On nonlocal equations of porous medium type

Survey of 4 papers

Jørgen Endal

Department of mathematical sciences
NTNU

03 May 2017

In collaboration with F. del Teso and E. R. Jakobsen.

DNA seminar, NTNU

Local and nonlocal diffusion

Diffusion is the act of “spreading out” – the movement from areas of high concentration to areas of low concentration.

Local and nonlocal diffusion

Let $u(x,t)$ be the probability for a particle to be at discrete $x \in h\mathbb{Z}, t \in \tau\mathbb{N} \cap [0,T]$.

Assume that we are only allowed to jump one point either to the left or to the right, each with probability $\frac{1}{2}$.

The probability of being at point x at time $t+\tau$ is then

$$u(x,t+\tau) = \frac{1}{2}u(x+h,t) + \frac{1}{2}u(x-h,t).$$

Local and nonlocal diffusion

Let $u(x,t)$ be the probability for a particle to be at discrete $x \in h\mathbb{Z}, t \in \tau\mathbb{N} \cap [0,T]$.

Assume that we are only allowed to jump one point either to the left or to the right, each with probability $\frac{1}{2}$.

Rearrange to get

$$u(x,t+\tau) - u(x,t) = \frac{1}{2}\big(u(x+h,t) + u(x-h,t) - 2u(x,t)\big).$$

Local and nonlocal diffusion

Let $u(x,t)$ be the probability for a particle to be at discrete $x \in h\mathbb{Z}, t \in \tau\mathbb{N} \cap [0,T]$.

Assume that we are only allowed to jump one point either to the left or to the right, each with probability $\frac{1}{2}$.

Choose $\tau = \frac{1}{2}h^2$ and divide by it to obtain

$$\frac{u(x,t+\tau)-u(x,t)}{\tau} = \frac{u(x+h,t)+u(x-h,t)-2u(x,t)}{h^2}.$$

Local and nonlocal diffusion

Let $u(x,t)$ be the probability for a particle to be at discrete $x \in h\mathbb{Z}, t \in \tau\mathbb{N} \cap [0,T]$.

Assume that we are only allowed to jump one point either to the left or to the right, each with probability $\frac{1}{2}$.

As $\tau, h \to 0^+$, we will later see that that u satisfies

$$\partial_t u = \Delta u \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R} \times (0,T)),$$

that is, u is a distributional solution of the heat equation.

A. Einstein. Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. *Annalen der Physik* (in German), 322(8): 549–560, 1905.

Local and nonlocal diffusion

Now, we change the rules: A particle can jump to any point with a certain probability, but the probability of jumping to the left or to the right is exactly the same.

Consider $K : \mathbb{R} \to [0, \infty)$ satisfying

(i) $K(y) = K(-y)$

(ii) $\sum_{k\in\mathbb{Z}} K(k) = 1.$

As before, the probability of being at point x at time $t + \tau$ is

$$u(x, t + \tau) = \sum_{k\in\mathbb{Z}} K(k)u(x + hk, t).$$

Local and nonlocal diffusion

Now, we change the rules: A particle can jump to any point with a certain probability, but the probability of jumping to the left or to the right is exactly the same.

Consider $K : \mathbb{R} \to [0, \infty)$ satisfying

(i) $K(y) = K(-y)$

(ii) $\sum_{k \in \mathbb{Z}} K(k) = 1$.

We use property (ii) to obtain

$$u(x, t + \tau) - u(x, t) = \sum_{k \in \mathbb{Z}} K(k)\big(u(x + hk, t) - u(x, t)\big).$$

Local and nonlocal diffusion

Now, we change the rules: A particle can jump to any point with a certain probability, but the probability of jumping to the left or to the right is exactly the same.

To continue, we choose K up to normalization factors as

$$K(y) = \begin{cases} \frac{1}{|y|^{1+\alpha}} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

for $\alpha \in (0,2)$.

Divide by τ it to obtain

$$\frac{u(x,t+\tau)-u(x,t)}{\tau} = \sum_{k\in\mathbb{Z}} \frac{K(k)}{\tau}\big(u(x+hk,t)-u(x,t)\big).$$

Local and nonlocal diffusion

Now, we change the rules: A particle can jump to any point with a certain probability, but the probability of jumping to the left or to the right is exactly the same.

