

REPRESENTATIONS OF PURE SYMMETRIC AUTOMORPHISM GROUPS OF RAAGS

JAVIER ARAMAYONA & CONCHITA MARTÍNEZ PÉREZ

ABSTRACT. We study representations of the pure symmetric automorphism group $\text{PAut}(A_\Gamma)$ of a RAAG A_Γ with defining graph Γ .

We first construct a homomorphism from $\text{PAut}(A_\Gamma)$ to the direct product of a RAAG and a finite direct product of copies of $F_2 \times F_2$; moreover, the image of $\text{PAut}(A_\Gamma)$ under this homomorphism is surjective onto each factor. As a consequence, we obtain interesting actions of $\text{PAut}(A_\Gamma)$ on non-positively curved spaces

We then exhibit, for connected Γ , a RAAG which property contains $\text{Inn}(A_\Gamma)$ and embeds as a normal subgroup of $\text{PAut}(A_\Gamma)$. We end with a discussion of the linearity problem for $\text{PAut}(A_\Gamma)$.

1. INTRODUCTION

Given a group G with a finite generating set X , the *pure symmetric automorphism group* $\text{PAut}(G)$ is the group of those automorphisms of G that send every generator in X to a conjugate of itself. In this paper, we will be interested in representations of pure symmetric automorphism groups of *right-angled Artin groups* (RAAGs) with respect to the standard generating set; see section 2 for definitions.

An important and motivating example is that of the free group F_n on n letters x_1, \dots, x_n . Humphries [14] proved that $\text{PAut}(F_n)$ is generated by the *partial conjugations* $x_j \mapsto x_i x_j x_i^{-1}$. This revealed a strong relation between $\text{PAut}(F_n)$ and braid groups, for Goldsmith [12] had previously showed that the subgroup of $\text{Aut}(F_n)$ generated by partial conjugations is isomorphic to the fundamental group of the configuration space of n unknotted, unlinked circles in \mathbb{S}^3 . Shortly afterwards, McCool [18] gave an explicit presentation of $\text{PAut}(F_n)$ in terms of partial conjugations, where every relation is a commutativity relation between (products of) generators.

For a general RAAG A_Γ , Humphries's result above was extended by Laurence [17], who proved that $\text{PAut}(A_\Gamma)$ is generated by partial conjugations. In addition, Toinet [23] proved that $\text{PAut}(A_\Gamma)$ has a presentation similar in spirit to McCool's presentation of $\text{PAut}(F_n)$. This was subsequently refined by Koban-Piggott [15], who showed that it suffices to consider a finite set of commutation relations between (products of) partial conjugations of A_Γ .

Date: September 17, 2018.

The first named author is partially supported by RYC-2013-13008 and the second one by Gobierno de Aragón and European Regional Development Funds. Both authors were funded by grant MTM2015-67781.

1.1. Subdirect images of $\text{PAut}(A_\Gamma)$. The first aim of this paper is to use Koban-Piggott's presentation [15] of $\text{PAut}(A_\Gamma)$ in order to produce a (non-injective) homomorphism of $\text{PAut}(A_\Gamma)$ to the product of a RAAG times a direct product of copies of $F_2 \times F_2$. Recall that a subgroup of a direct product of groups is said to be a *subdirect product* if its projection onto each direct factor is surjective. We will show:

Theorem 1.1. *For every graph Γ there exist a graph Δ , a number $N \geq 0$, and a homomorphism*

$$\text{PAut}(A_\Gamma) \rightarrow A_\Delta \times \prod_{i=1}^N (F_2 \times F_2)$$

whose image is a subdirect product. Moreover, the image of every partial conjugation has infinite order.

The homomorphism of Theorem 1.1 is constructed explicitly, and N and Δ are determined in terms of the graph Γ . In this direction, following [6, 13] we say that a pair of distinct vertices v, w of Γ form a *SIL-pair* if $v \notin \text{lk}(w)$ and $\Gamma \setminus \text{lk}(v) \cap \text{lk}(w)$ has a connected component Y with $v, w \notin Y$. Then the number N in Theorem 1.1 is equal to half the number of *SIL-type* relations in Koban-Piggott's presentation of $\text{PAut}(A_\Gamma)$ (see the comment after the proof of Theorem 1.1), and A_Δ is the RAAG obtained from $\text{PAut}(A_\Gamma)$ by killing every partial conjugation that appears in a SIL-type relation. In the specific case when $A_\Gamma = F_n$, every partial conjugation appears in a SIL-type relation, and thus we have $\Delta = \emptyset$ and $N = \frac{n(n-1)(n-2)}{2}$.

