On nonlocal equations of porous medium type

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23 May 2017

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A talk given at
Stability for Evolution Problems
as a part of
International Conference on Elliptic and Parabolic Problems

Local and nonlocal diffusion

Diffusion is the act of "spreading out" – the movement from areas of high concentration to areas of low concentration.

Let $u(x,t)$ be the probability for a particle to be at discrete $x \in h\mathbb{Z}, t \in \tau\mathbb{N} \cap [0,T]$.

Assume that we are only allowed to jump one point either to the left or to the right, each with probability $\frac{1}{2}$.

The probability of being at point x at time $t+\tau$ is then

$$u(x,t+\tau) = \frac{1}{2}u(x+h,t) + \frac{1}{2}u(x-h,t).$$

Local and nonlocal diffusion

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Assume that we are only allowed to jump one point either to the left or to the right, each with probability $\frac{1}{2}$.

Choose $\tau = \frac{1}{2}h^2$ and divide by it to obtain

$$\frac{u(x,t+\tau)-u(x,t)}{\tau} = \frac{u(x+h,t)+u(x-h,t)-2u(x,t)}{h^2}.$$

Let $u(x,t)$ be the probability for a particle to be at discrete $x \in h\mathbb{Z}$, $t \in \tau\mathbb{N} \cap [0,T]$.

Assume that we are only allowed to jump one point either to the left or to the right, each with probability $\frac{1}{2}$.

As $\tau, h \to 0^+$,

$$\partial_t u = \Delta u \qquad \text{in} \qquad \mathbb{R} \times (0,T),$$

that is, u is a solution of the heat equation.

A. Einstein. Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. *Annalen der Physik* (in German), 322(8): 549–560, 1905.

Local and nonlocal diffusion

Now, we change the rules: A particle can jump to any point with a certain probability, but the probability of jumping to the left or to the right is exactly the same.

We choose a density $K : \mathbb{R} \to [0, \infty)$ up to normalization factors as

$$K(y) = \begin{cases} \frac{1}{|y|^{1+\alpha}} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

for $\alpha \in (0, 2)$. It satisfies

(i) $K(y) = K(-y)$

(ii) $\sum_{k \in \mathbb{Z}} K(k) = 1$.

As before, the probability of being at point x at time $t + \tau$ is

$$u(x, t + \tau) = \sum_{k \in \mathbb{Z}} K(k) u(x + hk, t).$$

Local and nonlocal diffusion

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Then, for the choice $\tau = h^\alpha$,

$$\frac{u(x, t + \tau) - u(x, t)}{\tau} = \sum_{k \in \mathbb{Z} \setminus \{0\}} \big(u(x + hk, t) - u(x, t)\big) K(hk) h,$$

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$$K(y) = \begin{cases} \frac{1}{|y|^{1+\alpha}} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

for $\alpha \in (0, 2)$. It satisfies

(i) $K(y) = K(-y)$

(ii) $\sum_{k\in\mathbb{Z}} K(k) = 1$.

or

$$\frac{u(x, t + \tau) - u(x, t)}{\tau} = \int_{|z|>0} \big(u(x + z, t) - u(x, t)\big) \,\mathrm{d}\nu_h$$

with the measure $\nu_h(z) := \sum_{k\in\mathbb{Z}\setminus\{0\}} hK(hk)\delta_{hk}(z)$.

Now, we change the rules: A particle can jump to any point with a certain probability, but the probability of jumping to the left or to the right is exactly the same.

As $\tau, h \to 0^+$,

$$\begin{aligned}\partial_t u &= \int_{|z|>0} \left(u(x+z,t) - u(x,t) - z\partial_x u(x,t)\mathbf{1}_{|z|\leq 1}\right) \frac{c_{1,\alpha}}{|z|^{1+\alpha}}\,\mathrm{d}z \\ &= -(-\Delta)^{\frac{\alpha}{2}} u \qquad \text{in} \qquad \mathbb{R}\times(0,T)\end{aligned}$$

where $c_{1,\alpha} > 0$ and $-(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$ is the fractional Laplacian. We thus observe that u is a solution of the fractional heat equation.