Choose $\tau = h^{\alpha}$, and note that

$$\frac{K(k)}{\tau} = \frac{1}{h^{\alpha}|k|^{1+\alpha}} = \frac{h}{h^{1+\alpha}|k|^{1+\alpha}} = hK(hk).$$

Then

$$\frac{u(x,t+\tau)-u(x,t)}{\tau} = \sum_{k\in\mathbb{Z}\setminus\{0\}} \big(u(x+hk,t)-u(x,t)\big)K(hk)h.$$

Local and nonlocal diffusion

Now, we change the rules: A particle can jump to any point with a certain probability, but the probability of jumping to the left or to the right is exactly the same.

Choose $\tau = h^{\alpha}$, and note that

$$\frac{K(k)}{\tau} = \frac{1}{h^{\alpha}|k|^{1+\alpha}} = \frac{h}{h^{1+\alpha}|k|^{1+\alpha}} = hK(hk).$$

Or

$$\frac{u(x,t+\tau)-u(x,t)}{\tau} = \int_{|z|>0} \big(u(x+z,t)-u(x,t)\big)\,\mathrm{d}\nu_h$$

with the measure $\nu_h(z) := \sum_{k\in\mathbb{Z}\setminus\{0\}} hK(hk)\delta_{hk}(z)$.

Local and nonlocal diffusion

Now, we change the rules: A particle can jump to any point with a certain probability, but the probability of jumping to the left or to the right is exactly the same.

As $\tau, h \to 0^+$, we will later see that that u satisfies

$$\begin{aligned}\partial_t u &= \int_{|z|>0} \big(u(x+z,t) - u(x,t) - z\partial_x u(x,t)\mathbf{1}_{|z|\leq 1}\big)\frac{c_{1,\alpha}}{|z|^{1+\alpha}}\,\mathrm{d}z \\ &= -(-\Delta)^{\frac{\alpha}{2}} u \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}\times(0,T))\end{aligned}$$

where $c_{1,\alpha} > 0$ and $-(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$ is the fractional Laplacian. We thus observe that u is a distributional solution of the fractional heat equation.

E. Valdinoci. From the long jump random walk to the fractional Laplacian. *Bol. Soc. Esp. Mat. Apl. SeMA*, (49):33–44, 2009.

Generalized porous medium equations

We consider the following Cauchy problem:

$$\text{(GPME)} \qquad \begin{cases} \partial_t u - \mathcal{L}^\mu[\varphi(u)] = 0 & \text{in} \quad Q_T := \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

where

- $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, and
- $\mathcal{L}^\mu$ is a symmetric pure-jump Lévy operator (anomalous/nonlocal diffusion operator).

Main results:

- Uniqueness in L^∞.
- Existence in $L^1 \cap L^\infty$.
- Convergent numerical schemes in $L^1 \cap L^\infty$.

Selective summary of previous results

Local case: $\partial_t u = \Delta u$, $\partial_t u = \Delta u^m$, $\partial_t u = \Delta\varphi(u)$.

J. L. Vázquez. *The porous medium equation. Mathematical theory*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

Selective summary of previous results

Nonlocal case: $\partial_t u = \mathcal{L}^\mu[\varphi(u)]$.

- Well-posedness when $\mathcal{L}^\mu \equiv -(-\Delta)^{\frac{\alpha}{2}}$:

Many people: Vázquez, de Pablo, Quirós, Rodríguez, Brändle, Bonforte, Stan, del Teso, Muratori, Grillo, Punzo, …

- Well-posedness for other $\mathcal{L}^\mu$:

Bounded operators

F. Andreu-Vaillo, J. Mazón, J. D. Rossi, and J. J. Toledo-Melero. *Nonlocal diffusion problems*, volume 165 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010.

Fractional Laplace like operators (with some x-dependence)

A. de Pablo, F. Quirós, and A. Rodríguez. Nonlocal filtration equations with rough kernels. *Nonlinear Anal.*, 137:402–425, 2016.

Selective summary of previous results

Previous results (mostly) rely on:

- The porous medium nonlinearity $\varphi(u) = u^m$ with $m > 1$.
- A very restrictive class of Lévy operators.
- The use of L^1-energy solutions.

