L^1-CONTRACTION FOR BOUNDED (NONINTEGRABLE) SOLUTIONS OF DEGENERATE PARABOLIC EQUATIONS

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SUMMARY

We establish existence, uniqueness and continuous dependence estimates for bounded (nonintegrable) entropy solutions of degenerate parabolic equations with local and nonlocal diffusion. To do this, a sort of L^1_{loc}-contraction estimate is obtained. Many of the results are new in both the local and nonlocal case.

MAIN RESULTS

We consider bounded nonintegrable entropy solutions of the following degenerate parabolic equation:

$$\text{(DPE)}\qquad \begin{cases} \partial_t u + \operatorname{div} f(u) = \mathfrak{L}\varphi(u) & \text{in} \quad Q_T \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^d, \end{cases}$$

where $Q_T := \mathbb{R}^d \times (0,T)$, $u = u(x,t)$ is the unknown function, and div the x-divergence. The operator $\mathfrak{L}$ will be either the x-Laplacian $\triangle$ or the nonlocal anomalous diffusion operator $\mathcal{L}^\mu$ defined on $C_c^\infty(\mathbb{R}^d)$ as

$$\mathcal{L}^\mu[\phi](x) = \int_{|z|>0} \phi(x+z) - \phi(x) - z\cdot D\phi(x)\mathbf{1}_{|z|\leq 1}\,\mathrm{d}\mu(z).$$

We assume that

$$f \in W^{1,\infty}_{\text{loc}}(\mathbb{R},\mathbb{R}^d),$$

$$\varphi \in W^{1,\infty}_{\text{loc}}(\mathbb{R}) \text{ is nondecreasing,}$$

and

$$\mu \text{ is a nonnegative Radon measure on } \mathbb{R}^d\setminus\{0\} \text{ such that}$$

$$\int_{|z|\leq 1}|z|^2\,\mathrm{d}\mu(z) + \int_{|z|>1}\mathrm{d}\mu(z) < \infty.$$

Remark. In our most general local case, (DPE) has the right-hand side $\operatorname{div}(a(u)Du)$ where a is a symmetric, positive-definite and bounded matrix. To simplify the exposition, we have chosen, for $1 \leq i,j \leq d$,

$$\begin{cases} a_{ij}(u) = \varphi'(u) & \text{if} \quad i = j \\ a_{ij}(u) = 0 & \text{if} \quad i \neq j. \end{cases}$$

Theorem 1. Let u, v be entropy solutions of (DPE) with respective initial data $u_0, v_0 \in L^\infty(\mathbb{R}^d)$. Then there exists $\psi \in C([0,T];L^1(\mathbb{R}^d))$ such that for any $t \in [0,T]$

$$\text{(MR)}\qquad \begin{aligned} &\int_{|x-x_0|\leq R}(u-v)^+(x,t)\,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^d}(u_0-v_0)^+(x)\psi(x,t)\,\mathrm{d}x. \end{aligned}$$

The function ψ is roughly speaking the solution of the second order Hamilton-Jacobi-Bellman equation (HJB) which is a "dual equation" of (DPE). Our proof thus relies on a nonstandard L^1-estimate for that kind of equation. The above result is a sort of an L^1_{loc}-contraction result, and further consequences are

- L^1_{loc}-and BV_{loc}-bounds, and
- a comparison principle and L^∞-bounds.

We also obtain:

Theorem 2. There exists a unique entropy solution u of (DPE) when $u_0 \in L^\infty(\mathbb{R}^d)$. Moreover, $u \in L^\infty(Q_T) \cap C([0,T];L^1_{\text{loc}}(\mathbb{R}^d))$.

When $\mathfrak{L} = \mathcal{L}^\mu$, we improve Theorems 1 and 2 in [2] compared to [5] since we in the former consider general Lévy measures μ and a more "optimal" ψ (see Approaches 1 and 2). In the local isotropic case, [1] only provides an improvement of Theorem 1.

To our knowledge, Theorem 1, the L^1_{loc}-and BV_{loc}-bounds, and Theorem 2 are new in the local anisotropic case [1].