It is easy to see that the homomorphism given by Theorem 1.1 is not injective unless $\text{PAut}(A_\Gamma)$ is a RAAG itself; see Proposition 3.2. Hence a natural question is:

Question 1.2. *Determine the kernel of the homomorphism in Theorem 1.1.*

Applications to CAT(0)-actions. A result of Bridson [3, Theorem 1.1] asserts that, for n sufficiently large, if $\text{Aut}(F_n)$ acts by (semisimple) isometries on a complete CAT(0) space, then every transvection (and thus every partial conjugation also) is elliptic, i.e. it fixes a point. By considering induced actions, the same result holds for finite-index subgroups of $\text{Aut}(F_n)$ too.

However, the presence of transvections is crucial for these results. Indeed, Koban-Piggott's presentation of $\text{PAut}(F_n)$ [15] implies that $\text{PAut}(F_n)$ (which recall is generated by all partial conjugations) has non-trivial abelianization, and thus acts by hyperbolic isometries on a CAT(0) space X . However, the action so obtained is *abelian*, in the sense that the image of the homomorphism $\text{PAut}(F_n) \rightarrow \text{Isom}(X)$ is abelian.

In sharp contrast, composing the homomorphism ρ from Theorem 1.1 with a faithful representation of F_2 into (say) $\text{PSL}(2, \mathbb{Z})$, we obtain the following:

Corollary 1.3. *For every $n > 0$, $\text{PAut}(F_n)$ admits a non-abelian CAT(0)-action where every partial conjugation acts hyperbolically.*

1.2. Normal RAAG subgroups of $\text{PAut}(A_\Gamma)$. A well-known corollary of a result of Davis-Januszkiewicz [10] is that RAAGs are linear. As a

consequence, we may use the homomorphism constructed in Theorem 1.1 to construct a non-faithful linear representation of $\text{PAut}(A_\Gamma)$. In light of the analogy between $\text{PAut}(F_n)$ and braid groups alluded to above, a natural (open) question is whether $\text{PAut}(A_\Gamma)$ is linear. It is easy to see that the group of *inner automorphisms* $\text{Inn}(A_\Gamma)$ of A_Γ , which is isomorphic to the quotient of A_Γ by its center, is itself a RAAG that embeds as a normal subgroup of $\text{PAut}(A_\Gamma)$.

In the second part we construct, for connected Γ , a graph $\hat{\Gamma}$ such that $A_{\hat{\Gamma}}$ properly contains $\text{Inn}(A_\Gamma)$, and it still embeds in $\text{PAut}(A_\Gamma)$ thus providing a linear normal subgroup of $\text{PAut}(A_\Gamma)$. More concretely, we will show:

Theorem 1.4. *Let Γ be a connected graph. Then there exists a right-angled Artin group $A_{\hat{\Gamma}}$ and an injective map*

$$A_{\hat{\Gamma}} \rightarrow \text{PAut}(A_\Gamma)$$

whose image is normal and contains $\text{Inn}(A_\Gamma)$. Moreover if Γ has no SIL pair, this map is an isomorphism.

In [7, Section 3], Charney and Vogtmann consider a subgroup K of the *outer* automorphism group of A_Γ which is generated by cosets of certain partial conjugations, and which they prove is abelian. As it turns out, a similar reasoning to that of the proof of Theorem 1.4 implies that the lift \tilde{K} of K to $\text{Aut}(A_\Gamma)$ is a RAAG, and that it embeds as a normal subgroup of $\text{PAut}(A_\Gamma)$. However, in general \tilde{K} will be properly contained in the group $A_{\hat{\Gamma}}$ given by Theorem 1.4. See Remark 4.7 for more details.

The plan of the paper is as follows. In section 2 we will give the necessary background on right-angled Artin groups and their automorphism groups. Then in section 3 we proceed to construct the homomorphism of Theorem 1.1. Finally, in section 4 we prove Theorem 1.4.

Acknowledgements. These ideas sprouted from conversations during the 11th *Barcelona Weekend in Group Theory* in May 2016. The authors are grateful to the organizer of the conference, Pep Burillo. We would like to thank Vassilis Metaftsis for the comment [19], and the referee for useful comments and suggestions.

2. GRAPHS, RAAGS, AND THEIR AUTOMORPHISMS

In this section we will briefly review the necessary definitions needed for the proofs of our main results.

2.1. Graphs. Throughout, Γ will denote a finite simplicial graph. We will write $V(\Gamma)$ and $E(\Gamma)$, respectively, for the set of vertices and edges of Γ .

Given a vertex $v \in V(\Gamma)$, the *link* $\text{lk}_\Gamma(v)$ of v in Γ is the set of those vertices of Γ which are connected by an edge to v . The *star* of v is $\text{st}_\Gamma(v) := \text{lk}_\Gamma(v) \cup \{v\}$. If there is no risk of confusion, we will simply write $\text{lk}(v)$ and $\text{st}(v)$ in order to relax notation.

As mentioned in the introduction, we say that two distinct vertices v, w form a *SIL* if $v \notin \text{lk}(w)$ and $\Gamma \setminus \text{lk}(v) \cap \text{lk}(w)$ has a connected component which does not contain either v or w .