E. Valdinoci. From the long jump random walk to the fractional Laplacian. *Bol. Soc. Esp. Mat. Apl. SeMA*, (49):33–44, 2009.

Generalized porous medium equations

We consider the following Cauchy problem:

$$\text{(GPME)} \qquad \begin{cases} \partial_t u - \mathcal{L}^\mu[\varphi(u)] = 0 & \text{in} \quad Q_T := \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

where

- $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, and
- $\mathcal{L}^\mu$ is a symmetric pure-jump Lévy operator (anomalous/nonlocal diffusion operator).

Main results:

- Uniqueness in L^∞.
- Existence in $L^1 \cap L^\infty$.
- Convergent numerical schemes in $L^1 \cap L^\infty$.

Selective summary of previous results

Local case: $\partial_t u = \Delta u$, $\partial_t u = \Delta u^m$, $\partial_t u = \Delta\varphi(u)$.

J. L. Vázquez. *The porous medium equation. Mathematical theory*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

Selective summary of previous results

Nonlocal case: $\partial_t u = \mathcal{L}^\mu[\varphi(u)]$.

- Well-posedness when $\mathcal{L}^\mu = -(-\Delta)^{\frac{\alpha}{2}}$:

Many people: Vázquez, de Pablo, Quirós, Rodríguez, Brändle, Bonforte, Stan, del Teso, Muratori, Grillo, Punzo, ...

- Well-posedness for other $\mathcal{L}^\mu$:

Nonsingular operators

F. Andreu-Vaillo, J. Mazón, J. D. Rossi, and J. J. Toledo-Melero. *Nonlocal diffusion problems*, volume 165 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010.

Fractional Laplace like operators (with some x-dependence)

A. de Pablo, F. Quirós, and A. Rodríguez. Nonlocal filtration equations with rough kernels. *Nonlinear Anal.*, 137:402–425, 2016.

Selective summary of previous results

Previous results (mostly) rely on:

- The porous medium nonlinearity $\varphi(u) = u^m$ with $m > 1$.
- A very restrictive class of Lévy operators.
- The use of L^1-energy solutions.

In our case:

- Uniqueness is hard to prove because of a very weak solution concept (however, existence is then easier).
- The result we obtain is kind of different since we work in L^∞.
- We can handle very weak assumptions on φ and $\mathcal{L}^\mu$.

Lévy operator

$\mathcal{L}^\mu$ is a symmetric pure-jump Lévy operator (anomalous/nonlocal diffusion operator) defined, for smooth enough functions ψ, as e.g. the singular integral

$$\mathcal{L}^\mu[\psi](x) := \int_{\mathbb{R}^N\setminus\{0\}} \big(\psi(x+z) - \psi(x) - z \cdot D\psi(x)\mathbf{1}_{|z|\leq 1}\big)\,\mathrm{d}\mu(z).$$

Assumptions

Unless otherwise stated we always assume that

(A$_\varphi$) $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing,

and

(A$_\mu$) $\mu \geq 0$ is a symmetric Radon measure on $\mathbb{R}^N \setminus \{0\}$ satisfying

$$\int_{|z|\leq 1} |z|^2 \,\mathrm{d}\mu(z) + \int_{|z|>1} 1 \,\mathrm{d}\mu(z) < \infty.$$

Assumptions

The assumption

$$\varphi : \mathbb{R} \to \mathbb{R} \text{ is continuous and nondecreasing,}$$

includes nonlinearities of the following kind

- the porous medium,
- fast diffusion, and
- Stefan problem.