In our case:

- Uniqueness is hard to prove because of a very weak solution concept (however, existence is then easier).
- The result we obtain is kind of different since we work in L^∞.
- We can handle very weak assumptions on φ and $\mathcal{L}^\mu$.

Lévy operator

$\mathcal{L}^\mu$ is a symmetric pure-jump Lévy operator (anomalous/nonlocal diffusion operator) defined, for smooth enough functions ψ, as e.g. the singular integral

$$\mathcal{L}^\mu[\psi](x) := \int_{\mathbb{R}^N\setminus\{0\}} \big(\psi(x+z)-\psi(x)-z\cdot D\psi(x)\mathbf{1}_{|z|\leq 1}\big)\,\mathrm{d}\mu(z).$$

Note that it includes the previously mentioned fractional Laplacian by choosing $\mathrm{d}\mu(z)=\frac{c_{N,\alpha}}{|z|^{N+\alpha}}\,\mathrm{d}z$ for some $c_{N,\alpha}>0$.

Assumptions

Unless otherwise stated we always assume that

(A$_\varphi$) $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing,

and

(A$_\mu$) $\mu \geq 0$ is a symmetric Radon measure on $\mathbb{R}^N \setminus \{0\}$ satisfying

$$\int_{|z|\leq 1} |z|^2 \,\mathrm{d}\mu(z) + \int_{|z|>1} 1 \,\mathrm{d}\mu(z) < \infty.$$

Assumptions

The assumption

$$\varphi : \mathbb{R} \to \mathbb{R} \text{ is continuous and nondecreasing,}$$

includes nonlinearities of the following kind

- the porous medium,
- fast diffusion, and
- Stefan problem.

Assumptions

The assumption

$\mu \geq 0$ is a symmetric Radon measure on $\mathbb{R}^N \setminus \{0\}$ satisfying

$$\int_{|z|\leq 1} |z|^2 \,\mathrm{d}\mu(z) + \int_{|z|>1} 1 \,\mathrm{d}\mu(z) < \infty.$$

ensures that our $\mathcal{L}^\mu$

- is the most general (symmetric, linear) nonlocal operator preserving the maximum principle;
- is a pure jump symmetric Lévy operator;
- contains spatial discretizations of $\mathrm{tr}(\sigma\sigma^T D^2 \cdot) + \mathcal{L}^\mu[\cdot]$;
- is a Fourier multiplier $\mathcal{F}(\mathcal{L}^\mu[\psi]) = -\sigma_{\mathcal{L}^\mu}\mathcal{F}(\psi)$; and
- is relevant for applications (in finance, physics, biology, etc.).

Distributional solution

Definition

Under the assumptions (A_φ), (A_μ), and $u_0 \in L^\infty(\mathbb{R}^N)$, $u \in L^\infty(Q_T)$ is a distributional solution of (GPME) if

$$0 = \int_0^T \int_{\mathbb{R}^N} \Big(u(x,t)\partial_t\psi(x,t) + \varphi(u(x,t))\mathcal{L}^\mu[\psi(\cdot,t)](x) \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}^N} u_0(x)\psi(x,0)\,\mathrm{d}x$$

for all $\psi \in C_\mathrm{c}^\infty(\mathbb{R}^N \times [0,T))$.

Uniqueness

Theorem (Preuniqueness, [del Teso, JE, Jakobsen, 2017])

Assume (A_φ) *and* (A_μ). *Let* $u(x,t)$ *and* $\hat{u}(x,t)$ *satisfy*

$$u, \hat{u} \in L^\infty(Q_T),$$

$$u - \hat{u} \in L^1(Q_T),$$

$$\partial_t u - \mathcal{L}^\mu[\varphi(u)] = \partial_t \hat{u} - \mathcal{L}^\mu[\varphi(\hat{u})] \qquad \text{in} \qquad \mathcal{D}'(Q_T)$$

$$\operatorname*{ess\,lim}_{t\to0^+} \int_{\mathbb{R}^N} (u(x,t) - \hat{u}(x,t))\psi(x,t)\,\mathrm{d}x = 0 \quad \forall \psi \in C_\mathrm{c}^\infty(\mathbb{R}^N \times [0,T)).$$

Then $u = \hat{u}$ *a.e. in* Q_T.

