Instantaneous boundedness in linear and nonlinear diffusion equations, and related functional inequalities

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Recent trends in the analysis of non-local and non-linear equations

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Main results

$$\text{(GPME)} \qquad \begin{cases} \partial_t u + (-\mathfrak{L})[u^m] = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

(Think of $\mathfrak{L}$ as Δ or $-(-\Delta)^{\frac{\alpha}{2}}$, but it is also more general.)

- L^1–L^∞-smoothing.
- Their relation with functional inequalities like Gagliardo-Nirenberg-Sobolev:

$$\|f\|_{L^{2^*}(\mathbb{R}^N)} \lesssim Q_{-\mathfrak{L}}[f]^{\frac{1}{2}},$$

where

$$2^* > 2 \qquad \text{and} \qquad Q_{-\mathfrak{L}}[f,g] = \int f(-\mathfrak{L})[g].$$

($\mathfrak{L} = \Delta$ implies that $Q_{-\mathfrak{L}}[f] = \|\nabla f\|^2_{L^2(\mathbb{R}^N)}$.)

JE, M. Bonforte. Instantaneous boundedness in linear and nonlinear diffusion equations, and related functional inequalities. In preparation, 2022.

Linear case ($m = 1$). Scaling

Consider

$$
\text{(HE)} \qquad \begin{cases} \partial_t u + (-\Delta)[u] = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}
$$

Assume that we are searching for an estimate of the form

$$
\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-\sigma_1} \|u_0\|_{L^1(\mathbb{R}^N)}^{\sigma_2}.
$$

What are the only admissible values of σ_1, σ_2?

If u solves (HE), then

$$
\tilde{u}(x,t) := \kappa u(\Xi x, \Lambda t) \qquad \text{for } \kappa, \Xi, \Lambda > 0
$$

also solves (HE) with initial data $\tilde{u}_0(x) := \kappa u(\Xi x, 0)$ as long as $\Xi^2 = \Lambda$. Fix $\tilde{u}$ with mass $M = 1$, then $\kappa = M^{-1}\Xi^N$, and $\tilde{u}_0 \in L^1$.

Linear case ($m = 1$). Scaling

Consider

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\text{(HE)} \qquad \begin{cases} \partial_t u + (-\Delta)[u] = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}
$$

I.e.,

$$
\|\tilde{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-\sigma_1}\|\tilde{u}_0\|_{L^1(\mathbb{R}^N)}^{\sigma_2}
$$

$$
\begin{aligned}
\iff \|u(\Xi\cdot,\Lambda t)\|_{L^\infty(\mathbb{R}^N)} &\lesssim \kappa^{-1}\Lambda^{\sigma_1}(\Lambda t)^{-\sigma_1}\kappa^{\sigma_2}\|u(\Xi\cdot,0)\|_{L^1(\mathbb{R}^N)}^{\sigma_2} \\
&= \kappa^{\sigma_2-1}\Lambda^{\sigma_1}(\Lambda t)^{-\sigma_1}\Xi^{-N\sigma_2}\|u_0\|_{L^1(\mathbb{R}^N)}^{\sigma_2} \\
&= \kappa^{\sigma_2-1}\Xi^{2\sigma_1-N\sigma_2}(\Lambda t)^{-\sigma_1}\|u_0\|_{L^1(\mathbb{R}^N)}^{\sigma_2},
\end{aligned}
$$

with $\sigma_1 = (N/2)\sigma_2$ and $\sigma_2 = 1$.

Hence,

$$
\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-\frac{N}{2}}\|u_0\|_{L^1(\mathbb{R}^N)}.
$$

Consider

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I.e.,

$$\begin{aligned} \|\tilde{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} &\lesssim t^{-\sigma_1} \|\tilde{u}_0\|^{\sigma_2}_{L^1(\mathbb{R}^N)} \\ \iff \|u(\Xi\cdot,\Lambda t)\|_{L^\infty(\mathbb{R}^N)} &\lesssim \kappa^{-1}\Lambda^{\sigma_1}(\Lambda t)^{-\sigma_1}\kappa^{\sigma_2}\|u(\Xi\cdot,0)\|^{\sigma_2}_{L^1(\mathbb{R}^N)} \\ &= \kappa^{\sigma_2-1}\Lambda^{\sigma_1}(\Lambda t)^{-\sigma_1}\Xi^{-N\sigma_2}\|u_0\|^{\sigma_2}_{L^1(\mathbb{R}^N)} \\ &= \kappa^{\sigma_2-1}\Xi^{2\sigma_1-N\sigma_2}(\Lambda t)^{-\sigma_1}\|u_0\|^{\sigma_2}_{L^1(\mathbb{R}^N)}, \end{aligned}$$

with $\sigma_1 = (N/2)\sigma_2$ and $\sigma_2 = 1$.

