Lee and Vazquez [LV], see details in Section 18.

Extensions. We have also made progress in extending the asymptotic study to cover the following three important directions:

(i) Signed solutions for the signed PME: $u_t = \Delta(|u|^{m-1}u)$, cf. Theorem 20.1 taken from the paper [KV2].

(ii) Equations with forcing, where we present a new result, Theorem 20.2 which gives the asymptotic behaviour for

$$u_t = \Delta(|u|^{m-1}u) + f, \qquad f \in L^1(Q)\,. \tag{1.9}$$

This answers a question of Bénilan and is maybe the most natural extension of Theorem 1.1 in the spirit of his ideas.

(iii) Exponents m less than one (Fast Diffusion Equations). We introduce a new idea, *uniform convergence in relative error*, Theorem 21.1.

Though the theory has obvious counterparts for related equations, like the p-Laplacian equation, $u_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$, the general Filtration Equation $u_t = \Delta\Phi(u)$ with Φ monotone nondecreasing, and reaction-diffusion equations like $u_t = \Delta u^m - u^p$, we will not pursue these directions here. In the first case the similarity of methods and results is impressive. The basic convergence result is proved in [KV1]. Rates of convergence have been recently obtained, see [DP2]. The survey paper [Kl], though a bit old is very informative about the theory of nonlinear diffusion equations.

Finally, the porous media equation looks like a variation of the heat equation, and perturbative techniques could be expected to be the way to the analysis. It must be stressed that it is not so, and the study performed below relies on the ideas and machinery of nonlinear analysis. Indeed, the PME is a good benchmark for a number of important nonlinear techniques.

Notation. In the whole paper $n \geq 1$ is the space dimension, and the exponents α and β will be fixed to the values (1.3): $\alpha = n/(2+n(m-1)) > 0,\ \beta = \alpha/n$. Given a solution $u(x,t)$, we will often write $u(t)$ to denote the function $t \mapsto u(\cdot,t)$, as in $\|u(t)\|_p = \|u(\cdot,t)\|_{L^p(\mathbb{R}^n)}$.

2. Definitions and preliminary results

Problem (0.1)–(0.2) does not possess classical solutions for general data in the class $\mathcal{X}_0 = \{u_0 \in L^1(\mathbb{R}^n),\ u_0 \geq 0\}$ (or even in a smaller class, like the set of smooth nonnegative and rapidly decaying initial data). This is due to the fact that the equation is parabolic only where $u > 0$, but degenerates at the level $u = 0$. Therefore, we need to introduce a concept of generalized solution and make sure that the problem is well-posed in that class.

Concept of solution. (I) By a solution of equation (0.1) we will mean a nonnegative function $u(x,t)$, defined for $(x,t) \in Q$ such that: (i) viewed as a map $t \to u(\cdot,t) = u(t)$ we have $u \in C((0,\infty) : L^1(\mathbb{R}^n))$; (ii) the functions u^m, u_t and Δu^m belong to $L^1(t_1, t_2 : L^1(\mathbb{R}^n))$ for all $0 < t_1 < t_2$; and (iii) equation (0.1) is satisfied in the sense of distributions in Q.

(II) By a solution of problem (CP) we mean a solution of (0.1) such that the initial data are taken in the following sense:

$$u(t) \to u_0 \quad \text{in } L^1(\mathbb{R}^n) \quad \text{as } t \to 0. \tag{2.1}$$

In other words, $u \in C([0,\infty) : L^1(\mathbb{R}^n))$ and $u(0) = u_0$.

This definition is usually called in the literature a **strong solution**. It is suitable for our purposes since problem (CP) is well-posed in this setting, but it is not the unique choice; we could have used the concept of **weak solution**, where we merely ask u^m and $\nabla_x u^m$ to be locally integrable functions in $\mathbb{R}^n \times [0,\infty)$ and the equation is satisfied in the sense that

$$\int\int \{u\varphi_t - \nabla_x u^m \cdot \nabla_x \varphi\}\, dxdt + \int u_0(x)\varphi(x,0)\, dx = 0$$

holds for every smooth test function $\varphi \geq 0$ which vanishes for all large enough $|x|$ and t. See [V3] for a discussion of those equivalent alternatives. Viscosity solutions have been discussed in [CV].

THEOREM 2.1. *Problem (CP) is well-posed in the framework of strong solutions. Moreover, the maps $S_t : u_0 \mapsto u(t)$ are order-preserving contractions on $\mathcal{X}_0 = L^1_+(\mathbb{R}^n)$. More precisely,*

$$\|(u_1(t) - u_2(t))_+\|_{L^1(\mathbb{R}^n)} \leq \|(u_1(0) - u_2(0))_+\|_{L^1(\mathbb{R}^n)}, \tag{2.2}$$

where $(\cdot)_+$ denotes positive part, $\max\{\cdot, 0\}$. *In particular, plain L^1-contraction holds*

$$\|u_1(t) - u_2(t)\|_{L^1(\mathbb{R}^n)} \leq \|u_1(0) - u_2(0)\|_{L^1(\mathbb{R}^n)}. \tag{2.3}$$

The Maximum Principle also follows from property (2.2).

Property (2.2) is called T-contraction in Bénilan's papers. Actually, the existence of a semigroup of T-contractions is true in the more general framework of signed solutions for the generalized PM equation $u_t = \Delta(|u|^{m-1}u)$, discussed in Section 20. Here are some additional important properties of the solutions.

PROPERTY 1. The solutions of problem (CP) satisfy the law of mass conservation

$$\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} u_0(x)\,dx \tag{2.4}$$

When $u \geq 0$ this means $\|u(t)\|_1 = \|u_0\|_1$ for all $t > 0$.

The property will also be true for solutions of any sign, but then it does not imply conservation of L^1-norm. It is also true for $0 < m < 1$ if $m \geq (n-2)/n$, but not below that value.

PROPERTY 2. The solutions are bounded for $t \geq \tau > 0$. Moreover, there exists a constant $C = C(m,n) > 0$ such that

$$0 \leq u(x,t) \leq C\,\|u_0\|_1^{2\beta}\,t^{-\alpha}\,. \tag{2.5}$$

PROPERTY 3. Energy estimates. Another aspect of the regularization of the solutions in time is obtained by multiplying the equation by u^m and formally integrating by parts. We arrive at

$$\int_{\mathbb{R}^n} u^{m+1}(x,t)\,dx + (m+1)\int_\tau^t \int_{\mathbb{R}^n} |\nabla(u^m)|^2\,dxdt \leq \int_{\mathbb{R}^n} u^{m+1}(x,\tau)\,dx \tag{2.6}$$

for all $0 < \tau < t$. Since we know by the previous properties that $u(\tau) \in L^p(\mathbb{R}^n)$ for all $p > 1$, in particular $p = m+1$, we conclude that ∇u^m is uniformly bounded in

$L^2(\mathbb{R}^n \times (\tau, t))$ in terms of the mass of the initial data. The justification of the calculation can be found in [V4]. In the same spirit, multiplication by $(u^m)_t$ and integration by parts gives

$$\frac{8m}{(m+1)^2}\int_\tau^t \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial t} u^{(m+1)/2}(x,t)\right)^2 dxdt + \int_{\mathbb{R}^n} |\nabla u^m(x,t)|^2\, dx \le \int_{\mathbb{R}^n} |\nabla u^m(x,\tau)|^2\, dx\,. \tag{2.7}$$

Combining with the previous one, it gives a bound for $\partial(u^{(m+1)/2})/\partial t$ in $L^2(\mathbb{R}^n \times (\tau, t))$ in terms of the mass of the initial data, and a better bound for ∇u^m in $L^\infty(\tau, \infty : L^2(\mathbb{R}^n))$. These and other gradient estimates were developed by Bénilan, cf. [Be3].

The next estimate is due to Aronson-Bénilan [AB] and plays a big role in the study of the Cauchy problem for the PME.

PROPERTY 4. Fundamental regularity estimate and consequences. Any nonnegative solution of the Cauchy problem (CP) satisfies the estimate

$$\Delta(u^{m-1}) \ge -\frac{C}{t}, \qquad C = \frac{\alpha\,(m-1)}{m}. \tag{2.8}$$

This implies another interesting estimate: $u_t \ge -\alpha\, u/t$. Moreover, conservation of mass is equivalent to $\int u_t\, dx = 0$, so that the last estimate leads to

$$\int |u_t(x,t)|\, dx \le \frac{2\alpha}{t}\int u(x,t)\, dx\,. \tag{2.9}$$

The one-sided estimate (2.8) is exact precesily for the source-type solutions that play a key role in our theory. For a proof of all the above facts we refer to the text [V4] and its references. In the proof of better convergence we will use a further regularity result that can be found in [DB].

PROPERTY 5. Bounded solutions are uniformly Hölder continuous for $t \ge \tau > 0$.

Ph. Bénilan devoted much time to finding the best Hölder exponent and his Kentucky Notes are a proof of his efforts. In one dimension the answer is $\alpha = \min\{1, 1/(m-1)\}$, but the question is not completely settled for $n > 1$.

PROPERTY 6. Finite propagation property. If the initial function u_0 is compactly supported so are the functions $u(\cdot, t)$ for every $t > 0$. Under these conditions there exists a free boundary or interface which separates the regions $\{(x,t) \in Q : u(x,t) > 0\}$ and $\{(x,t) \in Q : u(x,t) = 0\}$. This interface is usually an n-dimensional hypersurface in $\mathbb{R}^{n+1}$.

Let us conclude this section by pointing out that the source-type solutions $\mathcal{U}(x, t; C)$ are strong solutions of (0.1), but, strictly speaking, they are **not** solutions of problem (CP) because they do not take L^1 initial data. Indeed, it is easy to check that $\mathcal{U}$ converges to a Dirac mass as $t \to 0$ (this is the reason for the name "source-type solutions"). We will use this fact strongly in the first proof of asymptotic convergence, cf. Section 5. The reader should note that the behaviour of a whole class of solutions of equation (0.1) is described in terms of a simple family of functions which *are not* solutions in the same class, but in a larger class. More precisely, the special solutions which represent the whole dynamics at the asymptotic level exhibit a *singularity* (at $x = 0$, $t = 0$). This is a curious and quite general feature in the theory of asymptotic analysis.

3. Outline of the next sections. Four-step method

Before proceeding with the actual proofs we explain the "four-step method" as a general plan to prove asymptotic convergence in the form written down by Kamin and Vazquez in 1988. The first three steps are the common basis of the different proofs of convergence using similarity techniques and are worked out in detail in Section 4.. They allow to produce out of the original orbit $u(t)$ a family of rescaled orbits $\widetilde{u}_\lambda(t)$, which upon passage to the limit $\lambda \to \infty$ produces one or several **limit orbits** or limit solutions, $U(t)$. These orbits represent the asymptotic dynamics that we want to study. The proof of the convergence results is then reduced to the identification of (a unique) limit orbit $U(t)$ and the description of the mode and rates of convergence.

The four-step method, which is a general procedure for the study of many asymptotic problems, can be described as follows.

STEP 1. Rescaling. It produces out of an orbit $u(t)$ a family of *zoomed orbits*, $\{\widetilde{u}_\lambda(t)\}$. The orbit is viewed as a map $u(t) : t \mapsto u(\cdot, t)$ from $[0, \infty)$ into a suitable functional space.

STEP 2. Estimates and compactness. Appropriate estimates allow to show that if we choose the correct rescaling in Step 1, we obtain a family of uniformly compact orbits (in suitable functional spaces). Failure of guessing the correct scaling size produces rescaled orbits that grow to infinity or go to zero. In either case the method stops.

STEP 3. Passing to the limit. We replace the limit $t \to \infty$ by the limit in the scaling parameter λ for fixed t. We obtain one or several limit orbits $U(t)$.

STEP 4. Identifying the limit. The function U obtained as a limit of the orbits in Step 3 has to be identified as a solution of an equation, usually (but not necessarily) the same equation we started with. Some additional characteristics allow to determine the limit function in a unique way.

There is a final mini-step, 5. Rephrasing the result, that consists in undoing the λ-transformation and stating the result in the original variables.

We will perform all these steps for the Cauchy problem for the PME. The first three steps are common to all proofs below and, actually, these steps are applicable to a number of problems sharing the two properties of scale-invariance and regularity (smoothness) of the orbits. It is very important to make clear at this moment that the limit obtained after Step 3 must be nontrivial: neither zero nor infinity. Otherwise (and this happens quite often in preliminary analyses), we are missing the correct size of the asymptotic process and (probably) no relevant information is obtained from the analysis.

It is at the level of Step 4 that we can find very different ways of identifying the limit obtained in Step 3. Each of the different techniques will be discussed separately.

In Sections 5, 6 below we give the first proof of Theorem 1.1 by using the characterization of the limit solutions $U(t)$ through their initial data, which is shown to be a Dirac mass. This idea was explained in the paper [KV1].

In Section 7 the basic L^1 asymptotic result is improved into uniform convergence, which is the second part of Theorem 1.1; the result was announced in [V5] and appears here with a detailed proof using the *local regularizing effect*. We then estimate the asymptotics of supports and interfaces for compactly supported solutions, Section 9.

In Section 10 we introduce the *continuous rescaling* as an alternative scaling method that is used sometimes with advantage over the standard *fixed-rate rescaling* (i.e., the λ-scaling). This part of the notes concludes with the proof of optimality of the convergence rates, a question that has been a debated issue among experts: for restricted classes of data the rates can be improved, but not for the whole class $\mathcal{X}_0$ (even if we take smooth data the answer is still negative, it depends on the behaviour at infinity).

Subsequent sections are devoted to present alternative proofs of the main result: first, we have two methods based on the existence of a Lyapunov functional and the use of the Invariance Principle. Lyapunov methods are probably the most effective and popular tools in the study of asymptotic problems. Section 14 presents the method of lower barriers used by Friedman and Kamin [FK], a kind of nonstandard Lyapunov functional.

Afterwards, we turn to the ideas of asymptotic symmetry which allow to reduce the study of asymptotic dynamics to conditions of radial symmetry in the spatial variable, Section 15. There are then methods which use the special properties of one-dimensional problems. We present two of them, the method of Concentration Comparison, Section 16, and the method of Intersection Comparison, Section 17. For radial solutions good rates of convergence are obtained, Theorems 16.4 and 17.3.

The rest of the notes contains the analysis of further progress as mentioned at the end of Section 1. New results are presented in the three outlined directions.