Suppose that v, w are not linked. The following terminology comes from [9] (see also [13]). A connected component of $\Gamma - \text{st}(v)$ that is also a connected component of $\Gamma - \text{st}(w)$ will be called a *shared* component. The unique component of $\Gamma - \text{st}(v)$ that contains w will be called the *dominant* component of v . Finally, a *subordinate component* of w is any connected component of $\Gamma - \text{st}(w)$ that is contained in the dominant component of v . By [9, Lemma 2.1], any connected component of $\Gamma - \text{st}(v)$ is of one of these types; moreover v and w form a SIL if and only if there is some shared component.

2.2. Right-angled Artin groups. Let Γ be a finite simplicial graph. The *right-angled Artin group* (RAAG, for short) defined by Γ is the group with presentation

$$A_\Gamma = \langle v \in V(\Gamma) \mid [v, w] = 1 \iff vw \in E(\Gamma) \rangle.$$

As usual, in order to keep the notation under control, we are blurring the distinction between vertices of Γ and the corresponding generators of A_Γ .

Notable examples are the extreme cases when Γ has no edges, so that A_Γ is a free group, and when Γ is a complete graph, in which case A_Γ is a free abelian group.

2.3. Automorphisms of RAAGs. Let A_Γ the RAAG defined by the graph Γ , and consider its automorphism group $\text{Aut}(A_\Gamma)$. In the light of the precedent paragraph, for a fixed number n of vertices of Γ , the group $\text{Aut}(A_\Gamma)$ interpolates between the cases of $\text{Aut}(F_n)$ and $\text{GL}(n, \mathbb{Z}) = \text{Aut}(\mathbb{Z}^n)$.

Laurence [17] and Servatius [21] gave an explicit finite set of generators for $\text{Aut}(A_\Gamma)$ for arbitrary Γ . We now briefly recall these generators:

- (i) *Graphic automorphisms.* These are the elements of $\text{Aut}(A_\Gamma)$ induced by the symmetries of Γ .
- (ii) *Inversions.* Given $v \in V(\Gamma)$, the inversion on v is the automorphism that maps v to v^{-1} and fixes the rest of generators of A_Γ .
- (iii) *Transvections.* Given vertices $v, w \in V(\Gamma)$ with $\text{lk}(v) \subset \text{st}(w)$, the transvection t_{vw} is the automorphism of A_Γ given by

$$\begin{cases} t_{vw}(v) = vw, \\ t_{vw}(z) = z, z \neq v. \end{cases}$$

- (iv) *Partial conjugations.* Given $v \in V(\Gamma)$ and a connected component A of $\Gamma - \text{st}(v)$, the partial conjugation of A by v is the automorphism $c_{A,v}$ of A_Γ given by

$$\begin{cases} c_{A,v}(w) = v^{-1}wv, w \in A \\ c_{A,v}(z) = z, z \notin A. \end{cases}$$

Laurence [17] and Servatius [21] proved:

Theorem 2.1 ([17, 21]). *$\text{Aut}(A_\Gamma)$ is generated by the four types of automorphisms described above. In particular, it is finitely generated.*

For the sake of completeness, although it will not be needed here, we remark that Day [8] subsequently proved that $\text{Aut}(A_\Gamma)$ is finitely presented, and gave an explicit presentation.

2.4. Pure symmetric automorphism groups. We will consider the subgroup $\text{PAut}(A_\Gamma)$ of $\text{Aut}(A_\Gamma)$ consisting of those automorphisms that send every generator of A_Γ to a conjugate of itself. Laurence proved:

Theorem 2.2 (Laurence [17]). *$\text{PAut}(A_\Gamma)$ coincides with the subgroup of $\text{Aut}(A_\Gamma)$ generated by partial conjugations.*

As mentioned in the introduction, the above theorem was proved earlier by Humphries [14] in the particular case when A_Γ is a free group. Observe that a consequence of Theorem 2.1 above is that if Γ does not have vertices v, w with $\text{lk}(v) \subset \text{st}(w)$, then $\text{PAut}(A_\Gamma)$ has finite index in $\text{Aut}(A_\Gamma)$, since in this case there are no transvections.

As mentioned in the introduction, McCool [18] gave an explicit presentation of $\text{PAut}(F_n)$. We record (an equivalent form of) this presentation next. Let v_1, \dots, v_n the standard basis of F_n , and denote by c_{ij} the automorphism of F_n given by conjugating v_i by v_j . We have:

Theorem 2.3 (McCool [18]). *$\text{PAut}(F_n)$ is generated by the set $\{c_{ij} \mid 1 \leq i, j \leq n\}$, and a complete set of relations is:*

- (i) $[c_{ij}, c_{kl}] = 1$, whenever $\{i, j\} \cap \{k, l\} = \emptyset$ or $j = l$, and
- (ii) $[c_{ij}c_{kj}, c_{ik}] = 1$ whenever i, j, k are pairwise distinct.