J. L. Vázquez. *Smoothing and decay estimates for nonlinear diffusion equations*. Oxford Lecture Series in Mathematics and its Applications, volume 33. Oxford University Press, Oxford, 2006.

Linear case ($m = 1$). Heat kernel

The function

$$u(x,t) = \int_{\mathbb{R}^N} u_0(y) H_{-\Delta}(x-y,t)\,\mathrm{d}y,$$

with

$$H_{-\Delta}(x-y,t) \eqsim t^{-\frac{N}{2}} \exp\Big(-\frac{|x-y|^2}{4t}\Big) \qquad \text{(the constant is } (4\pi)^{-\frac{N}{2}}\text{)},$$

is the solution of

$$\text{(HE)} \qquad \begin{cases} \partial_t u + (-\Delta)[u] = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

Since $0 \leq H_{-\Delta}(x-y,t) \lesssim t^{-\frac{N}{2}}$, we immediately have

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-\frac{N}{2}} \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Linear case ($m = 1$). Nash inequality

Assume (by L^p-interpo. and the Gagliardo-Nirenberg-Sobolev ineq.)

$$\|f\|_{L^2(\mathbb{R}^N)} \lesssim \|f\|_{L^1(\mathbb{R}^N)}^{\vartheta} \|\nabla f\|_{L^2(\mathbb{R}^N)}^{1-\vartheta}$$

with

$$\vartheta = \frac{1}{2}\frac{2^* - 2}{2^* - 1} \qquad \text{where } 2^* = 2N/(N-2).$$

Define $Y(t) := \|u(t)\|_{L^2(\mathbb{R}^N)}^2$, and consider

$$Y'(t) = \int \partial_t(u^2) = 2\int u\partial_t u = -2\int u(-\Delta)[u] = -2\|\nabla u\|_{L^2(\mathbb{R}^N)}^2$$
$$\lesssim -Y(t)^{\frac{1}{1-\vartheta}} \|u(t)\|_{L^1(\mathbb{R}^N)}^{-\frac{2\vartheta}{1-\vartheta}} \leq -\|u_0\|_{L^1(\mathbb{R}^N)}^{-2\frac{\vartheta}{1-\vartheta}} Y(t)^{1+\frac{\vartheta}{1-\vartheta}}.$$

Solving the differential inequality gives

$$\|u(t)\|_{L^2(\mathbb{R}^N)}^2 = Y(t) \lesssim t^{-\frac{1-\vartheta}{\vartheta}} \|u_0\|_{L^1(\mathbb{R}^N)}^2 = t^{-\frac{N}{2}} \|u_0\|_{L^1(\mathbb{R}^N)}^2.$$

Linear case ($m = 1$). Nash inequality

Assume

$$\|f\|_{L^2(\mathbb{R}^N)} \lesssim \|f\|_{L^1(\mathbb{R}^N)}^{\vartheta} \|\nabla f\|_{L^2(\mathbb{R}^N)}^{1-\vartheta}.$$

We have

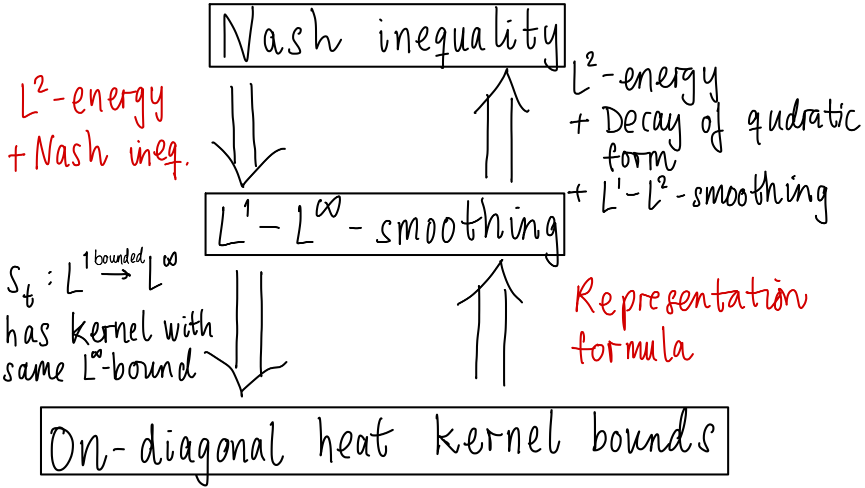
$$\|u(t)\|_{L^2(\mathbb{R}^N)} \lesssim t^{-\frac{N}{4}} \|u_0\|_{L^1(\mathbb{R}^N)},$$