4. The first steps

1. Rescaling. In order to observe the asymptotic behaviour of the orbit of problem (CP) we rescale it according to the Barenblatt exponents. Let us see the whole story of scaling transformations in some detail. Let $u = u(x, t)$ be a solution of (0.1). We apply the group of dilations in all the variables

$$u' = Ku, \qquad x' = Lx, \qquad t' = Tt, \tag{4.1}$$

and impose the condition that u' so expressed as a function of x' and t', i.e.,

$$u'(x', t') = Ku\left(\frac{x'}{L}, \frac{t'}{T}\right), \tag{4.2}$$

has to be again a solution of (0.1). Then:

$$\frac{\partial u'}{\partial t'} = \frac{K}{T}\frac{\partial u}{\partial t}\left(\frac{x'}{L}, \frac{t'}{T}\right), \qquad \Delta_{x'}(u')^m = K^m L^{-2}\Delta_x(u^m)\left(\frac{x'}{L}, \frac{t'}{T}\right).$$

Hence, (4.2) will be a solution if and only if $KT^{-1} = K^m L^{-2}$, i.e.,

$$K^{m-1} = L^2 T^{-1}. \tag{4.3}$$

We thus obtain a two-parametric transformation group acting on the set of solutions of (0.1). Choosing as free parameters L and T it can be written as

$$u'(x', t') = L^{\frac{2}{m-1}} T^{\frac{-1}{m-1}} u(x, t) = \left(\frac{L^2}{T}\right)^{\frac{1}{m-1}} u\left(\frac{x'}{L}, \frac{t'}{T}\right).$$

Using standard letters for the independent variables and putting $u' = \mathcal{T}u$, we get:

$$(\mathcal{T}u)(x, t) = L^{\frac{2}{m-1}} T^{-\frac{1}{m-1}} u\left(\frac{x}{L}, \frac{t}{T}\right). \tag{4.4}$$

Moreover, we can use one of the parameters to force $\mathcal{T}$ to preserve some important behaviour of the orbit. Here we recall that $\mathcal{U}_M(x, t)$ has a constant mass; actually, this characterizes uniquely the solution (which is the ideal orbit we want to approach). Imposing thus the condition of mass conservation at $t = 0$ we get

$$\int_{\mathbb{R}^n} (\mathcal{T}u_0)(x)\, dx = \int_{\mathbb{R}^n} u_0(x)\, dx, \tag{4.5}$$

namely,

$$\int_{\mathbb{R}^n} Ku_0\left(\frac{x}{L}\right) dx = \int_{\mathbb{R}^n} u_0(x)\, dx.$$

It easily follows that $KL^n = 1$. This and (4.2) give the expressions

$$K = T^{-\alpha}, \qquad L = T^{\beta}, \tag{4.6}$$

with the exponents given by (1.3). The transformation we are going to use is finally

$$(\mathcal{T}u)(x,t) = T^{-\alpha}u(x/T^{\beta}, t/T).$$

It is convenient to write the scaling factor in terms of $\lambda = 1/T$. Then, the solution is

$$\widetilde{u}_{\lambda}(x,t) = (\mathcal{T}_{\lambda}u)(x,t) = \lambda^{\alpha}u(\lambda^{\beta}x, \lambda t) \tag{4.7}$$

with initial data $\widetilde{u}_{0,\lambda}(x) = (\mathcal{T}_{\lambda}u_0)(x) = \lambda^{\alpha}u_0(\lambda^{\beta}x)$.

Visualization. The scaled orbits at $t = 1$ are just a **zoomed** version of the original orbit at $t = \lambda$.

Property. The source-type solutions are invariant under the λ-rescaling, i.e. $\mathcal{U}_M(t) = \mathcal{T}_{\lambda}(\mathcal{U}_M(t))$.

2a. Uniform estimates. The family $\widetilde{u}_{\lambda}(t)$, $\lambda > 0$, is uniformly bounded in $L^1(\mathbb{R}^n)$ for t positive:

$$\int_{\mathbb{R}^n} \widetilde{u}_{\lambda}(x,t)dx = \int_{\mathbf{R^n}} \lambda^{\alpha}u(\lambda^{\beta}x, \lambda t)\,dx = \int_{\mathbb{R}^n} u(y, \lambda t)\,dy = M < \infty. \tag{4.8}$$

Using now (2.5) we get

$$\|\widetilde{u}_{\lambda}(\cdot,1)\|_{\infty} = \lambda^{\alpha}\|u(\cdot,\lambda)\|_{\infty} \le \lambda^{\alpha}\frac{M^{2\alpha/n}}{\lambda^{\alpha}}C = CM^{2\alpha/n} \tag{4.9}$$

independently of λ, and in the same way

$$\|\widetilde{u}_{\lambda}(\cdot,t_0)\|_{\infty} = \lambda^{\alpha}\|u(\cdot,t_0\lambda)\|_{\infty} \le \lambda^{\alpha}\frac{M^{2\alpha/n}}{\lambda^{\alpha}t_0^{\alpha}}C = CM^{2\alpha/n}\,t_0^{-\alpha}. \tag{4.10}$$

Control of the norms $\|\cdot\|_1$ and $\|\cdot\|_{\infty}$ means control of all norms $\|\cdot\|_p$ for all $p \in [1,\infty]$. Thus, $\|\widetilde{u}_{\lambda}(\cdot,t)\|_p$ are equi-bounded for all $p \in [1,\infty]$.

Next, we take $t_0 > 0$ so that, by the regularizing effect, $u(t_0) \in L^{m+1}(\mathbb{R}^n)$. The energy estimates of Section 2 give

$$\int_{\mathbb{R}^n} |\nabla\widetilde{u}_{\lambda}^m(x,t)|^2dx \le C(t_0, \|\widetilde{u}_{\lambda}(x,t_0)\|_{L^{m+1}}) \tag{4.11}$$

for $t \ge t_0 > 0$. Now, the $\|\widetilde{u}_{\lambda}(t)\|_{L^{m+1}}$ are equi-bounded, hence $\|\nabla\widetilde{u}_{\lambda}^m(x,t)\|_{L^2}$ are equi-bounded (for $t \ge t_0 > 0$).

Moreover, by (2.9) of Property 5, Section 2,

$$\left\|\frac{\partial\widetilde{u}_{\lambda}}{\partial t}(t)\right\|_{L^1} \le C\frac{\|\widetilde{u}_{\lambda}(t)\|_{L^1}}{t} \tag{4.12}$$

Using the same argument we conclude that the norms $\|(\widetilde{u}_\lambda)_t(t)\|_{L^1}$ are equi-bounded if $t \geq t_0 > 0$.

2b. Compactness. Let us recall the Rellich-Kondrachov Theorem. *Let Ω be a bounded domain with C^1 boundary. Then*

$$p < N \Rightarrow W^{1,p}(\Omega) \subset L^q(\Omega) \text{ for all } q \in [1, p^*), \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N},$$
$$p = N \Rightarrow W^{1,p}(\Omega) \subset L^q(\Omega) \text{ for all } q \in [1, +\infty),$$
$$p > N \Rightarrow W^{1,p}(\Omega) \subset C(\overline{\Omega}).$$

All these injections are compact. In particular, $W^{1,p}(\Omega) \subset L^p(\Omega)$ with compact injection for all $p \geq 1$. In $\Omega = \mathbb{R}^n$, the above injections are compact in local topology (convergence on compact subsets).

Let us now recall our situation for the family $\{\widetilde{u}_\lambda\}_{\lambda \geq 1}$ for $t \geq t_0 > 0$:

$$\widetilde{u}_\lambda(x,t) \in L^\infty_{x,t} \subset L^1_{loc}, \quad \frac{\partial \widetilde{u}_\lambda}{\partial t}(x,t) \in L^\infty_t(L^1_x) \subset L^1_{x,t}\ (t \in (t_0, t_1)),$$

and

$$\nabla_x \widetilde{u}^m_\lambda \in L^2_{x,t} \subset L^1_{x,t}.$$

All spaces in time are local in the sense that they exclude $t = 0$.

PROPOSITION 4.1. *The family $\{\widetilde{u}_\lambda\}_{\lambda>1}$ is relatively compact locally in $L^1_{x,t}$. Also the family $\{\widetilde{u}^m_\lambda\}_{\lambda>1}$.*

3. Passage to the limit. We can now take a sequence $\lambda_k \to \infty$ and assert that $\widetilde{u}_{\lambda_k}$ converges in $L^1_{loc}(Q)$ to some function U:

$$\lim_{\lambda\to\infty} \widetilde{u}_\lambda(x,t) = U(x,t). \tag{4.13}$$

We need to study the properties of such **limit functions** $U(x,t)$.

LEMMA 4.2. *Any limit U is a nonnegative weak and strong solution of* (0.1) *satisfying uniform bounds in $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ for all $t \geq \tau > 0$.*

Proof. It is clear that, as a consequence of the passage to the limit U is nonnegative. Also, $U(t)$ is uniformly bounded in L^1 and L^∞ for $t \geq t_0 > 0$, according to formulas (2.4), (2.5). In order to check that it is a weak solution we review the sense in which $\widetilde{u}_\lambda$ is a weak solution:

$$\iint \{\widetilde{u}_\lambda \varphi_t - \nabla(\widetilde{u}^m_\lambda) \cdot \nabla\varphi\}\, dxdt + \int \widetilde{u}_{0\lambda}(x)\varphi(x,0)\, dx = 0$$

for all φ test. We have already remarked that our uniform estimates are not good near $t = 0$. In view of this, we restrict the test functions to the class

$$\varphi \in C_0^\infty(\mathbb{R}^n \times (0, \infty)),$$

so that φ vanishes in a neighborhood of $t = 0$. Then

$$\iint \{\widetilde{u}_\lambda \varphi_t - \nabla \widetilde{u}_\lambda^m \nabla \varphi\} dxdt = 0. \tag{4.14}$$

With our estimates

$$\begin{cases} \widetilde{u}_\lambda \to U & \text{locally in } L^1_{x,t} \\ \widetilde{u}_\lambda \to U & \text{weak}^* \text{ in } L^\infty_{loc} \\ \nabla \widetilde{u}_\lambda^m \to \nabla U^m & \text{in } L^2_{x;t,loc} \text{weak,} \end{cases}$$

we may pass to the limit in this expression (along a subsequence $\lambda_n \to \infty$) to get

$$\iint \{U \varphi_t - \nabla_x U^m \cdot \nabla_x \varphi\}\, dxdt = 0. \tag{4.15}$$

This means that U is a weak solution of equation (0.1). In fact, if $\tau > 0$

$$\int_\tau^\infty \int_{\mathbf{R^n}} \{U \varphi_t - \nabla_x U^m \cdot \nabla_x \varphi\}\, dxdt + \int U(x, \tau)\varphi(x, \tau)\, dx = 0.$$

□

5. Identification of the limit. Compact support

Thus far, we have posed the dynamics in the form of an initial value problem and we have introduced a method of rescaling which has allowed to obtain, after passage to the limit, one or several new solutions of the original problem. These solutions, that we call the **asymptotic dynamics**, form a special subset of the set of all orbits of our dynamical system and represent the (scaled) asymptotic behaviour of the original orbits. Their complete description becomes our main problem. The asymptotic dynamics turns out to be quite simple in the present case. The general framework in which this kind of ideas are studied is that of Dynamical Systems invariant under groups of transformations.

We proceed next with Step 4, i.e., the identification of the limit, in the form devised in [KV1]. Several other options will be examined later. We want to prove that the limit U along any sequence $\lambda_n \to \infty$ is necessarily $\mathcal{U}_M$. Both U and $\mathcal{U}_M$ are solutions of the PME for $t > 0$, enjoying a number of similar bounds. In order to identify them we only need to check their initial data and use a suitable uniqueness theorem for the Cauchy problem (CP). The necessary uniqueness theorem is available thanks to M. Pierre's work [Pi].

THEOREM 5.1. *Weak solutions of the PME in the class* $u \in C((0,\infty) : L^1(\mathbb{R}^n))$, $u \geq 0$, *which take a bounded and nonnegative measure* $\mu(x)$ *as initial data in the sense that*

$$\lim_{t\to 0}\int_{\mathbb{R}^n} u(x,t)\varphi(x)\,dx = \int_{\mathbb{R}^n}\varphi(x)\,d\mu(x), \tag{5.1}$$

for all $\varphi \in C_b(\mathbb{R}^n)$, $\varphi \geq 0$, *are uniquely determined by the initial measure.*

Let us then worry about the initial data. At first sight it looks easy:

LEMMA 5.2. *If* $\lambda \to \infty \Rightarrow \lim \widetilde{u}_{0,\lambda}(x) \to M\delta(x)$ *in the sense of bounded measures.*

Proof. As $\lambda \to \infty$, since $\alpha = n\,\beta > 0$,

$$\int_{\mathbb{R}^n}\widetilde{u}_{0\lambda}(x)\,\varphi(x)\,dx = \int_{\mathbb{R}^n}\lambda^{\alpha}u_0(\lambda^{\beta}x)\varphi(x)\,dx = \int_{\mathbb{R}^n}u_0(y)\varphi(y/\lambda^{\beta})\,dy,$$

which converges to $\int_{\mathbf{R^n}} u_0(y)\,\varphi(0)\,dy$ for all $\varphi \in C_c^{\infty}(\mathbb{R}^n)$, $\varphi \geq 0$. We have used the mass value: $\int_{\mathbb{R}^n} u_0(y)\,dy = M$. □

The problem of the double limit. Unfortunately, the fact that the initial data for $\widetilde{u}_\lambda$ converge to $M\,\delta(x)$ does not justify by itself that $U(t)$ takes initial data $M\,\delta(x)$, because we do not control the evolution of the $\widetilde{u}_\lambda$ near $t = 0$ in a uniform way and a discontinuity might be taking place near $t = 0$ in the limit $\lambda \to \infty$. This is a typical case of double limits,

$$\lim_{t\to 0}\lim_{\lambda\to\infty}\widetilde{u}_\lambda(x,t) = \lim_{\lambda\to\infty}\lim_{t\to 0}\widetilde{u}_\lambda(x,t)\,?$$

Preparing for a correct analysis, the first thing to do is to check that U and $\mathcal{U}_M$ have the same mass, i.e., that U has mass M. Since

$$\int_{\mathbb{R}^n}\widetilde{u}_\lambda(x,t)\,dx = M,$$

and $\widetilde{u}_{\lambda_k}$ converges to U in $L^1_{x,t}$-strong locally, we have $\widetilde{u}_{\lambda_k}(t) \to U(t)$ for a.e. t in $L^1_x(\mathbb{R}^n)$ locally and a.e. in $(x,t) \in Q$. By Fatou's Lemma

$$\int_{\mathbb{R}^n} U(t)\,dx \leq \lim_{k\to\infty}\int_{\mathbb{R}^n}\widetilde{u}_{\lambda_k}(x,t)\,dx = M,$$

hence the mass is equal or less. We have met again a difficulty. This difficulty is in principle essential. There are examples for rather simple equations in the nonlinear parabolic area where the initial data are not trivial but the whole solution disappears in the limit! Should such 'disaster' happen, we talk about an *initial layer of discontinuity*, an interesting object of study.

Compactly supported solutions. Here the only way the discontinuity can happen is by mass escaping to infinity, since there is only a mechanism at play, diffusion. In view of this difficulty we change tactics and try to establish the result under additional hypothesis:

• **Extra assumption.** *We take u_0 a bounded, $0 \le u_0 \le C$, and compactly supported function,* $\operatorname{supp}(u_0) \subset B_R(0)$.