Uniqueness

Corollary (Uniqueness, [del Teso, JE, Jakobsen, 2017])

Assume (A_φ), (A_μ), *and* $u_0 \in L^\infty(\mathbb{R}^N)$. *Then there is at most one distributional solution* u *of* (GPME) *such that* $u \in L^\infty(Q_T)$ *and* $u - u_0 \in L^1(Q_T)$.

Proof: Assume there are two solutions u and $\hat{u}$ with the same initial data u_0. Then all assumptions of Theorem Preuniqueness obviously hold ($\|u-\hat{u}\|_{L^1} \leq \|u-u_0\|_{L^1} + \|\hat{u}-u_0\|_{L^1} < \infty$), and $u = \hat{u}$ a.e. ☐

Uniqueness holds for $u_0 \notin L^1$, for example $u_0(x) = c + \phi(x)$ for $c \in \mathbb{R}$ and $\phi \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. However, periodic u_0 are not included because of the assumption $u - u_0 \in L^1$.

Existence

Theorem (Existence, [del Teso, JE, Jakobsen, 2017])

Assume (A_φ), (A_μ), *and* $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. *Then there exists a unique distributional solution* u *of* (GPME) *satisfying*

$$u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N)).$$

Proof: By convergence of numerical solution (as we will see later). □

The proof of Theorem Preuniqueness

Based on a proof by Brézis and Crandall.

H. Brézis and M. G. Crandall. Uniqueness of solutions of the initial-value problem for $u_t - \Delta\varphi(u) = 0$. *J. Math. Pures Appl. (9)*, 58(2):153–163, 1979.

1. Define $U := u - \hat{u}$ and $\Phi := \varphi(u) - \varphi(\hat{u})$, then U solves

$$\begin{cases} \partial_t U - \mathcal{L}^\mu[\Phi] = 0 & \text{in} \quad Q_T \\ U(x,0) = 0 & \text{on} \quad \mathbb{R}^N. \end{cases}$$

   Note that $U \in L^1 \cap L^\infty$ and $\Phi \in L^\infty$.

2. Consider

$$\varepsilon v_\varepsilon - \mathcal{L}^\mu[v_\varepsilon] = g \qquad \text{in} \qquad \mathbb{R}^N,$$

   and define $B_\varepsilon^\mu[g] := v_\varepsilon$, that is, $B_\varepsilon^\mu = (\varepsilon I - \mathcal{L}^\mu)^{-1}$ is the resolvent of $\mathcal{L}^\mu$.
   Note that this is a *linear* elliptic equation.

The proof of Theorem Preuniqueness

3. Define

$$h_\varepsilon(t) := \int_{\mathbb{R}^N} U B_\varepsilon^\mu[U] \,\mathrm{d}x = \int_{\mathbb{R}^N} (\varepsilon I - \mathcal{L}^\mu) B_\varepsilon^\mu[U] B_\varepsilon^\mu[U] \,\mathrm{d}x$$
$$= \varepsilon \| B_\varepsilon^\mu[U] \|_{L^2}^2 + \| (-\mathcal{L}^\mu)^{\frac{1}{2}} [B_\varepsilon^\mu[U]] \|_{L^2}^2 .$$

4. Show that $h_\varepsilon \to 0$ as $\varepsilon \to 0^+$.
5. $U = (\varepsilon I - \mathcal{L}^\mu) B_\varepsilon^\mu[U] \to 0$ as $\varepsilon \to 0^+$ by Steps 3 and 4.

The proof of Theorem Preuniqueness

The hardest part is to show that $h_\varepsilon \to 0$ as $\varepsilon \to 0^+$. Some important steps:

1. $\varepsilon B^\mu_\varepsilon[U] \to 0$ implies $h_\varepsilon \to 0$ as $\varepsilon \to 0^+$.
2. Enough to prove that $\varepsilon B^\mu_\varepsilon[\gamma] \to 0$ for all $\gamma \in C_c^\infty(\mathbb{R}^N)$. Note that $\Gamma_\varepsilon := \varepsilon B^\mu_\varepsilon[\gamma]$ solves
$$\varepsilon\Gamma_\varepsilon - \mathcal{L}^\mu[\Gamma_\varepsilon] = \varepsilon\gamma \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}^N).$$
3. A priori results and compactness give $\Gamma_\varepsilon \to \Gamma$ as $\varepsilon \to 0^+$.
4. (Liouville) If $\operatorname{supp}\mu \neq 0$, $\Gamma \in C_0$, and $\mathcal{L}^\mu[\Gamma] = 0$ in $\mathcal{D}'$, then $\Gamma \equiv 0$.