and then, by duality,

$$\begin{aligned}
\|u(t)\|_{L^\infty} &= \sup_{\|\phi\|_{L^1}=1} \left| \int u(t)\phi \right| = \sup_{\|\phi\|_{L^1}=1} \left| \int S_t[u_0]\phi \right| \\
&= \sup_{\|\phi\|_{L^1}=1} \left| \int S_{\frac{t}{2}}[S_{\frac{t}{2}}[u_0]]\phi \right| = \sup_{\|\phi\|_{L^1}=1} \left| \int S_{\frac{t}{2}}[u_0] S_{\frac{t}{2}}[\phi] \right| \\
&\leq \sup_{\|\phi\|_{L^1}=1} \|S_{\frac{t}{2}}[u_0]\|_{L^2} \|S_{\frac{t}{2}}[\phi]\|_{L^2} \lesssim t^{-\frac{N}{4}} \|S_{\frac{t}{2}}[u_0]\|_{L^2} \\
&\lesssim t^{-\frac{N}{2}} \|u_0\|_{L^1}.
\end{aligned}$$

E. H. Lieb and M. Loss. *Analysis*. Graduate Studies in Mathematics, volume 14. American Mathematical Society, Providence, RI, 2001.

Linear case ($m = 1$). Overview



Nonlinear case ($m > 1$). Introduction

$$(\text{GPME}) \qquad \begin{cases} \partial_t u + (-\mathfrak{L})[u^m] = 0 & \text{in } \mathbb{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

Cons:

- We do not have a representation formula.
- It is harder to find the correct functional set-up.

Pros:

- We still have scaling (always time-scaling).
- Some estimates are true in the nonlinear, but not true in the linear.

Nonlinear case ($m > 1$). A nice trick

Consider the operator $-\mathfrak{L} \mapsto I - \mathfrak{L}$, i.e.,

$$\partial_t u + (I - \mathfrak{L})[u^m] = 0 \qquad \Longleftrightarrow \qquad \partial_t u + (-\mathfrak{L})[u^m] = -u^m.$$

Then $t \mapsto Y(t)$ solves $Y'(t) = -Y(t)^{1+(m-1)}$, so

$$Y(t) \leq \left(\frac{1}{(m-1)t}\right)^{\frac{1}{m-1}}.$$

Moreover, comparison yields (with $Y(0) = \infty$)

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq Y(t) \leq \left(\frac{1}{(m-1)t}\right)^{\frac{1}{m-1}}.$$

Holds independently of the operator! But needs "good" nonlinearity.

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L. Véron. Effets régularisants de semi-groupes non linéaires dans des espaces de Banach. *Ann. Fac. Sci. Toulouse Math. (5)*, 1(2):171–200, 1979.

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Nonlinear case ($m > 1$). Tools

$$\text{(GPME)} \qquad \begin{cases} \partial_t u + (-\mathfrak{L})[u^m] = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{on } \mathbb{R}^N. \end{cases}$$

We need:

- $(-\mathfrak{L})^{-1}$ with kernel $\mathbb{G}_{-\mathfrak{L}}(x-y) = \int_0^\infty H_{-\mathfrak{L}}(x-y,t)\,\mathrm{d}t$.
- Time scaling. $u_\Lambda(x,t) := \Lambda^{\frac{1}{m-1}} u(x,\Lambda t)$ solution when u is.
- Comparison principle.
- L^p-bounds.

Nonlinear case ($m > 1$). Tools

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Nonlinear case ($m > 1$). Time-monotonicity

- Time scaling. $u_\Lambda(x,t) := \Lambda^{\frac{1}{m-1}} u(x, \Lambda t)$ solution when u is.
- Comparison principle.

Provides the well-known

$$\partial_t u \geq -\frac{u}{(m-1)t}$$

M. Crandall and M. Pierre. Regularizing effects for $u_t + A\varphi(u) = 0$ in L^1. *J. Funct. Anal.*, 45(2):194–212, 1982.