Then, $\operatorname{supp}(\widetilde{u}_{0\lambda}) \subset B_{R/\lambda^\beta}(0)$. Moreover, there exists a source-type solution of the form $V(x,t) = \mathcal{U}_{M'}(x, t+1)$ with $M' >> M$ such that $V(x,0) = \mathcal{U}_{M'}(x,1) \ge u_0(x)$. Then,

$$\widetilde{u}_\lambda(x,0) = \lambda^\alpha u_0(x\lambda^\beta, 0) \le \lambda^\alpha \mathcal{U}_{M'}(x\lambda^\beta, 1) = \mathcal{U}_{M'}\left(x, \frac{1}{\lambda}\right),$$

where in the last equality we have used the invariance of $\mathcal{U}$ under $\mathcal{T}_\lambda$. We conclude from the Maximum Principle that

$$\widetilde{u}_\lambda(x,t) \le \mathcal{U}_{M'}\left(x, t+\frac{1}{\lambda}\right), \tag{5.2}$$

and in the limit $U(x,t) \le \mathcal{U}_{M'}(x,t)$. The bound solves all our problems since it implies that the support of the family $\{\widetilde{u}_\lambda(t)\}$ is uniformly small for all λ large and t close to zero. Indeed, we observe the relation between the radii of the supports of a solution and its rescaling:

$$R_\lambda(t) = \frac{1}{\lambda^\beta} R(\lambda t), \tag{5.3}$$

It follows that the support of $\widetilde{u}_\lambda(t)$ is contained in a ball of radius

$$R = C\,(M')^{(m-1)\beta}\left(t+\frac{1}{\lambda}\right)^\beta \tag{5.4}$$

with $C = C(m,n)$. Now we can proceed.

LEMMA 5.3. *The limit U has mass M for all $t > 0$.*

This is a consequence of the Dominated Convergence Theorem since U is bounded above by a big source-type.

LEMMA 5.4. *Under the present assumptions on u_0 we have $U(x,t) \to M\delta(x)$ as $t \to 0$, i.e.,*

$$\lim_{t\to 0} \int_{\mathbb{R}^n} U(x,t)\varphi(x)\,dx = M\varphi(0) \tag{5.5}$$

for all test functions $\varphi \in C_c^\infty(\mathbb{R}^n)$.

Proof. Since $M = \int U(x,t)\,dx$ we have for $t > 0$

$$
\begin{aligned}
&|\textstyle\int U(x,t)\varphi(x) - M\varphi(0)\,dx| \le \int |U(x,t)\|\varphi(x) - \varphi(0)|dx \\
&\quad \le \textstyle\int_{|x|\le\delta} |U(x,t)\|\varphi(x) - \varphi(0)|dx + \int_{|x|>\delta} |U(x,t)\|\varphi(x) - \varphi(0)|dx \le (*)
\end{aligned}
$$

By continuity there exists $\delta > 0$ such that $|\varphi(x) - \varphi(0)| \le \varepsilon/2M$ if $|x| \le \delta$. Besides, φ is bounded so that

$$|\varphi(x) - \varphi(0)| \le 2C \quad (\varphi \in C_c^\infty)\,.$$

Since U vanishes for $|x| \ge \delta$ if t is small enough we get

$$(*) \le M\frac{\varepsilon}{2M} + 2C\int_{|x|>\delta} |U(x,t)| \le K\varepsilon\,.$$

□

Conclusion. Using the uniqueness result, Theorem 5.1, we can identify U. Hence, for $t = 1$ we have $\widetilde{u}_{\lambda_n}(x,1) \to \mathcal{U}_M(x,1)$ in $L^1_{loc}(\mathbb{R}^n)$. Now, the $\widetilde{u}_\lambda$ have compact support which is uniformly bounded in λ. It follows that

$$\widetilde{u}_{\lambda_n}(x,1) \to \mathcal{U}_M(x,1) \qquad \text{in } L^1\text{-strong.}$$

(We pass from local to global convergence). The limit is thus independent of the sequence λ_n. It follows that the whole family $\{\widetilde{u}_\lambda\}$ converges to $\mathcal{U}_{\mathcal{M}}$ as $\lambda \to \infty$.

5. Rephrasing the result. The argument has concluded, but we still have to write the conclusion in the original variables and scales. Let $F_M(x) = \mathcal{U}_M(x,1)$. We have just proved that

$$\lim_{\lambda\to\infty} \|\lambda^\alpha u(\lambda^\beta x, \lambda) - F_M(x)\|_{L^1} = 0,$$

which means with $y = \lambda^\beta x$ that

$$\lim_{\lambda\to\infty} \int \lambda^\alpha |u(y,\lambda) - \lambda^{-\alpha} F_M(y/\lambda^\beta)|\lambda^{-\beta n}\,dy = 0.$$

Noting that $\mathcal{U}_M(y,\lambda) = \lambda^{-\alpha} F_M(y/\lambda^\beta)$ and that $\alpha = \beta n$, we arrive at

$$\lim_{\lambda\to\infty} \int |u(y,\lambda) - \mathcal{U}_M(y,\lambda)|\,dy = 0,$$

i.e., replacing λ by t

$$\lim_{t\to\infty} \|u(y,t) - \mathcal{U}_M(y,t)\|_{L^1_y} = 0.$$

This is the asymptotic formula (1.5). It has been proved for the class of bounded and compactly supported initial data.

6. General initial data

We now extend the result from compactly supported initial data to the whole class of data u_0 satisfying (0.3) by a general density argument. Given $\varepsilon > 0$ we construct an approximation $\widetilde{u}_0$ which is bounded and compactly supported and such that

$$\|u_0 - \widetilde{u}_0\|_1 \le \varepsilon, \qquad \int_{\mathbb{R}^n} \widetilde{u}_0(x)\,dx = \widetilde{M}.$$

To prove formula (1.5) for $u(x,t)$ we only have to use the triangle formula plus the contraction property (2.3):

$$\|u(t) - \mathcal{U}_M(t)\|_1 \le \|u(t) - \widetilde{u}(t)\|_1 + \|\widetilde{u}(t) - \mathcal{U}_{\widetilde{M}}(t)\|_1 + \|\mathcal{U}_{\widetilde{M}}(t) - \mathcal{U}_M(t)\|_1.$$

Now, $|M - \widetilde{M}| \le \varepsilon$, hence $\|\mathcal{U}_{\widetilde{M}}(t) - \mathcal{U}_M(t)\|_1 \le \varepsilon$. By the contraction principle,

$$\|u(t) - \widetilde{u}(t)\|_1 \le \|u(0) - \widetilde{u}(0)\|_1 \le \varepsilon.$$

Thus, we get $\|u(t) - \mathcal{U}_M(t)\|_1 \le \varepsilon + \delta(t) + \varepsilon = 2\varepsilon + \delta(t)$, where $\delta(t) \to 0$ as $t \to \infty$ according to the result proved for the special solutions. As $t \to \infty$ we get

$$\lim_{t\to\infty} \|u(t) - \mathcal{U}_M(t)\|_1 \le 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary this completes the proof of the L^1 estimate in Theorem 1.1.

Comment on the method. The method we have used so far in the proof of Theorem 1.1 can be applied to different equations and systems as long as they possess good scaling properties that are relevant for the asymptotics, and as long as the identification step has some nice characteristic which enables us to determine the solution obtained as limit. We note that *no essential use is made of the Maximum Principle*, which is replaced as a main argument by compactness. This makes the method in principle well suited for systems and higher-order equations.

Alternative. In fact, we have used the maximum principle and the property of finite propagation in Step 4 in the case of compact support. This can be avoided with no much effort. We give next a direct proof of Step 4 suggested by the referee: the main point of the argument to show that the initial data are taken is to make sure that there is no mass escaping to infinity. Such a property is ensured by the following result.

LEMMA 6.1. *Let u be a solution with data $u_0 \in \mathcal{X}_0$. For $T, R > 0$ we have*

$$\lim_{R\to\infty} \sup_{\lambda\ge 1} \sup_{t\in(0,T)} \int_{\{|x|\ge R\}} \widetilde{u}_\lambda(x,t)\,dx = \lim_{t\to 0,\lambda\to\infty} \int_{\{|x|\ge R\}} \widetilde{u}_\lambda(x,t)\,dx = 0. \tag{6.1}$$

Proof. Take a C^∞ cutoff function $\varrho = \varrho(|x|)$ such that $0 \le \varrho \le 1$, $\varrho(x) = 0$ for $|x| \le 1$, $\varrho(x) = 1$ for $|x| \ge 2$. Let $\varrho_R(x) = \varrho(x/R)$. Multiplying the equation by ϱ_R and integrating by parts we get

$$\frac{d}{dt}\int \varrho_R \widetilde{u}_\lambda(t)\,dx \le \frac{1}{R^2}\|\Delta\varrho\|_\infty \|\widetilde{u}_\lambda\|_\infty^{m-1} M \le \frac{C(m,n,\varrho,M)}{R^2}\, t^{-\alpha(m-1)},$$

by Property 2 of Section 2. We have $\alpha(m-1) < 1$, hence integration in time from 0 to $t \le T$ gives

$$\int \varrho_R \widetilde{u}_\lambda(t)\,dx \le \int \varrho_R \widetilde{u}_\lambda(0)\,dx + \frac{C(m,n,\varrho,M)}{R^2}\, T^{2\beta}.$$

Moreover, for $\lambda \ge 1$, $\int \varrho_R(x)\widetilde{u}_\lambda(x,0)\,dx = \int \varrho(x/R\lambda^\beta)u_0(x)\,dx \le \int_{|x|\ge R\lambda^\beta} u_0(x)\,dx$, which goes uniformly to zero as $R\lambda^\beta \to \infty$. The two results follow. □

With this result it is easy to check that the limit U has mass M for all $t > 0$, and with it the identification of Step 4 follows.

7. Uniform convergence. Local regularizing effect

Once the basic L^1- convergence result is proved, we turn next to the question of improved convergence. We will show that the asymptotic convergence takes actually place in L^∞, i.e., uniformly. The improvement of convergence is a consequence of the smoothness of the flow, in more precise form, of the local regularity properties which apply to our problem.

We know that the family $\{\widetilde{u}_\lambda\}$ is bounded for all $t \ge t_0 > 0$. The local regularity theory [DB] says that

LEMMA 7.1. *Equi-bounded families of solutions of the PME are equi-continuous with a uniform Hölder modulus of continuity, i.e., they are bounded in some space $C^\varepsilon(Q)$, $\varepsilon > 0$.*

This is a local result. When we apply it to the family $\{\widetilde{u}_\lambda\}$ it says that it is a bounded family in $C^\varepsilon(\Omega)$ for every compact subdomain of Q with

$$\|\widetilde{u}_\lambda\|_{C^\varepsilon} \le C(\|\widetilde{u}_\lambda\|_\infty, \Omega) \tag{7.1}$$

(where $\|\cdot\|_{C^\varepsilon}$ is the Hölder norm of the space $C^\varepsilon(\Omega)$). The same result applies in a bounded space-time domain and then C depends on the distance from Ω to the boundary of the total domain. We recall the usual definition of the Hölder semi-norm in parabolic domains

$$[u]_\varepsilon = \sup_{(x,t),(x',t')\in\Omega} \frac{|u(x,t)-u(x',t')|}{\|x-x'\|^\varepsilon + |t-t'|^{\varepsilon/2}}$$

The space is also written as $C^{\varepsilon,\varepsilon/2}_{x,t}$. Returning to our family, boundedness in the Hölder space implies equi-continuity. By the Ascoli-Arzelà theorem this implies compactness in the uniform norm, but on compact domains. We thus have

$$\widetilde{u}_\lambda(x,1) \to \mathcal{U}_M(x,1) \tag{7.2}$$

uniformly on compact subsets of Q. We again divide the end of the proof in two cases, according to their difficulty.

I. If we consider as before the subclass of solutions with bounded and compactly supported data we know that for $t = 1$ the $\widetilde{u}_\lambda$ have uniformly bounded supports, hence the convergence (7.2) is uniform in $x \in \mathbb{R}^n$. Rephrasing the result as before, we get

$$t^\alpha \|u(t,y) - t^{-\alpha} F_M(t^{-\beta} y)\|_\infty \to 0, \ \ t \to \infty,$$

which is the asymptotic formula (1.6). The uniform convergence of Theorem 1.1 is proved in this case.

II. For general u_0 things are not so simple. Arguing as before we know that

(i) at $t = 1$ the rescaled family $\widetilde{u}_\lambda(x,1)$ converges in $L^1(\mathbb{R}^n)$ to $\mathcal{U}_M$,
(ii) this convergence takes also place in the uniform norm on any ball $B_R(0) \subset \mathbb{R}^n$,
(iii) the same happens for every $t \in (1/2, 2)$ uniformly in time.

As a consequence, we get the following picture. Take a very large radius $R_1 \gg 1$, in particular larger than the radius of the support of $\mathcal{U}_M(x,1)$. In the time interval $1/2 < t < 1$ we have uniform convergence of $\widetilde{u}_\lambda$ towards $\mathcal{U}_M$ in the ball of radius R_1. Now we have to examine the outer region, $\mathcal{O}_1 = \{|x| \ge R_1\}$. We know that $u_\lambda \ge \mathcal{U}_M$ because $\mathcal{U}_M$ vanishes identically there. Moreover, the mass $\int_{\mathcal{O}_1} \widetilde{u}_\lambda(x,t)\,dx$ is less than ε in that outer domain: the reason is that $\widetilde{u}_\lambda$ and $\mathcal{U}_M$ have the same total mass and we have shown that they are almost identical for $|x| \le R_1$. Clearly, $\varepsilon \to 0$ as $\lambda \to \infty$. Under these circumstances we want to prove that there is a function $C(\varepsilon)$ with $C(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\widetilde{u}_\lambda(x,1) \le C(\varepsilon) \qquad \text{for } |x| \ge R_1, \tag{7.3}$$

and the proof of Theorem 1.1 will be complete. In other words, we want to translate small L^1-norms into small L^∞-norms. The technical tool to do that is the following result that has an interest in itself.

LEMMA 7.2. (Local regularizing effect) *Let f be any nonnegative, smooth, bounded and integrable function in $B_1 = B_R(a) \subset \mathbb{R}^n$ such that*

$$\Delta(f^p) \ge -K$$

for some $p > 0$ *and* $K > 0$. *Let* $B_2 = B_{R/2}(a)$. *Then* $f \in L^\infty(B_2)$ *and* $\|f\|_{L^\infty(B_2)}$ *depends only on p, K, n, R and* $\|f\|_{L^1(B_1)}$. *If* $\|f\|_1$ *is very small compared with R it takes the form*

$$\|f\|_{L^\infty(B_2)} \le C(p,n)\, \|f\|^{\rho}_{L^1(B_1)}\, K^\sigma, \tag{7.4}$$

with $\rho = 2/(2+pn)$ *and* $\sigma = n/(2+pn)$.

The exact condition for (7.4) to hold is $\|f\|_1^{\rho\, p/2} \ll RK^{\rho/2}$. In the application B_1 is any ball contained in $\{|x| \ge R_1\}$, and we take

$$f = u(t), \quad p = m-1, \quad K = \frac{\alpha\,(m-1)}{mt}$$

in the case where the solution u is positive everywhere, hence smooth. The general case is done by approximation. This completes the proof of Theorem 1.1.

Observe that the LRE implies the regularizing effect (2.5) when in the application we take $R = \infty$.

8. A proof of the local regularizing effect

This is a kind of technical appendix that the reader may wish to jump in a first reading.