ENTROPY SOLUTIONS

As usual, we use the Kružkov entropy-entropy flux pairs $(u-k)^\pm$ and $\operatorname{sign}^\pm(u-k)(f(u)-f(k))$ for all $k \in \mathbb{R}$. The treatment of entropy solutions for $\partial_t u + \operatorname{div} f(u) = 0$ is rather classical, so, we only discuss the treatment of the diffusion operators.

In the local isotropic case, we have

$$\partial_t(u-k)^\pm + \operatorname{div}\big(\operatorname{sign}^\pm(u-k)(f(u)-f(k))\big) - \triangle(\varphi(u)-\varphi(k))^\pm \leq 0$$

in $\mathcal{D}'(Q_T)$ for all $k \in \mathbb{R}$, and $\varphi(u) \in L^2(0,T;H^1_{\text{loc}}(\mathbb{R}^d))$. A definition for the anisotropic case is more delicate, and we need a weak chain rule in addition to "energy".

In the nonlocal case, we argue by splitting $\mathcal{L}^\mu$ at some level $r > 0$, that is,

$$\mathcal{L}^\mu[\phi](x) = \mathcal{L}^\mu_r[\phi](x) + \mathcal{L}^{\mu,r}[\phi](x) + \operatorname{div}(b^{\mu,r}\phi)$$

where $b^{\mu,r} := -\int_{|z|>r} z\mathbf{1}_{|z|\leq 1}\,\mathrm{d}\mu(z)$. This results in different treatments of the singular and nonsingular parts of the operator. We obtain

$$\begin{aligned} &\partial_t(u-k)^\pm + \operatorname{div}\big(\operatorname{sign}^\pm(u-k)(f(u)-f(k)) - b^{\mu,r}(\varphi(u)-\varphi(k))^\pm\big) \\ &- \mathcal{L}^\mu_r[(\varphi(u)-\varphi(k))^\pm] - \operatorname{sign}^\pm(u-k)\mathcal{L}^{\mu,r}[\varphi(u)] \leq 0 \end{aligned}$$

in $\mathcal{D}'(Q_T)$ for all $k \in \mathbb{R}$. Note that we do not need any notion of "energy" here.

KATO INEQUALITY

To show the L^1-contraction for $(u-v)^+$, and hence, the uniqueness of entropy solutions, we will use the following inequality.

Let u and v be entropy solutions of (DPE) with respective initial data $u_0, v_0 \in L^\infty(\mathbb{R}^d)$. Then for all $0 \leq \phi \in C_c^\infty(Q_T)$,

$$\text{(Kato)}\qquad \begin{aligned} 0 \leq \iint_{Q_T} \Big(&(u-v)^+\partial_t\phi \\ &+ \operatorname{sign}^+(u-v)\,(f(u)-f(v))\cdot D\phi \\ &+ (\varphi(u)-\varphi(v))^+\mathfrak{L}^*\phi \Big)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

In the general anisotropic local setting, this result has been proved in [3] for $u, v \in L^\infty(0,T;L^1(\mathbb{R}^d)) \cap L^\infty(Q_T)$ (see also [4] for a "kinetic" argument). The same proof holds in our case [1], that is, when we only assume $u, v \in L^\infty(Q_T)$, and the proof itself is based on the Kružkov's doubling of variables technique [6]. In the isotropic nonlocal setting, the result is also based on Kružkov's technique and can be found in [5].

Now, choose $\phi(x,t) := \Theta(t)\Gamma(x,t)$ where $0 \leq \Theta \in C_c^\infty(0,T)$ and $0 \leq \Gamma \in C_c^\infty(Q_T)$, that is, (Kato) becomes

$$\begin{aligned} 0 \leq &\iint_{Q_T}(u-v)^+\Gamma\Theta'\,\mathrm{d}x\,\mathrm{d}t \\ &+ \iint_{Q_T}\Theta(u-v)^+\left[\partial_t\Gamma + L_f|D\Gamma| + L_\varphi\big(\mathfrak{L}^*\Gamma\big)^+\right]\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