Nonlinear case ($m > 1$). "Representation formula"

$$\partial_t u + (-\mathfrak{L})[u^m] = 0 \quad \iff \quad u^m = -(-\mathfrak{L})^{-1}[\partial_t u]$$
$$\iff \quad u^m = -\partial_t u *_x \mathbb{G}_{-\mathfrak{L}}$$
$$\iff \quad u^m \leq \frac{u}{(m-1)t} *_x \mathbb{G}_{-\mathfrak{L}}.$$

Hence,

$$\big(u(x,t)\big)^m \leq \frac{1}{(m-1)t} \int_{\mathbb{R}^N} u(y,t)\mathbb{G}_{-\mathfrak{L}}(x-y)\,\mathrm{d}y.$$

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Hence,

$$\big(u(x,t)\big)^m \leq \frac{1}{(m-1)t} \int_{\mathbb{R}^N} u(y,t) \underbrace{\mathbb{G}_{-\mathfrak{L}}(x-y)}_{=\int_0^\infty H_{-\mathfrak{L}}(x-y,t)\,\mathrm{d}t} \,\mathrm{d}y.$$

M. Bonforte and J. L. Vázquez. A priori estimates for fractional nonlinear degenerate diffusion equations on bounded domains. *Arch. Ration. Mech. Anal.*, 218(1):317–362, 2015.

Nonlinear case $(m > 1)$. Examples

$$\partial_t u + (-\mathfrak{L})[u^m] = 0$$

- $-\mathfrak{L} = (-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0, 2]$ gives

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-N\theta} \|u_0\|^{\alpha\theta}_{L^1(\mathbb{R}^N)} \qquad \text{where } \theta := (\alpha + N(m-1))^{-1}.$$

A. de Pablo, F. Quirós, A. Rodríguez, and J. L. Vázquez. A general fractional porous medium equation. *Comm. Pure Appl. Math.*, 65(9):1242–1284, 2012.

Nonlinear case ($m > 1$). Examples

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- $-\mathfrak{L} = (\kappa^2 I - \Delta)^{\frac{\alpha}{2}} - \kappa^\alpha I$ with $\kappa > 0$ and $\alpha \in (0, 2)$ gives

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-\frac{1}{m}}\|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{1}{m}}.$$

- $-\mathfrak{L} = \sum_{i=1}^N (-\partial^2_{x_i x_i})^{\frac{\alpha}{2}}$ with $\alpha \in (0, 2)$ gives

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-\frac{1}{m-1}} + \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Nonlinear does *not* imply linear

Consider

$$\partial_t u + (-\mathfrak{L})[u^m] = 0,$$

with

$$-\mathfrak{L}[\psi](x) = \psi(x) - \int_{\mathbb{R}^N} \psi(z) J(x-z)\,\mathrm{d}z = (I - J*_x)[\psi](x)$$

where $J \geq 0$ such that $\|J\|_{L^1(\mathbb{R}^N)} = 1$ and $J \in L^p(\mathbb{R}^N)$.

- If $m = 1$, then

$$u(x,t) = u_0(x)\mathrm{e}^{-t} + W(x,t),$$

where $W \geq 0$ is some smooth function. Hence, no smoothing.

F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi, J. Toledo-Melero. *Nonlocal diffusion problems*. Mathematical Surveys and Monographs, volume 165. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010.

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- If $m > 1$, then

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \lesssim t^{-\frac{1}{m-1}} + \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Thank you for your attention!

Nonlinear case ($m > 1$). Why not the Moser iteration?

Idea: $\lim_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}$.

The Stroock-Varopoulos inequality gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^p} = \int \partial_t(u^p) = p\int u^{p-1}\partial_t u = -p\int u^{p-1}(-\mathfrak{L})[u^m]$$
$$= -pQ_{-\mathfrak{L}}[u^{p-1}, u^m] \leq -\frac{4mp(p-1)}{(p+m-1)^2}Q_{-\mathfrak{L}}[u^{\frac{p+m-1}{2}}].$$

L^p-interpo. and the Gagliardo-Nirenberg-Sobolev inequality gives

$$\|f\|_{L^{\tilde{p}}(\mathbb{R}^N)} \leq C\|f\|^{\vartheta}_{L^{\tilde{q}}(\mathbb{R}^N)} Q_{-\mathfrak{L}}[f]^{\frac{1}{2}(1-\vartheta)},$$

where

$$2 \leq \tilde{p} < 2^*, \qquad 1 \leq \tilde{q} < \tilde{p}, \qquad \vartheta := \frac{\tilde{q}}{\tilde{p}}\frac{2^*-\tilde{p}}{2^*-\tilde{q}}.$$

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