(i) To begin with, we may use scaling to reduce the number of parameters. Thus, if f satisfies the assumptions in the ball B_1 of radius R and center a, with constant K and integral $\|f\|_{L^1(B_1)} \le M$, then

$$f_1(x) = A\, f(Rx+a), \quad A^p = \frac{1}{KR^2},$$

satisfies the same assumptions with $R = K = 1$ and $a = 0$. Therefore, only this case must be considered, bearing in mind the transformation and the fact that

$$\|f_1\|_{L^1(B_1(0))} \le M_1 = MK^{-1/p} R^{-(np+2)/p}.$$

(ii) Under those assumptions, let $g(x) = f^p(x)$. Then $\Delta g \ge -1$. Therefore, for every $x_0 \in \mathbb{R}^n$ the function

$$G(x) = g(x) + \frac{1}{2n}|x-x_0|^2$$

is subharmonic in $B_1 = B_1(0)$. It follows that

$$G(x_0) \le \fint_B G(x)\, dx,$$

where B is any ball $B_r(x_0)$, $r > 0$ contained in B_1, and $\fint_B$ denotes average on B. The argument will continue in a different way for $p > 1$ and for $0 < p \le 1$.

(iii) If $p \le 1$, we can use the last formula to estimate g at an arbitrary point $x_0 \in B_{1/2}(0)$ as follows. If $B = B_r(x_0)$ and $0 < r \le 1/2$ then,

$$g(x_0) \le \fint_B g(x)dx + \frac{1}{2n}\fint_B |x-x_0|^2 dx \le \left(\fint_B g^{1/p}\,dx\right)^p + \frac{r^2}{2(n+2)},$$
$$\le \|f\|^p_{L^1(B_1)}\left(\frac{1}{\omega_n\, r^n}\right)^p + \frac{r^2}{2(n+2)}.$$

where ω_n denotes the volume of the unit ball. Minimization of the last expression with respect to $0 < r < 1/2$ gives the bound. In particular, when $\|f\|_1$ is small enough the minimum happens at an interior point of the r-interval and then

$$g(x_0) \le C\,\|f\|_1^{\frac{2p}{pn+2}},$$

Recall that this holds under the simplifying hypothesis, so that for $K, R \ne 1$ we have to undo the transformation. We ask the reader to check that it is really equivalent to (7.4).

iv) For $p > 1$ we proceed in two steps. In the first one we use the following result which can be easily established by repeated (but finite) iteration of the Moser integration technique.

LEMMA 8.1. *Let $G \ge 0$ be a locally integrable function in the ball $B = B_R(0)$ such that $\Delta G \ge 0$ and $G^{1/p} \in L^1(B)$ for some $p > 1$. Then*

$$\fint_{B_{R/2}} G\,dx \le C\left(\fint_B G^{1/p}\,dx\right)^p, \tag{8.1}$$

where $C > 0$ depends on p and n.

Isf we prove it for $R = 1$ then it is easily translated to a ball of radius $R \ne 1$ by scaling. From this it follows that $g = f^p \in L^1(B_r)$ for all $r < 1/2$, and in fact

$$\fint_{B_r} f^p\,dx \le C\left(\fint_{B_{2r}} f\,dx\right)^p + Cr^2 \tag{8.2}$$

with another $C = C(p,n)$. Then, for every $x_0 \in B_{1/2}(0)$ and $B = B_r(x_0)$ and $0 < r \le 1/4$ we have

$$g(x_0) \le \fint_B f^p(x)\,dx + \frac{1}{2n}\fint_B |x-x_0|^2 dx \le Cr^{-np}\|f\|_1^p + Cr^2.$$

The end is as before.

A related version of the local regularizing effect is due to [BCP]. The idea of the present proof was announced in [V4], Chapter 3.

9. Convergence of supports and interfaces

We assume in this section that u_0 is compactly supported and describe the asymptotic shape and size of the support as $t \to \infty$. We may assume without loss of generality that u_0 is continuous and nontrivial and that 0 belongs to the positivity set of u_0. We introduce the minimal and maximal radius,

$$\begin{cases} r(t) = \sup\{r > 0 : \quad u(x,t) > 0 \text{ in } B_r(0)\}, \\ R(t) = \inf\{r > 0 : \quad \text{supp}(u(x,t)) \subset B_r(0)\}. \end{cases} \tag{9.1}$$

Since the source-type solution $\mathcal{U}_M(x,t)$ is given by formula (1.2)–(1.3), its support is the ball of radius

$$\mathcal{R}(t) = \xi_0(m,n)(M^{m-1}\,t)^{\beta} = C_0\,t^{\beta}. \tag{9.2}$$

THEOREM 9.1. *As $t \to \infty$ we have*

$$\lim_{t\to\infty} \frac{r(t)}{\mathcal{R}(t)} = \lim_{t\to\infty} \frac{R(t)}{\mathcal{R}(t)} = 1. \tag{9.3}$$

Proof. The fact that the limits in (9.3) are equal or larger than 1 is a direct consequence of the uniform convergence of Theorem 1.1. On the contrary, the fact that for large t

$$R(t) \le (1+\varepsilon)\;\mathcal{R}(t)$$

needs a proof. Of course, we know that a large Barenblatt solution with some delay is a super-solution, hence there is a constant $C > 1$ such that for all large t

$$R(t) \le C\mathcal{R}(t).$$

On the other hand, we know that the mass contained in exterior sets of the form

$$\Omega_\varepsilon = \{|x| > (1+\varepsilon)\mathcal{R}(t)\}$$

is less than ε for all large t. By Lemma 7.2 there is uniform convergence to 0 in this region as $t \to \infty$, hence if the support is larger than the support of $\mathcal{U}$ the excess region takes the form of a *thin tail*.

We will show that the possible tail must disappear as time grows by means of a comparison with slow traveling waves. This is done as follows: if we define the ratio $s(t) = R(t)/t^{\beta}$, we must prove that

$$\limsup_{t\to\infty} s(t) = C_0.$$

Assume by contradiction that this limit is $C > C_0$ and take a very large time t_1 for which the ratio $s(t_1) \ge C - \varepsilon$, with ε very small. By scaling (4.7) we can reduce that time to

$t_1 = 1$. Since the ratio has limsup C we have $R(1/2) \le (C+\varepsilon)/2^\beta = d < C$. On the other hand, by the uniform convergence $u \to \mathcal{U}$, we may also assume that $u \le \varepsilon$ for $|x| \ge \mathcal{R}(1) + \varepsilon = C_0 + \varepsilon$, and $t \in [1/2, 1]$. Let $d_1 = \max\{d, C_0\}$, which we may take such that $d_1 < C - 4\varepsilon$ for ε small. Now, we compare u with the explicit traveling wave solution $\widehat{u}$ with small speed ε defined as

$$\widehat{u}^{m-1} = \frac{m-1}{m}\,(\varepsilon(t-1/2) + \varepsilon + d_1 - x_1)_+$$

where x_1 is the first coordinate of x. Comparison takes place in the region: $\{t \in [1/2, 1], x_1 \ge d_1\}$. By inspecting the parabolic boundary, we easily show that $u \le \widehat{u}$ there. Since $\widehat{u}$ vanishes for $x_1 \ge d_1 + \varepsilon + \varepsilon(t-1/2)$ we conclude that u vanishes at $t = 1$ for $x_1 \ge d + 2\varepsilon$. We may rotate the axes in the previous argument, hence we conclude that $u(x, 1) = 0$ for $|x| \ge d_1 + 2\varepsilon$ and this is a contradiction with $R(1) \ge C - \varepsilon$. The tail is eliminated. □

Theorem 9.1 is a manifestation of the property of **asymptotic symmetrization**, which will be discussed in greater detail in Sections 15 and following.

10. Continuous rescaling and stationary solutions

A different way of implementing the scaling of the orbits of problem (CP) and proving the previous facts consists of using the *continuous rescaling*

$$\theta(\eta, \tau) = t^\alpha u(x, t), \quad \eta = x\, t^{-\beta}, \quad \tau = \log(t), \tag{10.1}$$

with α and β the standard similarity exponents given by (1.3). The new orbit $\theta(\tau)$ satisfies the equation

$$\boxed{\theta_\tau = \Delta(\theta^m) + \beta\eta \cdot \nabla\theta + \alpha\theta} \tag{10.2}$$

It is bounded uniformly in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. The source-type solutions transform into the stationary profiles F_M in this transformation, i.e., $F(\eta)$ solves the nonlinear elliptic problem

$$\Delta f^m + \beta\eta \cdot \nabla f + \alpha f = 0. \tag{10.3}$$

The boundedness and compactness arguments of Section 4.3 apply and we may pass to the limit and form the ω-limit, which is the set

$$\omega(\theta) = \{f \in L^1(\Omega) : \exists \tau_j \to \infty \text{ such that } \theta(\tau_j) \to f\}. \tag{10.4}$$

The convergence takes place in the topology of the functional space in question, here any $L^p(\Omega)$, $1 \le p < \infty$ (strong). We have written a quite detailed account of this process in the Notes on the Dirichlet problem, [V5].

The rest of the proof consists in showing that the ω-limit is just the Barenblatt profile F_M. The argument can be translated in the following way. Corresponding to the sequence of scaling factors λ_n of Section 4 we take a sequence of **delays** s_n and define

$$\theta_n(\eta,\tau)=\theta(\eta,\tau+s_n). \tag{10.5}$$

The family $\{\theta_n\}$ is precompact in $L^\infty_{loc}(0,\infty:L^1(\mathbb{R}^n))$ hence, passing to a subsequence if necessary we have

$$\theta_n(\eta,\tau)\to\widehat{\theta}(\eta,\tau). \tag{10.6}$$

Again it is easy to see that $\widehat{\theta}$ is a weak solution of (10.2) satisfying the same estimates. The end of the proof is identifying it as a stationary solution, which was done in the previous proof by the other scaling method (*discrete rescaling*).

Theorem 1.1 can now be used to characterize the stationary solutions.

THEOREM 10.1. *The profiles F_M can be characterized as the unique solutions of equation* (10.3) *such that $f\in L^1(\mathbb{R}^n)$, $f^m\in L^1_{loc}(\mathbb{R}^n)$ and $f\geq 0$. The conditions $f^m\in H^1(\mathbb{R}^n)$, $f\in C(\mathbb{R}^n)$ are true, but not needed in the proof.*

Proof. Any other solution f can be taken as initial data for the evolution equation (10.2) and then Theorem 1.1 proves that the corresponding solution of (10.2) converges to the source-type solution with the same mass, F_M. Now, the solution $u(x,t)=t^{-\alpha}f(x\,t^{-\beta})$ is an admissible solution of the PME which converges in the rescaling to f. Therefore, $f=F_M$. ☐

11. Optimality of the convergence rates in $\mathcal{X}_0$

We devote this section to a first exploration of the sharpness of the convergence rates, proving Theorem 1.3. A more precise formulation is as follows:

Counterexample. *Given any decreasing function $\rho(t)\to 0$, there exists a solution of the Cauchy Problem with integrable and nonnegative initial data of mass $M>0$ such that*

$$\lim_{t\to\infty} sub\ \frac{(u(0,t)-\mathcal{U}(0,t;M))\,t^\alpha}{\rho(t)}=\infty. \tag{11.1}$$

Moreover, we can also get

$$\lim_{t\to\infty} sub\ \frac{\|u(t)-\mathcal{U}(t;M)\|_1}{\rho(t)}=\infty. \tag{11.2}$$

We can also ask the solution to be radially symmetric with respect to the space variable.

Construction. (i) We recall that the proof need only be done for $M=1$ since the scaling transformation

$$\widehat{u}_c(x,t)=c^{m-1}\,u(x,c\,t) \tag{11.3}$$

reduces a solution of mass $M > 0$ to a solution of mass 1 if $c = M^{-1/(m-1)}$. We take an initial function of the form

$$u_0(x) = \sum_{k=1}^{\infty} c_k\, \chi_k(x - a_k),$$

where $\chi_k(x)$ is the characteristic function of the ball of radius r_k centered at 0. The sequences a_k, c_k and r_k have to be determined in a suitable way. In the first place, we impose the conditions $c_k, r_k \geq 0$ and $c_k\, r_k^n = 2^{-k}/\omega$ (where ω is the volume of the ball of radius 1). Then, $M = \omega \sum_1^\infty c_k\, r_k^n = 1$.

(ii) We construct solutions u_k with initial data of the form

$$u_k(x, 0) = \sum_1^k c_i\, \chi_i(x - a_i), \tag{11.4}$$

and we proceed to choose c_k and a_k in an iterative way. In any case the mass of u_k is $M_k = 1 - 2^{-k}$, and we observe that (by the main convergence result) for every $\varepsilon > 0$ there must be a time $t_k(\varepsilon)$ (which depends also on the precise choice of the initial data) such that

$$t^\alpha |u_k(0, t) - \mathcal{U}(0, t; M_k)| \leq \varepsilon$$

for all $t \geq t_k(\varepsilon)$. We now recall that $\mathcal{U}(0, t; M) = c\, M^{2\beta}\, t^{-\alpha}$, so that the difference between $t^\alpha \mathcal{U}(0, t; M)$ and $t^\alpha \mathcal{U}(0, t; M')$ is constant in time, and in fact it can be estimated as larger than

$$t^\alpha\, (\mathcal{U}(0, t; M) - \mathcal{U}(0, t; M')) \geq k_1\, (M - M')$$

with the same constant $k_1 > 0$ for all $1 \geq M > M' \geq 1/2$.