Inspired by the above theorem, and using Day's presentation of $\text{Aut}(A_\Gamma)$ [8], Toinet [23] computed an explicit finite presentation $\text{PAut}(A_\Gamma)$. This presentation was subsequently refined by Koban-Piggott [15]; we state their result here, slightly reformulated to suit our purposes:

Theorem 2.4 (Koban-Piggott [15]). *$\text{PAut}(A_\Gamma)$ is generated by all partial conjugations $c_{A,v}$, subject to the following relations:*

- (i) $[c_{A,v}, c_{B,w}] = 1$ if either $v = w$ or $v \in \text{lk}(w)$,
- (ii) $[c_{A,v}, c_{B,w}] = 1$ if $v \neq w$ and either $(A \cup \{v\}) \cap (B \cup \{w\}) = \emptyset$ or $(A \cup \{v\}) \subseteq B$ or $(B \cup \{w\}) \subseteq A$,
- (iii) $[c_{A,v}c_{B,v}, c_{A,w}] = 1$ whenever $\Gamma - \text{lk}(v) \cap \text{lk}(w)$ has a connected component A which contains neither v nor w , and $w \in B$.

As in [9], it is convenient to reformulate these relations using the terminology introduced at the end of Subsection 2.1. So we have that the partial conjugations $c_{A,v}$ and $c_{B,w}$ commute if and only if either at least one of the connected components A and B is subordinate or both are shared but distinct. And case (iii) happens if A is shared and B dominant (and in this case, the pair of vertices $v, w \in V(\Gamma)$ must form a SIL). We will thus refer to relations of type (iii) as *SIL-type* relations in $\text{PAut}(A_\Gamma)$.

Remark 2.5. Observe that, in the case of $A_\Gamma = F_n$, the presentations of $\text{PAut}(F_n)$ given by Theorems 2.3 and 2.4 coincide. In the case when $A_\Gamma = F_n$, any pair of vertices v_j and v_k form a SIL (for $n \geq 3$) with shared components of the form $\{v_i\}$ for any other i .

2.5. Arrow diagrams. Next, we present a useful combinatorial manner of visualizing $\text{PAut}(A_\Gamma)$, in terms of *arrow diagrams*, as we now introduce.

Given a finite simplicial graph Λ , as before we denote its vertex set by $V(\Lambda)$ and its edge set by $E(\Lambda)$.

Definition 2.6. An *arrow diagram* is a pair (Λ, \mathcal{A}) , where Λ is a finite simplicial graph and \mathcal{A} is a subset of $V(\Lambda) \times E(\Lambda)$. The elements of \mathcal{A} are called *arrows*. Given an arrow $(v, e) \in \mathcal{A}$, we say that v is its *initial vertex*. Finally, we call Λ the *graph underlying the arrow diagram*.

The motivation for the name “arrow diagram” is that we may represent an arrow diagram as a graph together with some arrows, see figure 1.

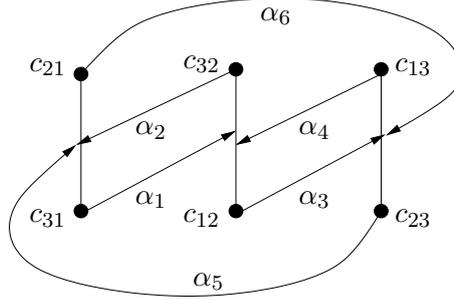


FIGURE 1. The graphical representation of an arrow diagram. In fact, the reader may check from Theorem 2.3 that this is the arrow diagram of $\text{PAut}(F_3)$.

We are now going to define a group from an arrow diagram. We choose to do this in full generality, hoping that the class of groups so defined may be of independent interest.

Definition 2.7. Let (Λ, \mathcal{A}) be an arrow diagram. The *arrow group* $G(\Lambda, \mathcal{A})$ defined by (Λ, \mathcal{A}) is the group generated by the vertices of Λ , subject to the following relations:

- (i) $[v, w] = 1$ if and only if $v \in \text{lk}(w)$.
- (ii) $[u, vw] = 1$ if and only if v, w span an edge e and $(u, e) \in \mathcal{A}$.

Observe that if $\mathcal{A} = \emptyset$, then $G(\Lambda, \mathcal{A}) = A_\Lambda$, the right-angled Artin group defined by Λ .

In the light of Theorem 2.4, the following result is immediate:

Proposition 2.8. *Let Γ be a finite simplicial graph. Then $\text{PAut}(A_\Gamma)$ is an arrow group.*

Note that the graph underlying the arrow diagram of $\text{PAut}(A_\Gamma)$ is, in general, not isomorphic to Γ .