Assumptions

The assumption

$$\mu \geq 0 \text{ is symmetric and satisfies } \int_{|z|>0} \min\{|z|^2, 1\} \, \mathrm{d}\mu(z) < \infty$$

ensures that our $\mathcal{L}^\mu$

- is the most general (symmetric, linear) nonlocal operator preserving the maximum principle;
- is a pure-jump symmetric Lévy operator;
- contains spatial discretizations of $\mathrm{tr}(\sigma\sigma^T D^2\cdot) + \mathcal{L}^\mu[\cdot]$;
- is relevant for applications (in finance, physics, biology, etc.);
- includes important examples:
  - the fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$;
  - relativistic Schrödinger type operators $m^\alpha I - (m^2 I - \Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$ and $m > 0$;
  - for the measure ν with $\nu(\mathbb{R}^N) < \infty$, $\mathcal{L}^\nu[\psi](x) = \int_{\mathbb{R}^N} \left(\psi(x+z) - \psi(x)\right) \mathrm{d}\nu(z)$;
  - for the function J with $\int_{\mathbb{R}^d} J(z) \, \mathrm{d}z = 1$, $\mathcal{L}^{J\,\mathrm{d}z}[\psi] = J * \psi - \psi$;
  - Fourier multipliers $\mathcal{F}(\mathcal{L}^\mu[\psi]) = -s_{\mathcal{L}^\mu}\mathcal{F}(\psi)$.

Distributional solution

Definition

Under the assumptions (A_φ), (A_μ), and $u_0 \in L^\infty(\mathbb{R}^N)$, $u \in L^\infty(Q_T)$ is a distributional solution of (GPME) if

$$0 = \int_0^T \int_{\mathbb{R}^N} \Big(u(x,t)\partial_t \psi(x,t) + \varphi(u(x,t))\mathcal{L}^\mu[\psi(\cdot,t)](x) \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}^N} u_0(x)\psi(x,0)\,\mathrm{d}x$$

for all $\psi \in C_c^\infty(\mathbb{R}^N \times [0,T))$.

Uniqueness

Theorem (Preuniqueness, [del Teso, JE, Jakobsen, 2017])

Assume (A_φ) *and* (A_μ). *Let* $u(x,t)$ *and* $\hat{u}(x,t)$ *satisfy*

$$u, \hat{u} \in L^\infty(Q_T),$$

$$u - \hat{u} \in L^1(Q_T),$$

$$\partial_t u - \mathcal{L}^\mu[\varphi(u)] = \partial_t \hat{u} - \mathcal{L}^\mu[\varphi(\hat{u})] \qquad \text{in} \qquad \mathcal{D}'(Q_T),$$

$$\operatorname*{ess\,lim}_{t\to 0^+} \int_{\mathbb{R}^N} (u(x,t) - \hat{u}(x,t))\psi(x,t)\,\mathrm{d}x = 0 \quad \forall \psi \in C_c^\infty(\mathbb{R}^N \times [0,T)).$$

Then $u = \hat{u}$ *a.e. in* Q_T.

Uniqueness

Corollary (Uniqueness, [del Teso, JE, Jakobsen, 2017])

Assume (A_φ), (A_μ), *and* $u_0 \in L^\infty(\mathbb{R}^N)$. *Then there is at most one distributional solution* u *of* (GPME) *such that* $u \in L^\infty(Q_T)$ *and* $u - u_0 \in L^1(Q_T)$.

Proof: Assume there are two solutions u and $\hat{u}$ with the same initial data u_0. Then all assumptions of Theorem Preuniqueness obviously hold ($\|u - \hat{u}\|_{L^1} \leq \|u - u_0\|_{L^1} + \|\hat{u} - u_0\|_{L^1} < \infty$), and $u = \hat{u}$ a.e. □

Uniqueness holds for $u_0 \notin L^1$, for example $u_0(x) = c + \phi(x)$ for $c \in \mathbb{R}$ and $\phi \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. However, periodic u_0's are not included because of the assumption $u - u_0 \in L^1$.

Existence

Theorem (Existence, [del Teso, JE, Jakobsen, 2017])

Assume (A_φ), (A_μ), *and* $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. *Then there exists a unique distributional solution* u *of* (GPME) *satisfying*

$$u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N)).$$

Proof: By convergence of numerical solution (as we will see later). □

The proof of Theorem Preuniqueness

Based on a proof by Brézis and Crandall.