(iii) The iterative construction of the u_k starts as follows. We may take c_1 as we like, e.g., $c_1 = 1$, then $r_1 = (2\omega)^{-1/n}$, and find the solution $u_1(x, t)$ with data $u_1(x, 0) = c_1\, \chi_1(x)$. Its mass is $M_1 = 1/2$ for all times. As said above, for sufficiently large times we have

$$t^\alpha |u_1(0, t) - \mathcal{U}(0, t; M_1)| \leq \varepsilon\,,$$

We can also find t_1 such that $\rho(t_1) < (1/2)k_1\, (M - M_1) = k_1/4$. Using the estimate for the difference of source-type solutions and the triangular inequality, and taking ε small enough($\varepsilon \leq k_1/4$), we get for all $t \geq t_1$

$$\begin{aligned} t^\alpha |u_1(0, t) - \mathcal{U}(0, t; 1)| &\geq t^\alpha |\mathcal{U}(0, t; 1) \\ &\quad - \mathcal{U}(0, t; M_1)| - t^\alpha |u_1(0, t) - \mathcal{U}(0, t; M_1)| \qquad (11.5) \\ &\geq k_1(1 - M_1) - \varepsilon \geq k_1/4 \geq \rho(t_1) \geq \rho(t)\,. \qquad (11.6) \end{aligned}$$

(iv) Iteration step. Assuming that we have constructed $u_2, \ldots, u_{k-1}$ by solving the equation with data (11.4), we proceed to choose c_k, and a_k and construct u_k as follows. We can take any $c_k > 0$, and then find a_k large enough so that the support of the solution v_k with initial data $v_k(x, 0) = \chi_k(x - a_k)$ does not intersect the support of u_{k-1} until a time $t_k > 2\,t_{k-1}$ (and we can even estimate how far a_k must be located for large t_k because we have a precise control of the support of u_{k-1} for large times, thanks to Theorem 9.1). Then, it is immediate to see that

$$u_k(x, t) = u_{k-1}(x, t) + v_k(x, t)$$

for all $x \in \mathbb{R}^n$ and $0 \le t \le t_k$ (i.e., superposition holds as long as the supports are disjoint). Indeed, this means that for all $0 \le t \le t_{k-1}$ we also have $u_k(0, t) = u_{k-2}(0, t)$, and by iteration we conclude that

$$u_k(0, t) = u_j(0, t) \quad \text{for all } 1 \le j < k \text{ and } 0 \le t \le t_{j+1}.$$

We now remark that t_k can be delayed as much as we like (on the condition of taking a_k far away). If we choose t_k large enough, the main asymptotic theorem implies the behaviour

$$t_k^\alpha u_k(0, t_k) = t_k^\alpha u_{k-1}(0, t_k) \sim t_k^\alpha \mathcal{U}(0, t_k; M_{k-1})$$

We want the error to be less than $k_1(1 - M_k)/2 = 2^{-(k+1)}k_1$. We also suggest to wait until $\rho(t_k) \le 2^{-(2k+1)}\,k_1$. Using again the triangle inequality: $|u_k - \mathcal{U}(M)| \ge |\mathcal{U}(M) - \mathcal{U}(M_{k-1})| - |u_k - \mathcal{U}(M_{k-1})|$ with $M = 1$, we get

$$t_k^\alpha |u_k(0, t_k) - \mathcal{U}(0, t_k; 1)| \ge 2^{-(k+1)}k_1 \ge 2^k \rho(t_k).$$

(v) In the final step we take the limit

$$u(x, t) = \lim_{k\to\infty} u_k(x, t).$$

By what was said before we may conclude that for $t \le t_k$ we have $u(0, t) = u_k(0, t)$, so that

$$\lim_{n\to\infty} t_k^\alpha \frac{u(0, t_k) - \mathcal{U}(0, t_k; 1)}{\rho(t_k)} = \infty.$$

This concludes the proof of the L^∞-estimate.

(vi) The construction can be easily modified so that the data u_0 are radially symmetric by defining χ_k to be the characteristic function of the annulus $A_k = \{x : a_k \le |x| \le a_k + r_k\}$ and imposing that c_k times the volume of A_k to equal 2^{-k}. The construction is repeated with the same attention to be given to a_k, i.e., to the far location of the A_k.

(vii) For the L^1 part we just observe that, taking t_k large enough we have at time $t = t_k$ and in a very large ball B_k (as large as we please by the iteration construction) the equality $u = u_k$ and the approximation

$$\|u_k - \mathcal{U}(x, t_k; 1)\| \ge \|u_k - \mathcal{U}(x, t_k; 1)\|_{L^1(\{x \notin B_k\})} \ge 2^{-k},$$

since the mass of u contained outside this ball is known (2^{-k}), and that of $\mathcal{U}$ is zero there. The result follows if the t_k have been chosen as before, $\rho(t_k)2^k \to 0$.

12. Proof of Theorem 1.1 by the Lyapunov method

We devote the next sections to derive alternative proofs of the main convergence result, Theorem 1.1. Two of them are based on standard implementations of the idea of *Lyapunov functional*. The third one introduces a non-standard version of this idea.

12.1. Lyapunov functional. Given an orbit $\{u(t)\}$ with mass $M > 0$ we introduce the functional

$$J_u(t) = \int_{\mathbb{R}^n} |u(x,t) - \mathcal{U}_M(x,t)|\, dx. \tag{12.1}$$

It is clear from the Contraction Property that J is nonincreasing in t. We get the following result.

LEMMA 12.1. *There exists the limit* $J_\infty = \lim_{t\to\infty} J(t) \ge 0$.

Note that $J(t)$ becomes zero only if $u(t)$ coincides with the source-type solution for some $t_1 > 0$ and then the equality holds for all $t \ge t_1$ and the asymptotic result is trivial. Otherwise $J(t) > 0$ for all $t > 0$. We have to examine this case.

12.2. Limit solutions. We perform Steps 1, 2 and 3 of the preceding proof to obtain a sequence $\lambda_k \to \infty$ such that

$$\widetilde{u}_{\lambda_k}(x,t) \to U(x,t) \tag{12.2}$$

in $L^1(\mathbb{R}^n \times (t_1, t_2))$. The limit U is again a solution of the PME. It is nontrivial and has mass M (this is easy for compactly supported solutions and then true for the rest by approximation, as we saw).

12.3. Invariance Principle. One of the key features of the use of Lyapunov functionals is the following Asymptotic Invariance Property.

LEMMA 12.2. *The Lyapunov functional is constant on limit orbits, i.e.,* J_U *does not depend on* t.

Proof. The Lyapunov functional is translated to the rescaled family $\widetilde{u}_\lambda$ by the formula

$$J_{\widetilde{u}_\lambda}(t) = \int_{\mathbb{R}^n} |\widetilde{u}_\lambda(x,t) - \mathcal{U}_M(x,t)|\, dx = J_u(\lambda t). \tag{12.3}$$

It follows that for fixed $t > 0$ we have

$$\lim_{\lambda\to\infty} J_{\widetilde{u}_\lambda}(t) = \lim_{\lambda\to\infty} J_u(\lambda t) = J_\infty.$$

On the other hand, we see that J depends in a lower-semicontinuous form on u. Moreover, it is continuous under the passage to the limit that we have performed. That means that for every $t > 0$ we have $J_U(t) = J_\infty$. □

12.4. A limit solution is a source-type. In order to identify U, the next result we need is the following.

LEMMA 12.3. *Let be an orbit $u(t)$ with mass $M > 0$ and with connected support for $t \geq t_0$. Then the function $J(t)$ is strictly decreasing in any time interval (t_1, t_2), $t_0 < t_1 < t_2$, unless $u = \mathcal{U}_M$ or both solutions have disjoint supports in that interval.*

Proof. We consider for $t \geq t_1 > 0$ the solution w of the PME with initial data at $t = t_1$

$$w(x, t_1) = \max\{u(t_1), v(t_1)\}, \tag{12.4}$$

where we put $v = \mathcal{U}_M$ for easier notation. Clearly, $w \geq u$ and $w \geq v$, hence

$$w(t) \geq \max\{u(t), v(t)\}, \quad t > t_1.$$

Moreover, we have $w(x, t_1) - u(x, t_1) = (v(x, t_1) - u(x, t_1))_+$ and $w(x, t_1) - v(x, t_1) = (u(x, t_1) - v(x, t_1))_+$ so that

$$J_u(t_1) = \int_{\mathbb{R}^n} (w(t_1) - u(t_1))\, dx + \int_{\mathbb{R}^n} (w(t_1) - v(t_1))\, dx,$$

while for general $t > t_1$

$$J_u(t) + 2\int_{\mathbb{R}^n} (w(t) - \max\{u(t), v(t)\})\, dx = \int_{\mathbb{R}^n} (w(t) - u(t))\, dx + \int_{\mathbb{R}^n} (w(t) - v(t))\, dx.$$

Both integrals on the right are nonincreasing in time by the contraction principle, hence constancy of J_u in an interval $[t_1, t_2]$ implies that

$$w(t_2) = \max\{u(t_2), v(t_2)\}. \tag{12.5}$$

In order to examine the consequences of this equality we use the Strong Maximum Principle. □

LEMMA 12.4. *Two ordered solutions of the PME cannot touch for $t > 0$ wherever they are positive.*

This is a standard result for classical solutions of quasilinear parabolic equations, cf. [LSU]. It follows that (12.5) is then possible on any connected open set Ω where $w(\cdot, t_2) > 0$ under three circumstances:

(i) $w(t_2) = u(t_2) > v(t_2)$, or
(ii) $w(t_2) = v(t_2) > u(t_2)$, or
(iii) $w(t_2) = u(t_2) = v(t_2)$.

Since the support of the source-type solution is a ball and the support of u is also connected, we conclude the result of Lemma 12.3.

Note. If M is not the mass of u there is still another possibility for constant J, namely that the solutions are different but ordered: either $u(t) \geq \mathcal{U}_M(t)$ or $u(t) \leq \mathcal{U}_M(t)$.

We may now conclude the proof of Theorem 3.1 by this method in the case where u_0 has compact support, so that by standard properties of the propagation of support, it is connected after a certain time t_0. Since the source-type solution penetrates into the whole space eventually in time and U has a non-contracting support, it follows that for large t the supports of U and $\mathcal{U}_M$ do intersect. Since both solutions cannot be ordered because they have the same mass, $J_U(t)$ must be zero since it is not strictly decreasing by Lemma 12.2. We have thus proved that $J_\infty = 0$ and

$$U = \mathcal{U}_M, \tag{12.6}$$

which identifies all possible limits of rescalings as the unique source-type solution with the same mass. This ends the proof (see Section 5). The extension to general data is done by density as before.

12.5. Continuous rescaling. One way of proving the previous facts is by using the continuous rescaling, formula (10.1). As explained in Section 10, taking a sequence of delays s_n we define

$$\theta_n(\eta, \tau) = \theta(\eta, \tau + s_n),$$

and passing to the limit

$$\theta_n(\eta, \tau) \to \widehat{\theta}(\eta, \tau). \tag{12.7}$$

Again it is easy to see that $\widehat{\theta}$ is a weak solution of (10.2) satisfying the same estimates. For θ the Lyapunov functional is translated into

$$J_\theta(t) = \int_{\mathbb{R}^n} |\theta(\eta, \tau) - F_M(\eta)|\, d\eta, \tag{12.8}$$

and we see that it is continuous under the passage to the limit we have performed. Let us examine now the situation when $J_\infty > 0$. Then $\widehat{\theta} \neq F_M$ and the orbit of $\widehat{\theta}$ has a strictly decreasing functional, so that for $\tau_2 = \tau_1 + h$ we have

$$J_{\widehat{\theta}}(\tau_1) - J_{\widehat{\theta}}(\tau_2) = c > 0.$$

Since $\widehat{\theta}$ is the limit of the θ_n we get for all large enough n

$$J_{\theta_n}(\tau_1) - J_{\theta_n}(\tau_1 + h) \geq c/2.$$

But this means that for all n large enough

$$J_\theta(\tau_1 + s_n) - J_\theta(\tau_1 + s_n + h) \geq c/2.$$

This contradicts the fact that J_θ has a limit. The proof is complete.

Comment. As we had announced, the proof of this section uses several steps of the former with a completely different end. It contains some fine regularity results that can make it difficult to apply in more general settings. However, some of these difficulties can be overcome by other means. Lasalle's Invariance Principle is a powerful tool in Dynamical Systems [Ls], worth knowing also in this context.

13. Another Lyapunov approach

A different Lyapunov approach was proposed in 1984 by Newman and developed in [N, R]. We can write the functional in continuously rescaled variables (cf. Section 10) as

$$\mathcal{J}_\theta(\tau) = \int_{\mathbb{R}^n} \{\theta(\eta,\tau)^m + \kappa\,|\eta|^2\theta(x,\tau)\}\,d\eta, \qquad \kappa = \frac{1}{2}\beta(m-1)\,, \tag{13.1}$$

where β is the similarity exponent. The proof of convergence in this instance will be based on the possibility of calculating the value of $d\mathcal{J}/d\tau$ along an orbit.

LEMMA 13.1. *Let $\mathcal{J}$ be the Newman functional given by* (13.1). *Then for every rescaled orbit of problem (CP) we have*

$$\frac{d\mathcal{J}}{d\tau} = -\frac{m^2}{m-1}\int \theta\,\{\nabla(\theta^{m-1} + k\,\eta^2)\}^2\,d\eta. \tag{13.2}$$

Proof. In order to analyze the evolution of $\mathcal{J}$ let us put for a moment

$$\mathcal{J}(\theta) = \int_{\mathbb{R}^n} \{\theta(\eta,\tau)^m + \lambda\,|\eta|^2\theta(\eta,\tau)\}\,d\eta,$$

with $\lambda > 0$. Let us perform the following formal computations:

$$\begin{aligned} d\mathcal{J}/d\tau &= \int (m\theta^{m-1} + \lambda\,\eta^2)\,\theta_\tau\,d\eta \\ &= \int (m\theta^{m-1} + \lambda\,\eta^2)(\Delta\theta^m + \beta\nabla\cdot(\eta\,\theta))\,d\eta \\ &= -\int \nabla(m\theta^{m-1} + \lambda\,\eta^2)\,(\nabla\theta^m + \beta\eta\theta)\,d\eta \\ &= -\int \theta\,\nabla(m\theta^{m-1} + \lambda\,\eta^2)\,\nabla\left(\frac{m}{m-1}\theta^{m-1} + \frac{\beta}{2}\eta^2\right)\,d\eta. \end{aligned}$$

In case $\lambda = \beta(m-1)/2$ we can write this quantity as (13.2), which proves that J is a Lyapunov functional, i.e., it is monotone along orbits.

These computations are easily justified for classical solutions which decay quickly at infinity. The result for general solutions is then justified by a density argument using the regularity of the solutions of the PME. cf. [V4] (but we can also restrict the Lyapunov analysis to the above mentioned class of solutions since the proof of convergence for general solutions is then completed by a density argument as in Section 6). □

Limit orbits and invariance. As in the previous section we pass to the limit along sequences $\theta_n(\tau) = \theta(\tau + s_n)$ to obtain limit orbits $\widehat{\theta}(\tau)$, on which the Lyapunov functional is constant, hence $d\mathcal{J}_{\widehat{\theta}}/d\tau = 0$.

Identification step. The proof of asymptotic convergence concludes in the present instance in a new way, by analyzing when $d\mathcal{J}/d\tau$ is zero. Here is the crucial observation that ends the proof: *the second member of* (13.2) *vanishes if and only if* θ *is a Barenblatt profile*.

The rate of convergence can be calculated by computing $d^2\mathcal{J}/dt^2$, which is not easy. We will continue with the subject in Section 19.

14. Proof of Theorem 3.1 using optimal barriers

14.1. We present still another kind of proof, based on the ideas of the paper of Friedman and Kamin [FK]. Given a solution $u(t)$ with mass $M > 0$ we consider for fixed $t > 0$ the set of source-type solutions which lie below $u(\cdot, t)$ and define a functional

$$\mathcal{M}(t; u) = \sup\{M' \geq 0 : \mathcal{U}_{M'}(x, t) \leq u(x, t)\} \tag{14.1}$$

Thus, $\mathcal{M}(t; u)$ is the mass of a certain *optimal barrier from below*, a source-type solution with mass $\mathcal{M}(t; u)$ which lies below u at time t. Indeed, the above definition is insufficient and we have to introduce a modification of the class of admissible barriers. For $\tau \in \mathbb{R}$ we define

$$\mathcal{M}(t; u, \tau) = \sup\{M' \geq 0 : \mathcal{U}_{M'}(x, t + \tau) \leq u(x, t)\} \tag{14.2}$$

(if $\tau < 0$ this definition applies for $t > -\tau$). The use of a delay τ is a tricky technicality involved in the argument of strict monotonicity which is essential in the proof.