2.5.1. Dual arrows. Viewed as an arrow group, $\text{PAut}(A_\Gamma)$ has a further interesting property, which plays an important rôle in the proof of Theorem 1.1. More concretely, we say that an arrow diagram (Λ, \mathcal{A}) has *property (D)* if for every arrow $\alpha = (v, e) \in \mathcal{A}$, there exists a unique arrow $\bar{\alpha} = (\bar{v}, \bar{e}) \in \mathcal{A}$ such that \bar{v} is a vertex of e , v is a vertex of \bar{e} , and the subgraph of Λ spanned by the vertices of e, \bar{e} has no other edges than e and \bar{e} . In this situation, we say that α and $\bar{\alpha}$ are *dual arrows*. For instance, the arrows α_1 and α_2 of Figure 1 are dual to each other. We deduce:

Proposition 2.9. *Let Γ be a finite simplicial graph. Then the arrow diagram of $\text{PAut}(A_\Gamma)$ has property (D).*

Proof. Consider an arrow $(v, e) \in \mathcal{A}$ in $\text{PAut}(A_\Gamma)$. By Theorem 2.4, it corresponds to a SIL-type relation

$$[c_{A,v}c_{B,v}, c_{A,w}] = 1,$$

for v, w a SIL-pair, where A is a shared component and B is dominant thus $w \in B$. Let C be the other dominant component, i.e., the connected component of $\Gamma - \text{lk}(w)$ that contains v . Then the partial conjugation $c_{C,w}$ is well-defined and we have

$$[c_{A,w}c_{C,w}, c_{A,v}] = 1,$$

as desired. Moreover, the only edges between $c_{A,w}$, $c_{C,w}$, $c_{A,v}$ and $c_{B,v}$ are the edges corresponding to $[c_{A,w}, c_{C,w}] = 1$ and $[c_{A,v}, c_{B,w}] = 1$. Thus property (D) holds. \square

Corollary 2.10. *Let Γ be a finite simplicial graph. Then the arrow diagram of $\text{PAut}(A_\Gamma)$ has an even number of arrows.*

Remark 2.11. Theorem 2.3 implies that the number of arrow pairs in the arrow diagram for $\text{PAut}(F_n)$ is equal to

$$\frac{n(n-1)(n-2)}{2}.$$

Indeed, whenever $\{i, j, k\}$ are pairwise distinct, there is an arrow $(c_{ik}, e(c_{ij}, c_{kj}))$, where $e(c_{ij}, c_{kj})$ is the edge between c_{ij} and c_{kj} . This arrow has dual arrow $(c_{ij}, e(c_{jk}, c_{ik}))$. Therefore, for each choice of $i \neq j$ we have $(n-2)/2$ pairs.

3. PROOF OF THEOREM 1.1

Before proceeding to proving Theorem 1.1, we need a final piece of notation. Let Γ be a finite simplicial graph and (Λ, \mathcal{A}) the arrow diagram of $\text{PAut}(A_\Gamma)$. We say that a vertex $w \in V(\Gamma)$ is *not involved in any arrows* if the following two conditions hold

- i) there is no arrow of the form (w, e) ,
- ii) there is no arrow of the form (v, e) where w is a vertex of e .

Proof of Theorem 1.1. As above, denote by (Λ, \mathcal{A}) the arrow diagram of $\text{PAut}(A_\Gamma)$. In the light of Corollary 2.10, we have that $|\mathcal{A}| = 2N$, for some $N \in \mathbb{N}$. We choose an indexing

$$\mathcal{A} = \{\alpha_1, \dots, \alpha_{2N}\}$$

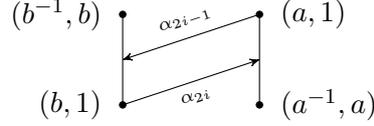
so that $\alpha_{2i} = \bar{\alpha}_{2i-1}$ for all $i = 1, \dots, N$. Now, let Δ be the subgraph of Γ spanned by those vertices of Λ that are not involved in any arrows, and A_Δ the associated RAAG. Choose a free basis a, b of F_2 , and write G for the direct product of N copies of $F_2 \times F_2$. We define a map

$$\rho : V(\Lambda) \rightarrow A_\Delta \times G$$

as follows:

- (i) If $v \in V(\Lambda)$ is not involved in any arrows, then we set $\rho(v) = (v, 1)$.
- (ii) Otherwise, we set $\rho(v) = (1, g)$, where $g = (g_1, \dots, g_N)$, with the i -th entry $g_i \in F_2 \times F_2$ being equal to $(1, 1)$ unless:

- (a) Suppose v is the initial vertex of the arrow α_{2i-1} (resp. $\alpha_{2i} = \bar{\alpha}_{2i-1}$). In this case, set $g_i = (a, 1)$ (resp. $g_i = (b, 1)$).
- (b) Suppose $v \in \text{lk}(w)$, where w is the initial vertex of the arrow α_{2i-1} , and the edge $e = \{v, w\}$ is the edge of the dual arrow $\bar{\alpha}_{2i-1}$ (resp. w is the initial vertex of the arrow $\alpha_{2i} = \bar{\alpha}_{2i-1}$, and the edge $e = \{v, w\}$ is the edge of the dual arrow $\bar{\alpha}_{2i} = \alpha_{2i-1}$). In this case, we set $g_i = (a^{-1}, a)$ (resp. $g_i = (b^{-1}, b)$).