H. Brézis and M. G. Crandall. Uniqueness of solutions of the initial-value problem for $u_t - \Delta\varphi(u) = 0$. *J. Math. Pures Appl. (9)*, 58(2):153–163, 1979.

1. Define $U := u - \hat{u}$ and $\Phi := \varphi(u) - \varphi(\hat{u})$, then U solves

$$\begin{cases} \partial_t U - \mathcal{L}^\mu[\Phi] = 0 & \text{in} \quad Q_T \\ U(x,0) = 0 & \text{on} \quad \mathbb{R}^N. \end{cases}$$

   Note that $U \in L^1 \cap L^\infty$ and $\Phi \in L^\infty$.

2. Consider

$$\varepsilon v_\varepsilon - \mathcal{L}^\mu[v_\varepsilon] = g \qquad \text{in} \qquad \mathbb{R}^N,$$

   and define $B_\varepsilon^\mu[g] := v_\varepsilon$, that is, $B_\varepsilon^\mu = (\varepsilon I - \mathcal{L}^\mu)^{-1}$ is the resolvent of $\mathcal{L}^\mu$.
   Note that this is a *linear* elliptic equation.

The proof of Theorem Preuniqueness

3. $U = \varepsilon B_\varepsilon^\mu[U] - \mathcal{L}^\mu[B_\varepsilon^\mu[U]]$.
4. Define

$$h_\varepsilon(t) := \int_{\mathbb{R}^N} U B_\varepsilon^\mu[U] \,\mathrm{d}x = \int_{\mathbb{R}^N} (\varepsilon I - \mathcal{L}^\mu) B_\varepsilon^\mu[U] B_\varepsilon^\mu[U] \,\mathrm{d}x$$
$$= \varepsilon \|B_\varepsilon^\mu[U]\|_{L^2}^2 + \|(\mathcal{L}^\mu)^{\frac{1}{2}}[B_\varepsilon^\mu[U]]\|_{L^2}^2.$$

5. Show that $h_\varepsilon \to 0$ as $\varepsilon \to 0^+$.

The proof of Theorem Preuniqueness

The hardest part is to show that $h_\varepsilon \to 0$ as $\varepsilon \to 0^+$. Some important steps:

1. $\varepsilon B_\varepsilon^\mu[U] \to 0$ implies $h_\varepsilon \to 0$ as $\varepsilon \to 0^+$.
2. Enough to prove that $\varepsilon B_\varepsilon^\mu[\gamma] \to 0$ for all $\gamma \in C_c^\infty(\mathbb{R}^N)$. Note that $\Gamma_\varepsilon := \varepsilon B_\varepsilon^\mu[\gamma]$ solves

$$\varepsilon \Gamma_\varepsilon - \mathcal{L}^\mu[\Gamma_\varepsilon] = \varepsilon\gamma \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}^N).$$

3. A priori results and compactness give $\Gamma_\varepsilon \to \Gamma$ as $\varepsilon \to 0^+$.
4. (Liouville) If $\operatorname{supp}\mu \neq \emptyset$, $\Gamma \in C_0$, and $\mathcal{L}^\mu[\Gamma] = 0$ in $\mathcal{D}'$, then $\Gamma \equiv 0$.

Note that a general Liouville result do not hold for $\mathcal{L}^\mu$: Take $\mu(z) = \delta_{2\pi}(z) + \delta_{-2\pi}(z)$, then $\mathcal{L}^\mu[\cos](x) = 0$, but this function is not constant.

Further uniqueness results

By similar methods, we obtain uniqueness in L^∞ for

(l/nlGPME)
$$\begin{cases} \partial_t u - \big(\operatorname{tr}(\sigma\sigma^T D^2\varphi(u)) + \mathcal{L}^\mu[\varphi(u)]\big) = 0 & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

and

$$u - \big(\operatorname{tr}(\sigma\sigma^T D^2\varphi(u)) + \mathcal{L}^\mu[\varphi(u)]\big) = g \qquad \text{in} \qquad \mathbb{R}^N.$$

Extensions and related results

We consider the following Cauchy problem:

(x-GPME)
$$\begin{cases} \partial_t u - A^\lambda[\varphi(u)] = 0 & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

where

- $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, and
- A^λ is a x-dependent generalization of $\mathcal{L}^\mu$.