It is clear from the Maximum Principle that

LEMMA 14.1. *For fixed* u *and* τ *the function* $\mathcal{M}(t) = \mathcal{M}(t; u, \tau)$ *is positive for every* $t > 0$ *and* $\mathcal{M}(t) \leq M$. *It is also nondecreasing in* t *so that there exists the limit*

$$\mathcal{M}_\infty(u) = \lim_{t\to\infty} \mathcal{M}(t; u, \tau), \quad 0 < \mathcal{M}_\infty(u) \leq M. \tag{14.3}$$

The monotonicity in time makes $\mathcal{M}(t)$ a kind of Lyapunov functional. Obviously, since $u(t)$ has mass M, the equality $\mathcal{M}(t) = M$ only happens if u is just the source-type solution $\mathcal{U}_M$ (with a delay). Otherwise, $0 < \mathcal{M}(t) < M$. It is also easy to prove that though $\mathcal{M}(t)$ depends on τ the limit $\mathcal{M}_\infty$ does not (hint: source-type solutions with same mass but some delay are very similar for large t).

14.2. Repeating the process of rescaling, compactness and passage to the limit of Section 4. on the orbit $u(t)$, solution of problem (CP), we have that along a sequence $\lambda_n \to \infty$

$$\widetilde{u}_{\lambda_n} \to U, \tag{14.4}$$

and U is a solution of equation (0.1). Note that the convergence is uniform on compact sets of space-time, which will imply at the end of the proof (Step 5 of the general plan) uniform convergence in sets of the form $|x| \le Ct^\beta$.

LEMMA 14.2. *$\mathcal{M}(t; U, 0)$ is constant and equals $\mathcal{M}_\infty(u)$.*

Proof. Let us put $\widehat{\mathcal{M}}(t) = \mathcal{M}(t; U, 0)$, the mass of the optimal lower barrier for U. Using the identity $\mathcal{M}(t; u, \tau) = \mathcal{M}(t/\lambda; \widetilde{u}_\lambda, \tau/\lambda)$, it is immediate from the passage to the limit in the rescalings and that $\widehat{\mathcal{M}}(t)$ is equal or larger than $\mathcal{M}_\infty(u)$ for all $t > 0$. Note that the delays scale like τ/λ, so that in passing to the limit the delay tends to 0.

Consider now for some $t > 0$ the barrier $\mathcal{U}_{\widehat{\mathcal{M}}(t)}(x, t)$ for $U(t)$. Since the convergence (14.4) is uniform on compact sets we conclude by approximation that for large λ_n there exists a lower barrier with mass $\widehat{\mathcal{M}}(t) - \varepsilon$. But the barrier for $\widetilde{u}_{\lambda_n}$ is approximately $\mathcal{M}_\infty(u)$. Hence, $\mathcal{M}_\infty(u) \ge \widehat{\mathcal{M}}(t)$. □

14.3. We have to introduce the argument of strict monotonicity after which Lemma 14.2 cannot be true unless U is a source-type. We need first a lemma about the way source-type solutions evolve in time.

LEMMA 14.3. *Let $R(t) = R(M)t^\beta$ be the radius of the support of the source-type solution $u = \mathcal{U}_M(x, t)$. There exists a radius R_* smaller than $R(M)$ such that*

$$u_t < 0 \quad \text{if } |x| < R_* t^\beta, \qquad u_t > 0 \quad \text{if } R_* t^\beta < |x| < Rt^\beta. \tag{14.5}$$

The proof of this result is a simple calculation with the explicit formula (1.2)–(1.3). We can now prove the technical result which was a main point in the [FK] paper.

LEMMA 14.4. *Let $\mathcal{U}_{\mathcal{M}_1}(x, t + \tau_1)$ be a lower barrier for a solution $u(x, t)$ at a time $t_1 > 0$. Then either both functions coincide at $t = t_1$ or we can improve the barrier in the following way: given a sufficiently close delay $\tau \le \tau_1$ and a time $t_2 > t_1$ we can improve the mass to some $\mathcal{M} > \mathcal{M}_1$ and $\mathcal{U}_{\mathcal{M}}(x, t + \tau)$ is a lower barrier for u at all times $t \ge t_2$.*

Proof. The inequality

$$u(x,t) \geq U_1(x,t) = \mathcal{U}_{\mathcal{M}_1}(x, t+\tau_1)$$

holds for all $t \geq t_1$. Let $t_2 > t_1$. Let B_2 be the support of the solution U_1 at $t = t_2$ and let R_2 be its radius. Since $u(x,t_2) \geq U_1(x,t_2) > 0$ in B_2 the Strong Maximum Principle which holds for classical solutions of nonlinear parabolic equations implies that u must be strictly larger than U_1 at t_2 inside B_2.

We construct a barrier with a larger mass at t_2 as follows. If the support of $u(\cdot, t_2)$ contains a larger ball than B_2 then it is immediate that there exists a lower barrier of the form

$$\mathcal{U}_{\mathcal{M}}(x, t_2+\tau_1)$$

for some $\mathcal{M} > \mathcal{M}_1$. In case the ball cannot be expanded we have to perform the trick of changing the delay. Using Lemma 14.3 we see that a small decrease in the delay (to a value $\tau < \tau_1$) produces a source-type (Barenblatt) profile that is larger in the middle and smaller near the boundary with smaller support. This means that if τ is close enough to τ_1 then $\mathcal{U}_{\mathcal{M}}(x, t_2+\tau)$ will be less than $u(x,t_2)$ inside the support the former. Hence, we can safely increase slightly the mass to some $\mathcal{M} > \mathcal{M}_1$ and the resulting function will still be a lower barrier

$$u(x,t_2) \geq \mathcal{U}_{\mathcal{M}}(x, t_2+\tau). \tag{14.6}$$

□

COROLLARY 14.5. *We have* $\mathcal{M}_\infty = M$ *and* $U = \mathcal{U}_M$.

Proof. Suppose that this is not true and $\mathcal{M}_\infty < M$. Then we are in the situation where the lower barrier for U can be increased from $\mathcal{M}_\infty$ to $\mathcal{M} = \mathcal{M}_\infty + \varepsilon$ for $t \geq t_2 > 0$ with some delay $\tau \leq 0$. The uniform convergence of $\widetilde{u}_\lambda \to U$ on compact subsets means then that for large enough λ the function $\widetilde{u}_\lambda$ admits a barrier of mass $\mathcal{M}' \geq \mathcal{M}_\infty + (\varepsilon/2)$ with the same delay. In terms of u it means a barrier of mass $\mathcal{M}'$ with a much larger delay τ. But this would mean that

$$\lim_{t\to\infty} \mathcal{M}(t; u, \tau) > \mathcal{M}_\infty,$$

a contradiction since $\mathcal{M}_\infty$ does not depend on τ. □

After some rephrasing, Theorem 1.1 is proved.

15. Asymptotic symmetry

We now turn to a different trend of ideas based on the exploitation of the special properties of the asymptotic dynamics, resuming the discussion started at the end of Section 4. The most general idea about asymptotic properties can be phrased as the obtention of **symmetries** possessed by the equation but absent from the initial data. We recall the wide sense given by physicists to the word 'symmetry'. Thus, our equation is invariant under plane symmetries and rotations in the space variable. This 'symmetry' is not true for general orbits because of the influence of the initial data. However, we have asymptotic symmetry which derives from the following **monotonicity lemma**, cf. [CVW].

LEMMA 15.1. *Let u a solution of the Cauchy problem* (CP) *with initial data supported in the ball $B_R(0)$, $R > 0$. Then for every x such that $|x| > 2R$ and every $r < |x| - 2R$, $r > 0$, we have*

$$u(x,t) \leq \inf_{|y|=r} u(y,t) \tag{15.1}$$

Proof. We use Alexandrov's Reflection Principle. We draw the hyperplane H which is mediatrix between the points x and y in the above situation. It is easy to see that H divides the space $\mathbb{R}^n$ into two half-spaces, one Ω_1 which contains y and the support of u_0 and another one, Ω_2, which contains x and where $u_0 = 0$. We consider now the initial and boundary-value problem in $\widehat{Q} = \Omega_1 \times (0, \infty)$. Two particular solutions of this problem are compared: one of them is u_1, the restriction of u to $\widehat{Q}$, another one is

$$u_2(z,t) = u(\pi(z), t), \quad z \in \Omega_1.$$

where π is the specular symmetry with respect to the hyperplane H. Thus, if we orient the coordinate axes so that $H = \{x_1 = 0\}$ then

$$\pi(x_1, \ldots, x_n) = (-x_1, \ldots, x_n).$$

Clearly, u_1 and u_2 are solutions of (0.1) in $\widehat{Q}$. Besides, $u_1 = u_2$ on the lateral boundary, $\Sigma = H \times (0, \infty)$. Finally, $u_1 \geq u_2$ for $t = 0$ since $u_2 = 0$ in Ω_1. By comparison for the mixed problem we have

$$u_1(z,t) \geq u_2(z,t) \qquad z \in \Omega_1,\ t > 0.$$

Putting $z = y$ we have $\pi(z) = x$ so that $u(y,t) \geq u(x,t)$ as desired. □

The conclusion for the asymptotic orbits is immediate.

THEOREM 15.2. *Let U be a solution of problem* (CP) *obtained as limit of the rescaling discussed in Section* 4. *Then U must be radially symmetric in the space variable, $U = U(r,t)$, $r = |x|$. It is also a nonincreasing function of r for fixed t.*

Proof. By density we may argue with an original orbit $u(x,t)$ with initial data u_0 supported in the ball of radius $R > 0$. But the use of Lemma 15.1 implies that for every $x \in \mathbb{R}^n$, $x \neq 0$ we have

$$u_\lambda(x,t) \le \inf_{|y|=r} u_\lambda(y,t)$$

as long as $|x| \ge 2R/\lambda^\beta$ and $0 < r < |x| - (2R/\lambda^\beta)$. In the limit $\lambda_n \to \infty$ we get

$$U(x,t) \le \inf_{|y|=r} U(y,t), \qquad 0 < r < |x|. \tag{15.2}$$

The conclusion follows. The L^1 continuity allows to extend the result to general data. □

Therefore, the asymptotic dynamics takes place under the conditions of radial symmetry and monotonicity in the radial variable. Of course, the reader will object that in our problem we already know, after Section 5, that U must be a source-type. There are two points we want to make in this connection: (i) the proof of Section 5 was based on a strong uniqueness result for the PME; (ii) the asymptotic property of Theorem 15.2 is a general fact that can be established under quite general assumptions and can be the base for alternative convergence proofs, applicable when uniqueness is not available in the form used in Section 5.

Let us also warn the reader that the influence of the initial data, though indirect, still exists in the form of class of data for which this process is true, which is given by the restriction (0.3). It is easy to see that other initial classes are incompatible with the process of asymptotic symmetry in this sense. For instance, data which are monotone in one direction preserve this property in time, and this is incompatible with radial symmetry.

Let us also mention another immediate consequence of Alexandrov's Reflection Principle.

LEMMA 15.3. *With the assumptions and notations of Section* 9*, the difference of maximal and minimal support radius, $R(t) - r(t)$, remains bounded in time.*

16. Convergence rates for radially symmetric solutions. Concentration comparison

16.1. The study of radially symmetric solutions $u = u(r,t)$ has an interest because we can use special techniques which produce error estimates with relative size $O(1/t)$. Combined with the idea of asymptotic symmetry of Section 15, it provides an alternative proof of the convergence results of Section 3. The ideas have been first explained in the paper [KV1]. It works as follows:

End of proof of Theorem 1.1 by this method. The proof starts as in Section 4 and proceeds through Steps 1, 2 and 3 so that we pass to the limit $\lambda_n \to \infty$ and obtain an asymptotic solution, U, which is radially symmetric by the results of Section 15. Assume that

Theorem 1.1 is proved for such solutions. Then we have convergence of U towards the source-type solution $\mathcal{U}_M(x,t)$. The triangle inequality gives then convergence along a subsequence

$$u(x,t_n) - \mathcal{U}_M(x,t_n) \to 0 \qquad \text{in } L^1(\mathbb{R}^n).$$

But the L^1-contraction implies that the whole family $\{u(t)\}_{t>0}$ converges. The proof is complete.

16.2. Therefore, we consider in the sequel the asymptotic behaviour of solutions of the Cauchy problem (CP) under the assumptions (0.3) plus

$$u_0 \text{ is bounded, radially symmetric and compactly supported.} \tag{16.1}$$

A density argument allows to dispense with the later condition as in Section 6. The main technical tool that we shall use is the comparison principle introduced in [V3] (see also [V1]) and called **Concentration Comparison**. In the restricted form that we need here it reads as follows

THEOREM 16.1. *Let u_i, $i = 1, 2$ be a pair of radially symmetric solutions of problem (CP) with initial data u_{0i} in the class* (0.3). *Assume that for every $r > 0$*

$$\int_{B_r(0)} u_{01}(x)\,dx \le \int_{B_r(0)} u_{02}(x)\,dx. \tag{16.2}$$

Then for every time $t > 0$ and radius $r > 0$

$$\int_{B_r(0)} u_1(x,t)\,dx \le \int_{B_r(0)} u_2(x,t)\,dx. \tag{16.3}$$

In [V1], pp. 522, the notation $u_{02} \succ u_{01}$ is introduced to represent the situation of (16.2), and it is read as u_{02} *is more concentrated than* u_{01}. The result can then be phrased as saying that *the concentration relation $\succ$ is conserved in the evolution* (Note: the general formulation, [V3], allows to compare a non-symmetric solution $u_1(x,t)$ with a symmetric solution $u_2(x,t)$, after symmetrizing the first one, but such generality is not needed here).

As an immediate consequence of this result we obtain comparison of supports.

LEMMA 16.2. *Let u_1 and u_2 two solutions as before and let us assume that they are compactly supported and have the same mass. If $R_1(t)$ and $R_2(t)$ are the radii of their respective supports we have*

$$R_1(t) \ge R_2(t), \qquad \text{for every } t \ge 0. \tag{16.4}$$

We only need to observe that if the common mass is M, then

$$R_i(t) = \sup\left\{ r > 0 : \int_{B_r(0)} u_i(x,t)\,dx < M \right\}.$$

16.3. Let us now proceed with the analysis of the long-time behaviour of solutions and supports for compactly supported, radially symmetric solutions. To begin with, we have the following improved asymptotics for the free boundary.

THEOREM 16.3. *Let $R(t)$ and $\mathcal{R}(t)$ the radii of the supports of $u(t)$ and $\mathcal{U}_M(t)$. Then there exists $t_2 > 0$ such that $\mathcal{R}(t) \le R(t) \le \mathcal{R}(t+t_2)$. Therefore,*

$$R(t) = \mathcal{R}(t)\left(1 + O\left(\frac{1}{t}\right)\right). \tag{16.5}$$

Proof. It is easy to see that given an initial datum u_0 as assumed in (16.1) with mass $M > 0$ and such that u_0 is positive near $r = 0$, there exist times $0 < t_1 < t_2$ such that

$$\mathcal{U}_M(x, t_1) \succ u_0(x) \succ \mathcal{U}_M(x, t_2). \tag{16.6}$$

The result $\mathcal{U}_M(x, t+t_1) \succ u(x, t) \succ \mathcal{U}_M(x, t+t_2)$ follows. Moreover, t_1 can be taken as small as we please. In case u_0 is not positive at $r = 0$ we have to wait for a certain time until the positivity set spreads to cover that point. According to Section 9 this happens after some time t_3. Then relation (16.6) holds with u_0 replaced by $u(x, t_3)$. □

We derive next an L^1-rate of convergence for the profiles.