We claim that the map ρ so constructed defines a homomorphism

$$\rho : G(\Lambda, \mathcal{A}) = \text{PAut}(A_\Gamma) \rightarrow A_\Delta \times G,$$

using a slight abuse of notation. In order to do so, we need to verify that ρ preserves the relations of $\text{PAut}(A_\Gamma)$ given in Theorem 2.4. We prove this in two separate claims:

Claim 1. Let $v, w \in V(\Delta)$ be linked vertices. Then

$$[v, w] = 1 \Rightarrow [\rho(v), \rho(w)] = 1.$$

Proof of Claim 1. First, suppose v is not involved in any arrows, in which case $\rho(v) = (v, 1)$. If w is not involved in any arrows either, then $\rho(w) = (w, 1)$. Otherwise, $\rho(w) = (1, g)$ for some $g \in G$. In either case, $[\rho(v), \rho(w)] = 1$ and we are done.

Therefore, we have to consider the case when v and w are both involved in some pair of arrows. By the definition the map ρ above, the only non-trivial case occurs when such arrows are dual to each other. Property (D) implies that, up to interchanging v and w , we may suppose that v is the initial vertex of α_{2i-1} or $\alpha_{2i} = \bar{\alpha}_{2i-1}$. Assume, for the sake of concreteness, that we are in the former case, as the other one is totally analogous. In this situation, we have that $\rho(v) = (1, g)$ and $\rho(w) = (1, g')$, where the i -th entry of g (resp. g') is $(a, 1)$ (resp. (a^{-1}, a)). In particular, $[\rho(v), \rho(w)] = 1$, as desired. \square

Claim 2. For vertices $v, w \in V(\Delta)$,

$$[u, vw] = 1 \Rightarrow [\rho(u), \rho(v)\rho(w)] = 1.$$

Proof of Claim 2. In this case we know that v and w span an edge $e \in E(\Lambda)$ with $(u, e) \in \mathcal{A}$, so $(u, e) = \alpha_{2i-1}$ or $\bar{\alpha}_{2i-1}$ for some i . Suppose, again for concreteness, that we are in the former case, so that u is the initial vertex of α_{2i-1} . Up to interchanging v and w , we may assume that v is the initial vertex of the arrow $\bar{\alpha}_{2i-1}$. In particular, we write $\bar{\alpha}_{2i-1} = (v, \bar{e})$, where

$\bar{e} \in E(\Lambda)$ with $u \in \bar{e}$. Let z be the vertex of \bar{e} which is not equal to u . From the definition of ρ , we have:

$$\begin{aligned}\rho(u) &= (1, g_u) \\ \rho(v) &= (1, g_v) \\ \rho(w) &= (1, g_w) \\ \rho(z) &= (1, g_z),\end{aligned}$$

for some $g_u, g_v, g_w, g_z \in G$ whose i -th entries are, respectively, equal to $(a, 1)$, $(b, 1)$, (b^{-1}, b) , and (a^{-1}, a) . In particular we see that, when restricted to the i -th component of ρ , the relation $[\rho(u), \rho(v)\rho(w)] = 1$ is preserved. Since the components of ρ correspond precisely to pairs of dual arrows, we obtain the result. This finishes the proof of Claim 2. \square

In the light of Claims 1 and 2, we know that ρ defines a homomorphism that we also denote ρ

$$\rho : \text{PAut}(A_\Gamma) \rightarrow A_\Delta \times G,$$

where G is the direct product of N copies of $F_2 \times F_2$. It remains to check that the composition ρ_i of ρ with the projection with the i -th direct factor of G is surjective, for every $i = 1, \dots, N$. To this end, consider the arrows α_{2i-1} and $\alpha_{2i} = \bar{\alpha}_{2i-1}$. By definition, α_{2i-1} corresponds to a relation of the form $[u, vw] = 1$; thus, up to renaming v and w , $\bar{\alpha}_{2i-1}$ corresponds to the relation $[w, uz] = 1$. Hence, in the i -th copy of $F_2 \times F_2$, the entries of $\rho(u)$ and $\rho(w)$ are $(a, 1)$ and $(b, 1)$, and therefore they generate a free group on two generators. Similarly, the entries $\rho(v)$ and $\rho(z)$ are (b^{-1}, b) and (a^{-1}, a) , respectively. In particular we see that, in this copy of $F_2 \times F_2$, the group generated by $\rho(u)$, $\rho(v)$, $\rho(w)$, and $\rho(z)$ is isomorphic to $F_2 \times F_2$, as desired. \square