Main results:

- Uniqueness in $L^1 \cap L^\infty$.
- Energy solutions $\iff$ distributional solutions with finite energy.

Important observations

$$\Delta_h\psi(x) := \frac{\psi(x+he_i)+\psi(x-he_i)-2\psi(x)}{h^2}$$
$$= \int_{\mathbb{R}^N} \big(\psi(x+z)-\psi(x)\big)\,\mathrm{d}\nu_h(z) =: \mathcal{L}^{\nu_h}[\psi](x)$$

where

$$\nu_h(z) := \frac{1}{h^2}\sum_{i=1}^{N} \delta_{he_i}(z) + \delta_{-he_i}(z)$$

satisfies $\nu_h(\mathbb{R}^N) < \infty$.

Important observations

By now, there exist several spatial discretizations of $\mathcal{L}^\mu$ (e.g. quadrature and spectral methods).

Y. Huang and A. Oberman. Finite difference methods for fractional Laplacians. Preprint, arXiv:1611.00164v1 [math.NA], 2016.

Our contribution is to note and exploit that (some of) the discretizations of $\mathcal{L}^\mu$ is again a Lévy operator.

Numerical schemes for (GPME)

Recall that our Cauchy problem was given as
(l/nlGPME)

$$\begin{cases} \partial_t u - \big(\operatorname{tr}(\sigma\sigma^T D^2\varphi(u)) + \mathcal{L}^\mu[\varphi(u)]\big) = 0 & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

Our numerical scheme can then take the following form

$$\text{(NumGPME)} \quad \begin{cases} \frac{U_\beta^j - U_\beta^{j-1}}{\Delta t} = G_{\Delta x}(U_\beta^j, U_\beta^{j-1}) & \text{in} \quad \Delta x\mathbb{Z}^N \times \Delta t\mathbb{N}, \\ \text{“}U_\beta^0 = u_0\text{”} & \text{in} \quad \Delta x\mathbb{Z}^N. \end{cases}$$

Numerical schemes for (GPME)

In our most general case, we have that

$$G_{\Delta x}(U_\beta^j, U_\beta^{j-1}) := \mathcal{L}^{\nu_{1,\Delta x}}[\varphi_1(U_\beta^j)] + \mathcal{L}^{\nu_{2,\Delta x}}[\varphi_2(U_\beta^{j-1})]$$

where $\nu_{1,\Delta x}, \nu_{2,\Delta x}$ satisfy $\nu_{1,\Delta x}(\mathbb{R}^N), \nu_{2,\Delta x}(\mathbb{R}^N) < \infty$.

Thus our framework includes

- a mixture of implicit and explicit schemes (θ-methods);
- the possibility of discretizing the singluar and nonsingular parts of $\mathcal{L}^\mu$ in different ways; and
- combinations of the above.

Note that by our previous observations, we are, in fact, able to approximate local operators of the form

$$\mathrm{tr}(\sigma\sigma^T D^2 \cdot).$$

Convergence of the numerical schemes

The scheme defined by (NumGPME) is

- monotone,
- (conservative if the φ's involved are Lipschitz)
- L^p-stable, and
- consistent.

Theorem (Convergence, [del Teso, JE, Jakobsen, 2017])

Assume $\nu_{1,\Delta x}(\mathbb{R}^N), \nu_{2,\Delta x}(\mathbb{R}^N) < \infty$, φ_1, φ_2 satisfy (A_φ), *and "$U_\beta^0 = u_0$" $\in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then, for the interpolant U, we have*

$$U \to u \qquad \text{in} \qquad C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N)) \qquad \text{as} \qquad \Delta x, \Delta t \to 0^+$$

where $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ is a distributional solution of (l/nlGPME).

References

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F. del Teso, JE, E. R. Jakobsen. On distributional solutions of local and nonlocal problems of porous medium type. Preprint, 2017.

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Thank you for your attention!