If Γ is a classical solution of

$$\text{(HJB)}\qquad \partial_t\Gamma + L_f|D\Gamma| + L_\varphi\left(\mathfrak{L}^*\Gamma\right)^+ \leq 0,$$

where L_f, L_φ are the respective Lipschitz constants of f, φ, and

$$\Theta(t) = \Theta_\varepsilon(t) := \int_{-\infty}^t \omega_\varepsilon(s-t_1) - \omega_\varepsilon(s-t_2)\,\mathrm{d}s$$

for $0 < t_1 < t_2 < T$ with $t_1 \to 0^+$ and $t_2 \to T^-$, then we obtain

$$\text{(PreMR)}\qquad \begin{aligned} &\int_{\mathbb{R}^d}(u-v)^+(x,T)\Gamma(x,T)\,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^d}(u_0-v_0)^+(x)\Gamma(x,0)\,\mathrm{d}x. \end{aligned}$$

There are now (at least) two possible approaches:

APPROACH 1: We solve (HJB) **indirectly** by viscosity solutions and regularization procedures. This is done in [5].

APPROACH 2: We solve (HJB) **directly** by viscosity solutions and regularization procedures. This is done in [1, 2].

APPROACH 1 (CF. [5])

In this approach, we find solutions of (HJB) by considering (super)solutions of two related problems.

Consider the respective nonnegative classical solutions Ψ, Φ of

$$\text{(Conv)}\qquad \partial_t\Psi + L_f|D\Psi| \leq 0 \quad \text{in} \quad Q_T,$$

$$\text{(Diff)}\qquad \partial_t\Phi + L_\varphi(\mathfrak{L}^*\Phi)^+ \leq 0 \quad \text{in} \quad Q_T,$$

and define

$$\Gamma(x,t) := \Psi(\cdot,t) *_x \Phi(\cdot,t)(x),$$

then $0 \leq \Gamma$ is a classical solution of (HJB).

Equation (Conv) has the C_c^∞-solution

$$\Psi_{\tilde\delta}(x,t) := \left[\mathbf{1}_{(-\infty,M]} * \omega_\varepsilon\right]\left(\sqrt{\tilde\delta^2 + |x-x_0|^2} + L_f t\right),$$

where L_f is the Lipschitz constant of f, $M > L_f T + 1$, $\tilde\delta > 0$, $x_0 \in \mathbb{R}^d$, and ω_ε is a mollifier.

To find a classical solution of (Diff), we mollify (in space and time) the viscosity solution of the same equation. Denote this solution by Φ_δ. It is not trivial to demonstrate that Φ_δ is L^1 in space for all $t \in [0,T]$. Our approach is to find a classical L^1-supersolution of (Diff), and then conclude by the comparison principle that $\Phi(\cdot,t) \in L^1(\mathbb{R}^d)$.

Now, we insert $\Gamma_{\delta,\tilde\delta}(x,t) := \Psi_\delta(\cdot,t) *_x \Phi_{\tilde\delta}(\cdot,t)(x)$ into (PreMR), and after several limit procedures we obtain (MR) with

$$\psi(x,t) = \Phi(\cdot, L_\varphi t) *_x \mathbf{1}_{|x-x_0|\leq R+1+L_f t}(x).$$

It seems a bit unnatural that the $+1$-factor is present in the above function. This is due to the fact that the initial data for (Diff) is not a Dirac's delta. We have not been able to choose this initial data for two reasons: i) There is no well-posedness theory for equations like (Diff) with measure initial data, and ii) the L^1-bound for Φ is obtained by comparison with a particular L^1-supersolution.

APPROACH 2 (CF. [1, 2])

In this approach, we work directly with (HJB). That is, we consider the problem

$$\begin{cases} \partial_t\Gamma + L_f|D\Gamma| + L_\varphi\left(\mathfrak{L}^*\Gamma\right)^+ = 0 & \text{in} \quad Q_T \\ \Gamma(x,T) = \mathbf{1}_{|x-x_0|\leq R}(x) & \text{on} \quad \mathbb{R}^d. \end{cases}$$

Recall that the standard (viscosity) theory for Hamilton-Jacobi-Bellman equations usually assumes the terminal data to be continuous and bounded, and there is no general L^1-theory for these equations. In our case, we thus, need to consider the following difficulties:

- the terminal data is merely bounded, and
- under which assumptions can the solution Γ be integrable.

In our works in progress, we build unique viscosity solutions of Hamilton-Jacobi-Bellman equations for merely bounded terminal data. Moreover, we give sufficient conditions on the terminal data in order to make sure that the viscosity solution is integrable, that is, we get (MR) with integrable $\psi = \Gamma$ solving the above problem.

References

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