Note that a general Liouville result do not hold for $\mathcal{L}^\mu$: Take $\mu(z) = \delta_{2\pi}(z) + \delta_{-2\pi}(z)$, then $\mathcal{L}^\mu[\cos](x) = 0$, but this function is not constant.

Further uniqueness results

By similar methods, we obtain uniqueness in L^∞ for

$$\begin{cases} \partial_t u - \big(\operatorname{tr}(\sigma\sigma^T D^2\varphi(u)) + \mathcal{L}^\mu[\varphi(u)]\big) = 0 & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

and

$$u - \big(\operatorname{tr}(\sigma\sigma^T D^2\varphi(u)) + \mathcal{L}^\mu[\varphi(u)]\big) = g \qquad \text{in} \qquad \mathbb{R}^N.$$

Important observations

$$\begin{aligned}\Delta_h\psi(x) &:= \frac{\psi(x+he_i)+\psi(x-he_i)-2\psi(x)}{h^2}\\ &= \int_{\mathbb{R}^N}\big(\psi(x+z)-\psi(x)\big)\,\mathrm{d}\mu_h(z) =: \mathcal{L}^{\mu_h}[\psi](x)\end{aligned}$$

where

$$\mu_h(z) := \frac{1}{h^2}\sum_{i=1}^{N}\delta_{he_i}(z)+\delta_{-he_i}(z)$$

satisfies (A_μ).

Important observations

By now, there exist several spatial discretizations of $\mathcal{L}^{\mu}$ (e.g. quadrature and spectral methods).

Y. Huang and A. Oberman. Finite difference methods for fractional Laplacians. Preprint, arXiv:1611.00164v1 [math.NA], 2016.

Our contribution is to note and exploit that (some of) the discretizations of $\mathcal{L}^{\mu}$ is again a Lévy operator.

Numerical schemes for (GPME)

Recall that our Cauchy problem was given as

$$\text{(GPME)} \qquad \begin{cases} \partial_t u - \mathcal{L}^{\mu}[\varphi(u)] = 0 & \text{in} \quad Q_T := \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N. \end{cases}$$

Our numerical scheme can then take the following form

$$\text{(NumGPME)} \qquad \begin{cases} \frac{U^j - U^{j-1}}{\Delta t} = G_{\Delta x}(U^j, U^{j-1}) & \text{in} \quad \Delta x \mathbb{Z}^N, \\ U^0 = u_0 & \text{in} \quad \Delta x \mathbb{Z}^N. \end{cases}$$

Numerical schemes for (GPME)

In our most general case, we have that

$$G_{\Delta x}(U^j, U^{j-1}) := \mathcal{L}^{\nu_{1,\Delta x}}[\varphi_1(U^j)] + \mathcal{L}^{\nu_{2,\Delta x}}[\varphi_2(U^{j-1})]$$

where $\nu_{1,\Delta x}, \nu_{2,\Delta x}$ satisfy (A_μ).

Thus our framework includes

- a mixture of implicit and explicit schemes (θ-methods);
- the possibility of discretizing the singluar and nonsingular parts of $\mathcal{L}^\mu$ in different ways; and
- combinations of the above.

Note that by our previous observations, we are also able to approximate local operators of the form

$$\mathrm{tr}(\sigma\sigma^T D^2\cdot).$$

Convergence of the numerical schemes

The scheme defined by (NumGPME) is

- monotone,
- (conservative if the φ's involved are Lipschitz)
- L^p-stable, and
- consistent.