THEOREM 16.4. *As $t \to \infty$ we have*

$$\|u(t) - \mathcal{U}_M(t)\|_1 = O\left(\frac{1}{t}\right). \tag{16.7}$$

Proof. We take for fixed $t \gg 1$ the source-type function $\widetilde{U}(x, t)$ with interface at $R(t)$. According to Theorem 16.3 and the formula for the free boundary of the source-type solution, it has a mass

$$\widetilde{M}(t) = M(1 + O(1/t))^{1/(\beta(m-1))} = M(1 + O(1/t)).$$

We have $\widetilde{U} \ge \mathcal{U}_M$. It is clear that $u(t)$ and $\widetilde{U}(t)$ have the same support at time t, the ball $\widetilde{B}$ of radius $R(t)$. Let $\phi(x) = u^{m-1}(x, t) - \widetilde{U}^{m-1}(x, t)$. According to the fundamental estimate we have $\Delta\phi \ge 0$ in $\widetilde{B}$, i.e., ϕ is subharmonic. Since ϕ is zero on the boundary of the ball, we conclude that $u \le \widetilde{U}$. Therefore,

$$\|(u(t) - \mathcal{U}_M(t))_+\|_1 \le \|(\widetilde{U}(t) - \mathcal{U}_M(t))_+\|_1 = \widetilde{M} - M = O\left(\frac{1}{t}\right).$$

Since u and $\mathcal{U}_M$ have the same mass, estimate (16.7) follows. □

17. The technique of intersection comparison

Another technique that can be used in the study of radially symmetric problems is the extended version of the Maximum Principle which is called **Intersection Comparison** and consists in counting the number of sign changes of the difference of two solutions.

DEFINITION. Given two radially symmetric solutions, u_1 and u_2, of problem (CP) and for every time $t \geq 0$ we consider finite sequences of radii $0 < r_0 < r_1 < \cdots < r_n$ (of length n) such that

$$(u_1(r_i) - u_2(r_i))\,(u_1(r_{i+1}) - u_2(r_{i+1})) < 0 \tag{17.1}$$

for every $i = 0, \ldots, n-1$. Then we define

$$N(t; u_1, u_2) = \max\{n : n \text{ is the length of a sequence satisfying (17.1)}\} \tag{17.2}$$

It is also important to record the list of signs of $u_1 - u_2$. The main result of this theory is

THEOREM 17.1. *For the solutions of the Cauchy problem for quasilinear parabolic equations of the type of equation PME posed in the whole space the counter N is non-increasing in time.*

This result stems from Sturm [St] and has been extended in recent years to wide classes of nonlinear parabolic equations by a number of authors, see the book [S4] and its references. The principle is also known as **lap number** theory, cf. [Ma]. More precisely, the decrease in N happens through loss of sign changes either at the boundary of the domain (if there is one) or inside (in this case two sign changes are lost at least at every time a change happens). The list of signs is conserved as long as N does not decrease.

By a careful geometrical inspection of our situation for the PME and using the regularity of the solutions, we get the following result.

LEMMA 17.2. *Given a radial solution u of problem* (CP) *with bounded, continuous and compactly supported data $u_0 \geq 0$, there exist time delays $0 \leq t_1 < t_2$ such that if $u_i(x,t) = \mathcal{U}_M(x, t + t_i)$, $i = 1, 2$, we have*

$$N(t, u, u_1) = N(t, u, u_2) = 1\,. \tag{17.3}$$

Moreover, the list of signs of $u - u_1$ is $-+$, while the list for $u - u_2$ is $+-$. Therefore, for all $r \approx 0$ we have for all large t

$$u_2(r,t) < u(r,t) < u_1(r,t)\,. \tag{17.4}$$

Note that the counter $N(t, u, u_1)$ cannot decrease to 0 since that would imply the ordering $u \le u_1$, and by equality of mass $u = u_1$. Even if this happens a small change in delay would restore the value $N = 1$. The same applies to $u_2 - u$.

As an immediate consequence of these estimates we obtain the support radius estimates of Theorem 16.3. The proof of Theorem 16.4 remains unchanged. Moreover, the technique of Intersection Comparison allows for a strong improvement of Theorem 16.4.

THEOREM 17.3. *As $t \to \infty$ we have*

$$u^{m-1}(x,t) = \mathcal{U}_M^{m-1}(x,t) + O(t^{-\gamma}), \qquad \gamma = \alpha(m-1) + 1. \tag{17.5}$$

Proof. Remark that $O(t^{-\gamma}) = \|\mathcal{U}_M(\cdot,t)\|_\infty^{m-1}\, O(t^{-1})$, therefore this is another estimate with relative error of size $O(1/t)$. Note that the delayed source-type solutions show this error, hence the stated convergence rate is optimal.

As for the proof, the estimate from above has been already established in the comparison $u \le \widetilde{\mathcal{U}}$ of Theorem 16.4. In the estimate from below it is useful to write the error as $e(x,t) = u^{m-1}(x,t) - \mathcal{U}_M^{m-1}(x,t)$. For $r \approx 0$ and large t, the estimate $e \le O(t^{-\gamma})$ follows from (17.4). To extend it to all r we use the Aronson-Bénilan estimate that implies that $\Delta e \ge 0$ as long as $\mathcal{U}_M > 0$. Since the function is radially symmetric, $e = e(r,t)$, we have for every fixed $t > 0$

$$e(0,t) = 0, \quad e_r(0,t) = 0 \quad \text{and} \quad (r^{n-1} e_r)_r \ge 0 \text{ for } 0 < r < \mathcal{R}(t).$$

This implies that $e(r,t) \ge e(0,t)$ for $0 < r < \mathcal{R}(t)$. This and the estimate for $R(t)$ end the result. □

No uniform rate can be given for general solutions without a decay estimate on u_0 at infinity, as we have already shown in Section 11, which applies even with radial symmetry. On the other hand, the convergence rates for radially symmetric solutions with compact support obtained in these two sections are optimal.

One dimension without radial symmetry. The problem without radial symmetry is rather well known in dimension $n = 1$, where we can still use the techniques of concentration (mass) comparison and intersection comparison used in these two last sections. As a first example of one-dimensional result, under the assumptions that $u_0 \in \mathcal{X}_0$ is compactly supported, not radially symmetric, it is proved in [V1] that there is an improved rate of convergence if we first calculate the *center of mass*, which is an invariant of the motion,

$$x_c = \frac{1}{M}\int_{\mathbb{R}} x\, u_0(x)\, dx = \frac{1}{M}\int_{\mathbb{R}} x\, u(x,t)\, dx,$$

where as usual $M = \int u_0(x)\, dx > 0$. Then, we have

THEOREM 17.4. *The left and right interfaces of u,* $r_1(t) = \inf\{x : u(x,t) > 0\}$, $r_2(t) = \sup\{x : u(x,t) > 0\}$, *satisfy*

$$r_i(t) = (-1)^i \mathcal{R}(t) + x_c + o(1) \tag{17.6}$$

as $t \to \infty$. *This implies an asymptotic error in the pressures of the form*

$$u^{m-1}(x,t) - \mathcal{U}_M^{m-1}(x - x_c, t) = O(t^{-m/(m+1)}). \tag{17.7}$$

18. Concavity and smooth convergence

The question of large time behaviour can be combined with the question of asymptotic geometry to obtain better asymptotic results. The first paper in that direction for the PME seems to be due to Bénilan and the author [BV], who prove that in dimension one and for compactly supported solutions of the PME concavity of the pressure is preserved in time (if $v = u^{m-1}$ and $v_{0,xx} \le 0$ where $v_0 > 0$, then $v_{xx} \le 0$ where $v > 0$). Aronson and the author then proved that *all* compactly supported solutions become eventually pressure-concave, which allows for better convergence estimates in 1D, [AV], Section 4:

$$\|u^{m-1}(x,t) - U_M^{m-1}(x - x_c, t)\|_{L^\infty(\mathbb{R}^n)} = O(t^{-2m/(m+1)}). \tag{18.1}$$

The result is inspired by the fact that in terms of the variable $v = u^{m-1}$ the source-type solution is an inverted parabola. A more precise asymptotics says that

$$v_{xx}(x,t) + \frac{m-1}{m(m+1)t} = O\left(\frac{1}{t^2}\right). \tag{18.2}$$

A simple geometrical argument shows that if such a function has the same mass as the parabola of the source-type solution pressure the error in size must be of order $O(1/t)$ times the height of the parabola, and this gives $O(t^{-2m/(m+1)})$ error for u^{m-1}. The error in the interface size is also of the same order, hence

$$r_i(t) = (-1)^i \mathcal{R}(t) + x_c + O(t^{-m/(m+1)}). \tag{18.3}$$

This reproduces in 1D the results for radially symmetric solutions of previous sections. A study of asymptotic geometry in 1D for more general diffusion equations is done in [GV] using Intersection Comparison.

The recent work of Lee and Vazquez [LV] extends these ideas to all dimensions $n \ge 1$: after a finite time the pressure of any solution with compact support becomes a concave function in the space variable, and it converges to all orders of differentiability to a truncated parabolic shape, Barenblatt's source-type profile. The assumptions on the initial data are u_0 nonnegative, compactly supported and what is called non-degenerate initial data, a technical condition. In particular, for large times the support of the solution is a convex subset of $\mathbb{R}^n$ which converges to a ball, and the convergence is smooth. Estimates are optimal. Here is a typical result from [LV].

THEOREM 18.1. *There is $t_o > 0$ such that $v(x,t)$ is concave in $\Omega(t) = \{x : v(x,t) > 0\}$ for $t \ge t_o$. More precisely,*

$$\lim_{t\to\infty} t\,\frac{\partial^2 v}{\partial x_i^2} = -\frac{(m-1)\beta}{m} \tag{18.4}$$

for any coordinate direction x_i, uniformly in $x \in supp(v)$. The results also hold for the heat equation (where the pressure is defined as $v = \log(u)$ and we get asymptotic log-concavity) and fast diffusion (with $v = 1/u^{1-m}$, $(n-2)/n < m < 1$, and we get asymptotic pressure convexity). In these cases $\operatorname{supp}(v(t)) = \mathbb{R}^n$.

19. Improved rates by the entropy method

Obtaining better convergence rates than the plain ones given by the main result needs assumptions on the data, and this is the purpose of an extensive literature. We have seen the rates of convergence obtained under the assumptions of radial symmetry and compact support in Sections 16, 17 and 18. Without those assumptions, the Lyapunov approach of Newman and Ralston [N, R] has been improved by means of ideas of entropy and entropy-dissipation into a tool to obtain rates of convergence by a number of authors in [CT, C5, DP, Ot] for data in the class $\mathcal{X}_2 \subset \mathcal{X}_0$ of initial data having also finite second moment

$$\int |x|^2\, u_0(x)\, dx < \infty. \tag{19.1}$$

THEOREM 19.1. *For every $u_0 \in \mathcal{X}_2$ we have*

$$\|u(x,t) - U_M(x,t)\|_{L^1(\mathbb{R}^n)} = O(t^{-\kappa}), \tag{19.2}$$

where κ equals the Barenblatt exponent β for $1 < m \le 2$, while $\kappa = 2\beta/m$ is obtained for $m > 2$.

This theorem has been proved in the above-mentioned references using different techniques: entropy dissipation technique, mass transportation arguments, Riemannian calculus, variational calculus. By checking the error committed by a source-type solution displaced in space we can see that $\kappa = \beta$ is the optimal rate to be expected (for all $m \ge 1$). The argument holds also for $m < 1$ (fast diffusion, to be treated below) if $m > \max\{(n-1)/n, n/n+2\}$. The extension to the lower part of the range, $(n-1)/n > m > (n-2)/n$, $n \ge 2$, has been done recently in [CVa], with rate $\kappa = 1/2$, thus completing the range of exponents m where Barenblatt's source-type solutions play a role. We refer to these works for further references and to Section 21 for a general treatment of fast diffusion.

20. Extensions. Sign change and forcing

The generalization of the porous medium equation and other simple models into a general mathematical model was one of the leitmotives of Bénilan's mathematical activity, since in his opinion an analyst must always strive for the general ideas valid for wide classes of differential operators acting in general spaces, in an effort to discover what is the powerful mathematical idea and result. In that respect, he was always inclined to see the porous medium equation in the form $u_t = \Delta\Phi(u)$, and Φ must be allowed to be an arbitrary monotone function, or even a maximal monotone graph, like in the famous book [Br]. On the other hand, this and more general forms, like $u_t = \operatorname{div} a(x, t, u, Du)$, turned out to be of interest for the applications.

Signed solutions. Going back to the simple porous medium, this trend of ideas favors the elimination of the sign restriction on the solutions, so that the whole L^1 may enter the picture. In order to tackle negative values of u one must adapt the power in a convenient way so that the equation continues to be degenerate parabolic. The preferred version has a symmetrical nonlinearity:

$$u_t = \Delta(|u|^{m-1}u)\,. \tag{20.1}$$

(There is no a priori reason to do that, apart from mathematical simplicity, which is a strong reason in itself). This equation is treated by Bénilan and Crandall [BC2] in the framework of m-accretive operators in L^1 to obtain a mild solution, and the integral conditions derived by Bénilan in his thesis. The asymptotic behaviour of integrable solutions has been studied by Kamin and Vazquez [KV2] and the result is

THEOREM 20.1. *Let $u_0 \in L^1(\mathbb{R}^n)$ and let $M = \int u_0(x)\,dx$. If $M > 0$ then*

$$\lim_{t\to\infty} \|u(t) - \mathcal{U}(t; M)\|_{L^1(\mathbb{R}^n)} = 0, \tag{20.2}$$

If $M = 0$ the limit applies with $U = 0$. If $M = -M' < 0$ we put $\mathcal{U}(x, t; M) = -\mathcal{U}(x, t; M')$ and the result holds.

However, when the integral is zero the asymptotics is trivial in our scale, and the actual scaling where the asymptotics is nontrivial depends on the subclass of initial data; this is partially studied in [KV2] and [BHV], where a beautiful problem of *anomalous exponents* appears.

Equations with forcing. Another interesting result concerns the influence of a forcing term in the form of an integrable right-hand side in the equation.

