Kolmogorov-Smirnov Test

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Let $X_1, \ldots, X_n$ be a simple random sample drawn from a distribution F. The goal of the Kolmogorov-Smirnov goodness of fit test is to test the null hypothesis $H_0 : F = F_0$, where F_0 is a fixed continuous distribution function. The test was introduced by Kolmogorov (1933) and studied afterwards by Smirnov (1939a,b) in detail. The basic idea of the test is to compare the theoretical distribution function under the null hypothesis, F_0, with the empirical distribution function corresponding to the sample, $F_n(x) = \sharp\{i : X_i \leq x\}/n$[1]. The comparison is carried out using the Kolmogorov-Smirnov statistic,

$$K_n \doteq \sup_x |F_n(x) - F_0(x)| = \|F_n - F_0\|_\infty,$$

that is, the biggest difference between both distribution functions. If the null hypothesis is true, then by Glivenko-Cantelli Theorem, we have that $K_n \to 0$, as $n \to \infty$, almost surely. Therefore, it is reasonable to reject H_0 whenever K_n is large enough for a fixed significance level.

To establish which values of K_n are large enough for rejection we need to study the distribution of K_n under H_0. Fortunately, it can be shown that this distribution is the same for any continuous distribution F_0. As a consequence, the critical values $K_{n,\alpha}$ such that

$$\mathrm{P}_{H_0}(K_n > K_{n,\alpha}) = \alpha$$

do not depend on F_0, and we can define the critical region of the test as $R = \{K_n > K_{n,\alpha}\}$, so that the significance level is α *for any* F_0. Many textbooks provide tables including $K_{n,\alpha}$ for selected values of n and α, so that the effective application of the test is quite simple.

When the sample size is large, we can also construct an approximate critical region using the asymptotic distribution of $\sqrt{n}K_n$, derived by Smirnov. In fact, $\sqrt{n}K_n$ converges in distribution to K, where

$$\mathrm{P}(K > x) = 2\sum_{j=1}^{\infty}(-1)^{j+1}\exp(-2j^2x^2).$$

Comparing with other goodness of fit tests, as those based on the χ^2 distribution, the Kolmogorov-Smirnov test presents the following advantages: a) the critical region is exact, it

[1] $\sharp A$ stands for the number of elements of the set A

is not based on asymptotic results, and b) it does not require to arrange the observations in classes, so that the test makes an efficient use of the information in the sample. On the other hand, its main limitations are: a) it can only be applied to continuous distributions, and b) the distribution F_0 must be completely specified. For instance, the critical region that we have defined above would not be valid –without further modification– to test that the observations come from a generic normal distribution (with arbitrary expectation and variance).

References

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