Example 3.1. In the particular case of $\text{PAut}(F_3)$, it is easy to give an explicit description of the generating set under the group homomorphism ρ above. This standard generating set has six elements, and is represented by the arrow diagram of figure 1. In the rest of the example, we will refer to the labelling of that figure. Then one checks that the images of the vertices of Λ under ρ are

$$\begin{aligned}\rho(c_{21}) &= (a^{-1}, a, 1, 1, a, 1) \\ \rho(c_{31}) &= (a, 1, 1, 1, a^{-1}, a) \\ \rho(c_{32}) &= (b, 1, a^{-1}, a, 1, 1) \\ \rho(c_{12}) &= (b^{-1}, b, a, 1, 1, 1) \\ \rho(c_{13}) &= (1, 1, b, 1, b^{-1}, b) \\ \rho(c_{23}) &= (1, 1, b^{-1}, b, b, 1).\end{aligned}$$

Observe that $\rho([c_{31}, c_{32}]) = ([a, b], 1, 1, 1, 1, 1)$ and this element commutes with $\rho(c_{13})$. Therefore

$$\rho([[c_{31}, c_{32}], c_{13}]) = 1$$

so ρ is not injective.

An enhanced version of this example gives:

Proposition 3.2. *The homomorphism*

$$\rho : \text{PAut}(F_n) \rightarrow \prod_{i=1}^N F_2 \times F_2$$

constructed above is not injective for $n \geq 3$.

Proof. We claim that for any i, j, k pairwise distinct

$$\rho([c_{ij}, c_{ik}], c_{ji}) = 1.$$

Consider first $\rho(c_{ij})$. Its component on each pair of dual arrows is trivial except in the following cases

- (1)_s for $1 \leq s \leq n$, $s \neq i, j$: arrow from c_{sj} with dual from c_{si} ,
- (2)_s for $1 \leq s \leq n$, $s \neq i, j$: arrow from c_{ij} with dual from c_{is} .

Now, doing the same for $\rho(c_{ik})$ one sees that for each pair of arrows, one of $\rho(c_{ij})$ or $\rho(c_{i,k})$ is trivial, except in the case (2)_s above for $s = k$. Therefore $\rho([c_{ij}, c_{ik}])$ is trivial everywhere except for that precise pair of arrows. As $\rho(c_{ji})$ is trivial on that pair we get the claim. \square

Remark 3.3. In fact there is no hope to be able to represent $\text{PAut}(F_n)$ faithfully as a subdirect product of a product of free groups of any rank. Indeed, since $\text{PAut}(F_n)$ is of cohomological type FP_∞ [2], a result of Bridson-Howie-Miller-Short [4] implies that $\text{PAut}(F_n)$ would be virtually a product of free groups, which is not the case.

4. NORMAL RAAG SUBGROUPS OF $\text{PAut}(A_\Gamma)$

In this section, we will prove Theorem 1.4. As stated in the introduction, the map of Theorem 1.1 gives a linear representation of $\text{PAut}(A_\Gamma)$ thus we have a linear quotient of this group. In Theorem 1.4 we identify a linear normal subgroup $A_{\hat{\Gamma}}$ of $\text{PAut}(A_\Gamma)$; more precisely a normal subgroup of $\text{PAut}(A_\Gamma)$ which its itself a RAAG. Observe that $\text{PAut}(A_\Gamma)$ contains the subgroup $\text{Inn}(A_\Gamma) \cong A_\Gamma/Z(A_\Gamma)$ of inner automorphisms as a normal subgroup. Moreover, $\text{Inn}(A_\Gamma)$ is well known to be a RAAG, namely the RAAG associated to the graph $\Gamma - \Lambda$ where Λ is the set of vertices of Γ which are linked to every vertex other than themselves. There are other ways to find canonical normal subgroups of $\text{PAut}(A_\Gamma)$ which are normal and RAAGs (see [5] and Remark 4.7 below) but our subgroup $A_{\hat{\Gamma}}$ is the biggest possible in a certain sense. For example, $\text{PAut}(A_\Gamma)$ is a RAAG itself if and only if there is no SIL (the if direction was proven in [6], the only if in [15]) and this happens if and only if $A = \text{PAut}(A_\Gamma)$.

As we will see, the RAAG $A_{\hat{\Gamma}}$ will be constructed explicitly via a graph $\hat{\Gamma}$ built from Γ . Before explaining all this, we need some preliminary work. First, we give the following local characterization of a graph containing no SILs. Given $n \geq 2$ by an n -claw we mean the complete bipartite graph $K_{1,n}$. Given an n -claw, its unique vertex of valence n is called the *center*, and the other vertices are called the *leaves*. We have:

Proposition 4.1. *Let Γ be a connected graph. The following conditions are equivalent:*

- (i) For each 3-claw embedded as full subgraph, there is a labelling v_1, v_2, v_3 of the leaves so that v_2 and v_3 are connected in $\Gamma - \text{st}(v_1)$ and v_1 and v_2 are connected in $\Gamma - \text{st}(v_3)$.
- (ii) Γ has no SIL.

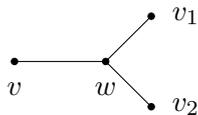
Proof. We first prove (i) \Rightarrow (ii). Assume, for contradiction, that Γ has a SIL pair v_1, v_2 . We claim that then v_1 and v_2 are leaves of some 3-claw embedded in Γ as full subgraph.