THEOREM 20.2. *Let u be the mild solution of the Cauchy problem for*

$$u_t = \Delta(|u|^{m-1}u) + f, \tag{20.3}$$

with initial data $u_0(x) \in L^1(\mathbb{R}^n)$ *and* $f \in L^1(Q)$ *(no sign restriction is imposed). Then,*

$$\lim_{t\to\infty} \|u(t) - \mathcal{U}_{M'}(t)\|_1 = 0\,, \tag{20.4}$$

where

$$M' = \int u_0(x)\,dx + \int\int_Q f\,dxdt. \tag{20.5}$$

Proof. We consider the solution $u_n(x,t)$ of the problem with same initial data and forcing term f_n such that

$$f_n(x,t) = f(x,t) \quad \text{if } t < n, \qquad f_n(x,t) = 0 \quad \text{if } t \ge n\,.$$

The mild solution becomes a standard weak continuous solution for $t \ge n$. We can think of this solution as having initial data at $t = n$ of the form $u_n(x,n) = \phi_n(x)$. According to Bénilan's analysis [Be]

$$M_n = \int \phi_n(x)\,dx = \int u_0(x)\,dx + \int_0^n \int f(x,t)\,dxdt.$$

Hence, for $t \gg n$ large enough, u_n approaches the source-type profile with mass

$$M_n = \int u_n(x,t)\,dx$$

(which is constant for $t \ge t_n$) with the modification explained above if $M_n \le 0$. This means that

$$\lim_{t\to\infty} \|u_n(t) - \mathcal{U}_{M_n}(t)\|_1 = 0.$$

On the other hand, $\lim_{t\to\infty} M_n = M'$. Finally, the contraction property for mild solutions (or *bonnes solutions* in the sense of [Be]) implies that

$$\|u - u_n\|_1 \le \int_0^t \int |f - f_n|\,dxdt = \int_n^\infty \int_{\mathbb{R}^n} |f|\,dxdt \to 0$$

as $n \to \infty$. The proof is complete. □

21. Improved convergence for fast diffusion

The extension of the asymptotic results proved above to exponent $m = 1$ gives as a consequence results that are well-known for the classical heat equation. It is interesting to remark that the proof given here applies (with inessential minor changes), and is very different from the usual proofs based on the representation formula.

We can even go below $m = 1$ and prove similar results for some so-called **fast-diffusion equations** which are just equation (0.1) with $0 < m < 1$. To start with we need two basic ingredients.

(a) A theory of well-posedness for the Cauchy problem. The results of Section 1 apply also in this case with minor easy changes. The main novelty is that solutions are positive everywhere and C^∞-smooth, which is rather good news in this context.

(b) The second ingredient is the model of asymptotic behaviour. The source-type solution exists just for $m > m_c = (n-2)/n$ and it can be conveniently written in the form

$$\mathcal{U}_M(x,t) = \left(\frac{Ct}{|x|^2 + At^{2\beta}}\right)^{1/(1-m)} = \frac{K\,t^{-\alpha}}{(A + (x\,t^{-\beta})^2)^{1/(1-m)}} \tag{21.1}$$

where $\beta = (2 - n(1-m))^{-1}$ is positive precisely in that range, $\alpha = n\beta$, $C = 2m/\beta(1-m)$ is a fixed constant, $K = C^{1/(1-m)}$, and $A > 0$ is an arbitrary constant that can be determined as a decreasing function of the mass $M = \int \mathcal{U}(x,t)\,dx$, $A = k(m,n)\,M^{-\gamma}$ with $\gamma = 2(1-m)\beta$.

In dimensions $n = 1, 2$ the whole range $0 < m < 1$ is covered. However, the *critical exponent*, $m_c = 1 - (2/n)$, is larger than zero for $n \geq 3$. It is then proved that for $0 < m < m_c$ no solution of the Barenblatt type exists (i.e., self-similar with constant positive mass). The value $m_c = (n-2)/n$ is related to the Sobolev embedding exponents as the reader will easily realize.

The authors of [FK] claim that their result of uniform convergence uniformly on sets of the form $|x| \leq c\,t^\beta$, is also true for $m < 1$ in the range $m_c < m < 1$, where the Barenblatt solution exists. Indeed, *the convergence results of Theorem* 1.1 *hold true for* $m > m_c$ and the proofs given above are true but for minor details.

Relative error convergence. However, the fact that the solutions of the fast diffusion equation do not have the property of conserving compact supports, but develop tails at infinity of a certain form gives rise to a very interesting estimate formulated in terms of *relative error*, or in other words, as *weighted convergence*, that we present next. It requires a suitable behaviour of the initial data as $|x| \to \infty$ (similar in decay to the Barenblatt solution).

THEOREM 21.1. *Under the assumption that* u_0 *is bounded and* $u_0(x) = O(|x|^{-2/(1-m)})$ *as* $|x| \to \infty$, *we have the asymptotic estimate*

$$\lim_{t\to\infty} \frac{|u(x,t) - \mathcal{U}(x,t;M)|}{\mathcal{U}(x,t;M)} \to 0 \tag{21.2}$$

uniformly in $x \in \mathbb{R}^n$. *The condition on the initial data can be weakened into the integral estimate*

$$\int_{|y-x|\leq|x|/2} |u_0(y)|\,dy = O(|x|^{n-\frac{2}{1-m}}) \quad as \ |x| \to \infty\,. \tag{21.3}$$

In particular, we have for $\|u(t) - \mathcal{U}(t; M)\|_1 \to 0$ as $t \to \infty$ (like case $p = 1$ of Theorem 1.1), and $t^\alpha |u(x,t) - \mathcal{U}(x,t;M)| \to 0$, as $t \to \infty$ uniformly in x (case $p = \infty$), but estimate (21.2) is much more precise because the convergence is uniform with weight

$$\rho = (|y|^2 + c)^{1/(1-m)}, \qquad y = x\, t^{-\beta}.$$

Proof. (i) The standard theorem implies that we have uniform convergence

$$t^\alpha |u(x,t) - \mathcal{U}(x,t;M)| \to 0,$$

uniformly on sets of the form $|x| \le C\, t^\beta$. The problem is then to extend this estimate to the outer region $\{|x| \ge C\, t^\beta\}$ by means of a so-called *tail analysis.* This analysis uses two properties of the Barenblatt solutions. The first is the effect at infinity of a delay in time. Thus, it is easy to see that $\mathcal{U}(x, t+\tau; M)$ grows with τ for large values of $y = x\, t^{-\beta}$, and precisely

$$\mathcal{U}(x, t+\tau; M) - \mathcal{U}(x,t;M) = \frac{\tau}{1-m}\mathcal{U}(x,t)\, t^{-1}(1 + O(t^{-1})),$$

uniformly for $|y| \gg 1$ (on the contrary, for small values of y the variation is opposite, $\partial U/\partial\tau < 0$). Next, we recall the fact that the asymptotic behaviour as $|x| \to \infty$ for fixed t of the source-type solutions is independent of the mass of the solution

$$\mathcal{U}(x,t;M) = (C\, t\, |x|^{-2})^{1/(1-m)}(1 + O(M^{-1/\gamma} y^{-2})).$$

Next, we note that this kind of universal behaviour of some solutions of fast diffusion for large $|x|$ can be generalized to a general class of solutions $u(x,t)$.

(ii) The tail analysis goes as follows. We examine first the existence of an upper bound using the assumption in the strong form, $u_0(x) = O(|x|^{-2/(1-m)})$.

We have to use the trick of comparing our initial data with a source-type solution with negative value of A. Then the denominator of (21.1) vanishes for some value of $|y| = y_0(A)$, and the solution becomes infinite there (blow-up). It is finite for $|x| \ge y_0\, t^\beta$, and indeed, a smooth classical solution in that region. If we take any $\tau > 0$ then there exists a large value of A (hence a large value of y_0), such that we have the comparison

$$u_0(x) \le \mathcal{U}(x, \tau; -A) \qquad |x| \ge y_0\, \tau^\beta.$$

By comparison we have for all $t > 0$, $|x| \ge y_0\,(t+\tau)^\beta$

$$u(x,t) \le \mathcal{U}(x, t+\tau; -A).$$

In view of the behaviour of $\mathcal{U}$ as $|y| \to \infty$ independently of A and letting $\tau \to 0$ we get the estimate

$$\limsup_{t\to\infty}\, u^{1-m}\frac{x^2}{t} \le C + \varepsilon,$$

uniformly for $|x|\,t^{-\beta} = |y| \geq y_\varepsilon \gg 1$. In view of the form of $\mathcal{U}$, the reader will be able to check that this is the upper bound needed in the tail part of (21.2).

(iii) Thanks to the L^1-L^∞ regularizing effect the general assumption (21.3) is converted into the previous assumption for $t_1 > 0$. Indeed, it is known that

$$u(x, t_1) \leq C\, t_1^{-\alpha} \int_{B_{|x|/2}} u_0(x)\, dx$$

Therefore, under assumption (21.3) we have $u(x, t_1) = O(|x|^{-2/(1-m)})$, and we can take this function as initial function after displacing the axis of times.

(iv) We now turn to the lower bound at the tail. We will take a small $\tau > 0$ and large $a > 0$ and make a comparison of $u(x, t)$ and $\mathcal{U}(x, t-\tau; M')$ in the region

$$\mathcal{R} = \{(x, t) :\ \tau < t < 2\tau,\ |x| \geq a\}.$$

It is clear that $u(x, \tau) \geq \mathcal{U}(x, 0; M')$ for $|x| \geq a$ since the last quantity is zero. On the other hand, by the continuity of u we can get $u \geq \mathcal{U}$ on the parabolic boundary $|x| = a$ if the mass M' is chosen small enough. By the Maximum Principle we conclude that $u(x, t) \geq \mathcal{U}(x, t-\tau; M')$ in $\mathcal{R}$. In particular,

$$u(x, 2\tau) \geq \mathcal{U}(x, \tau; M') \sim (C\tau/x^2)^{1/(1-m)} \qquad x \to \infty.$$

But the positivity of u at $\tau > 0$ implies that the inequality is true for all $x \in \mathbb{R}^n$ if M' is small enough. Hence, the same comparison holds for all later times

$$u(x, t) \geq \mathcal{U}(x, t-\tau; M'), \quad x \in \mathbb{R}^n\,.$$

The uniform behaviour of $\mathcal{U}$ as $y \to \infty$ independently of M' and the fact that τ is arbitrary allow us to conclude much as before that

$$\liminf_{t\to\infty}\ u^{1-m}\,\frac{x^2}{t} \geq C - \varepsilon\,,$$

uniformly for $|x|\,t^{-\beta} = |y| \geq y_\varepsilon \gg 1$. □

We remark that our result implies a uniform behaviour at the space far-field for all large t. More precisely,

COROLLARY 21.2. *Under the above assumptions we have the asymptotic limit as* $|y| \to \infty$, $t \to \infty$:

$$\lim\, (|x|^2/t)^{1/(1-m)}\, u(x, t) = K = (2m/\beta(1-m))^{1/(1-m)}.$$

Further improvements. When u_0 has an exact decay at infinity, $u_0(x) \sim a\,|x|^{-2/(1-m)}$, we have a precise x-asymptotics.

LEMMA 21.3. *If* $\lim_{|x|\to\infty} |x|^{2/(1-m)} u_0(x) = a \ge 0$, *then u satisfies*

$$\lim_{|x|\to\infty} x^2 u^{1-m}(x,t) = C\,(t+T), \tag{21.4}$$

locally uniformly in time $t > 0$, *with* $T = a^{1-m}/C$ *and C as in* (21.1). *Moreover, there exists* $t_o > 0$ *such that* $v(x,t) = u^{m-1}$ *is convex as a function of x for* $t \ge t_o$.

This is proved in [LV]. On the other hand, for radial data in the class of the Theorem, Carrillo and Vazquez [CVa] show a relative rate of decay $O(t^{-1})$ in the results of Theorem 21.1, extending the results of Sections 16, 17 to fast diffusion.

Subcritical range. Let us briefly point out that the breakdown of the asymptotic model implies in this case a complete change of asymptotic behaviour. Thus, it is proved that for the critical exponent, $m = m_c$, mass is conserved but the asymptotic behaviour is quite involved, cf. [Ki, GPV]. For $0 < m < m_c$ we get an even more drastic phenomenon: bounded integrable solutions disappear (vanish identically) in finite time and the relevant asymptotic profiles are self-similar solutions of the form

$$\mathcal{U}(x,t) = (T-t)^{\alpha} F(x\,(T-t)^{\beta}), \tag{21.5}$$

cf. [GP] and [PS]. We realize at once the crucial role played in the dynamics of these nonlinear heat equations by the existence of certain types of particular solutions.

On the other hand, in one space dimension, the standard theory of fast diffusion can be extended to the extra range $-1 < m \le 0$ on the condition of writing the equation in the modified form $u_t = (u^{m-1}u_x)_x$ (the original formula is not parabolic!). Source-type solutions still exist and Theorems 1.1 and 21.3 hold on the condition of working with the special class of *maximal solutions,* for which the problem is well posed. We remind the reader that in this range of exponents the theory of the Cauchy Problem is quite special because of non-uniqueness: there are other classes of solutions determined by nontrivial Neumann conditions at infinity, cf. [ERV, RV]. For those other solutions different asymptotic descriptions have to be found.

22. Final comments on the literature and results

The PME was derived by Boussinesq in 1903/4 in connection with flows in porous media, [Bo]. Around 1950 Zel'dovich and his group studied this equation as a model for heat propagation in plasma [ZR]; the source-type solutions were constructed in a particular case by Zel'dovich and Kompaneets [ZK] and in all generality by Barenblatt [B1], 1952. They were re-discovered in the West by Pattle [Pa] in 1959. The first proof of existence and uniqueness of a generalized solution was done by Oleinik and collaborators [OKC] in 1958 (in one space dimension). There are many known applications of the PME: Muskat considered it for the flow of gases in porous media in 1937 [Mu], it was studied by Gurtin and McCamy [GMC] in a model in population dynamics, and it appears in thin viscous films moving under gravity studied by Buckmaster [Bu]. We refer to [Ar, Pe, V4, W4] for details about the mathematical theory.

The basis of this paper is taken from the Course Notes [V5]. The scheme of the proof of the main result follows [KV1], where convergence is proved in L^1, i.e., as in (1.5), and uniformly for solutions with compact support. This paper contains still another proof, based on asymptotic symmetry, detailed in Section 16. Newman's Lyapunov approach is studied in [N, R]. The simpler Lyapunov approach of Section 12 using the L^1-norm is more or less folklore, but I have not found it written. The convergence of the supports for compactly supported solutions is stated in [KV2]. This paper contains the proof of convergence for solutions with changing sign.

Barrier methods, like the one used by [FK] have proved to be very suitable in the study of nonlinear diffusion problems without a standard Lyapunov functional: a very interesting application occurs in the study of the equation of elastoplastic filtration, in [KPV], where no other method seems to be work by lack of known invariants.

Better rates of convergence for radial solutions are discussed in [V1] and [AV], but the main results of Sections 16, 17 are new. In one space dimension finer asymptotic results are known, cf. [ZB, V1, AV, An].

New developments include the optimality of the general rates in $\mathcal{X}_0$ and the weighted convergence for fast diffusion. Full proofs are given for both. We mention the new geometrical and smoothness developments in Section 18, the work by the energy method of Section 19. The study of complexity in L^∞ is only mentioned in passing. A number of less related results have been left out by lack of space.

Acknowledgment

The author is grateful to a number of colleagues for comments and suggestions, and very especially to the referee whose positive criticism has resulted in a clear improvement of the text. He also wants to apologize for unintended omissions of results.

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