Theorem (Convergence, [del Teso, JE, Jakobsen, 2017])

Assume $\nu_{1,\Delta x}, \nu_{2,\Delta x}$ *satisfy* (A_μ), φ_1, φ_2 *satisfy* (A_φ), *and* $U^0 = u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. *Then, for the interpolant* U, *we have*

$$U \to u \qquad \text{in} \qquad C([0,T]; L^1_{\text{loc}}(\mathbb{R}^N)) \qquad \text{as} \qquad \Delta x, \Delta t \to 0^+$$

where $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0,T]; L^1_{\text{loc}}(\mathbb{R}^N))$ *is a distributional solution of* (GPME).

Extensions and related results

We consider the following Cauchy problem:

(x-GPME)
$$\begin{cases} \partial_t u - A^\lambda[\varphi(u)] = 0 & \text{in} \quad Q_T, \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

where

- $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, and
- A^λ is a x-dependent generalization of $\mathcal{L}^\mu$.

Main results:

- Uniqueness in $L^1 \cap L^\infty$.
- Energy solutions $\iff$ distributional solutions with finite energy.

Uniqueness

Theorem (Uniqueness, [del Teso, JE, Jakobsen, 2017])

Assume (A_φ), $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, *and "*$\lambda$ *satisfies the x-dependent version of* (A_μ)*". Then there is at most one energy solution u of* (x-GPME) *in*

$$\{u \in L^1(Q_T) \cap L^\infty(Q_T) \, : \, \varphi(u) \in X\}.$$

- Under some regularity assumptions on λ, we have

$$X \cap L^2(Q_T) = L^2(Q_T) \cap L^\infty(Q_T) \cap \{\text{"finite energy"}\}.$$

- When λ is bounded above and below by the x-independent measure corresponding to $-(-\Delta)^{\frac{\alpha}{2}}$, then

$$X = L^\infty(Q_T) \cap \{\text{"finite energy"}\}$$

{"finite energy"}

$$\{\text{``finite energy''}\} := \left\{ F \,:\, \frac{1}{2}\int_0^T \int_{\mathbb{R}^N} \int_{|z|>0} \left| F(x+z) - F(x) \right|^2 \lambda(x, \mathrm{d}z)\, \mathrm{d}x\, \mathrm{d}t < \infty \right\}$$

Special case $\lambda(x, \mathrm{d}z) = \mu(\mathrm{d}z)$

Let us specialize to the case $\lambda(x, \mathrm{d}z) = \mu(\mathrm{d}z)$, that is, $A^\lambda = \mathcal{L}^\mu$.

Theorem (Existence, [del Teso, JE, Jakobsen, 2017])

Assume (A_φ), (A_μ), *and* $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. *Then there exists a distributional solution of* (GPME) *satisfying*

(i) $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$; *and*

(ii) $\varphi(u) \in \{$*"finite energy"*$\}$.

Why?

$$\frac{1}{2}\int_0^T \int_{\mathbb{R}^N} \int_{|z|>0} \left|\varphi(u(x+z)) - \varphi(u(x))\right|^2 \mu(\mathrm{d}z)\,\mathrm{d}x\,\mathrm{d}t \\ \leq \|\varphi(u_0)\|_{L^\infty(\mathbb{R}^N)} \|u_0\|_{L^1(\mathbb{R}^N)}$$

Final remarks

- Under certain conditions (e.g., $u, u_0 \in L^\infty$), energy solutions $\iff$ distributional solutions with finite energy.
- Energy solutions are distributional solutions with finite energy, and hence, by our first uniqueness result, they are unique without any further requirements on φ.
- Remember that in the second result we needed $\varphi(u) \in X$. Thus, the first result is more robust in the x-independent case, and the second more general in the x-dependent case.

References

F. del Teso, JE, E. R. Jakobsen. Uniqueness and properties of distributional solutions of nonlocal equations of porous medium type. *Adv. Math.*, 305:78–143, 2017.

F. del Teso, JE, E. R. Jakobsen. On the well-posedness of solutions with finite energy for nonlocal equations of porous medium type. To appear in *EMS Series of Congress Reports*, 2017.

F. del Teso, JE, E. R. Jakobsen. A note on the well-posedness of distributional solutions of nonlocal (and local) equations of porous medium type. Preprint, 2017.

F. del Teso, JE, E. R. Jakobsen. Numerical analysis and methods for distributional solutions of nonlocal (and local) equations of porous medium type. Preprint, 2017.

Thank you for your attention!