In order to see this, we first check that there is a vertex w linked to both v_1 and v_2 . (We remark that this argument appears in [13, Lemma 6.2], although we include it here for completeness.) Let C be a shared component of the SIL; recall this means that C is both a connected component of $\Gamma - \text{st}(v_1)$ and of $\Gamma - \text{st}(v_2)$. As Γ is connected, there is some path in Γ connecting v_1 to C . But the fact that C is a connected component of $\Gamma - \text{st}(v_1)$ implies that this path has length 2; in particular, there exists some vertex w linked to v_1 and to some vertex v_3 in C . Now, since C is also a connected component of $\Gamma - \text{st}(v_2)$ we deduce that w is linked to v_2 too. At this point, we have our embedded 3-claw with center w and leaves v_1, v_2 and v_3 ; it remains to check that it is a full subgraph. Observe first that v_1 and v_2 cannot be linked in Γ by the definition of SIL. Neither can v_1 be linked to v_3 because of the choice of C ; the same applies to v_2 and v_3 . We claim that (i) cannot hold for the 3-star formed by v_1, v_2, v_3, w . Otherwise, we may assume by symmetry that (say) v_2 and v_3 are connected in $\Gamma - \text{st}(v_1)$, which is again impossible by the choice of C .

Conversely, assume that (i) fails. Up to relabeling we may assume that there is a full subgraph of Γ which is a 3-claw with leaves v_1, v_2 and v_3 , in such way that v_1 and v_2 are not connected in $\Gamma - \text{st}(v_3)$ and v_1 and v_3 are not connected in $\Gamma - \text{st}(v_2)$. Let C be the connected component of $\Gamma - \text{st}(v_3)$ that contains v_1 . Note that $v_2 \notin C$, hence C cannot be the dominant component of $\Gamma - \text{st}(v_3)$ with respect to v_2 . Assume that C is a subordinate component; in other words, $C \subseteq L_2$ where $L_2 \subseteq \Gamma - \text{st}(v_2)$ is the dominant component with respect to v_3 . Then, as $v_1 \in C$ and $v_3 \in L_2$, we would deduce that v_1 and v_3 are connected in $\Gamma - \text{st}(v_2)$, which is a contradiction. At this point, C must be a shared component, and thus v_2 and v_3 form a SIL pair. \square

A consequence of the proof of Proposition 4.1 is that, if Γ is connected, then for each SIL pair v, u there is a 3-claw with v and u as leaves that fails to satisfy the property in (i). This motivates the following definition.

Definition 4.2. Fix a vertex $v \in \Gamma$, and w a vertex linked to v . We say that w is v -SIL irrelevant if for any 3-claw $\Delta \subseteq \Gamma$, with center w and v a leaf, and which is a full subgraph of Γ , Δ satisfies condition (i) of Proposition 4.1. In other words, for any full subgraph with form



we have (possibly after interchanging v_1 and v_2) that v and v_1 are connected in $\Gamma - \text{st}(v_2)$ and at least one of the following holds: either v_1 and v_2 are

connected in $\Gamma - \text{st}(v)$ or v and v_2 are connected in $\Gamma - \text{st}(v_1)$. The *slink* of v is now defined as the set of vertices in $\text{lk}(v)$ which are v -SIL-irrelevant, that is:

$$\text{slk}(v) = \{w \in \text{lk}(v) \mid w \text{ is } v\text{-SIL irrelevant}\}.$$

Also, set

$$\text{sst}(v) = \text{slk}(v) \cup \{v\}.$$

Note that if a vertex v is not part of any SIL-pair, then $\text{st}(v) = \text{sst}(v)$. In fact, the name ‘‘SIL-irrelevant’’ in the definition above refers to the fact that the removal of w does not create new SIL-type relations (i.e. as in Theorem 2.4 (iii)) between the partial conjugations by v in the corresponding connected components (this will be clear below).

Example 4.3. Consider the following graphs



In the first graph, w_2 is v -SIL irrelevant, but w_1 is not. Thus $\text{sst}(v) = \{v, w_2\}$. In the second graph, w_4 and w_5 are both v -SIL irrelevant, while w_3 is not. Hence $\text{sst}(v) = \{v, w_4, w_5\}$.

Our next result relates irrelevant components with the connected components of the complement of a SIL:

Lemma 4.4. *Let Γ be a connected graph, and suppose u, v form a SIL-pair. Let L_v be the connected component of $\Gamma - \text{sst}(v)$ that contains u . Then, for any connected component D of $\Gamma - \text{st}(v)$ which is either a shared or the dominant component of the SIL, we have $D \subseteq L_v$. In other words, whenever*

- (i) *either D is also a connected component of $\Gamma - \text{st}(u)$,*
- (ii) *or $u \in D$,*

we have $D \subseteq L_v$.

Proof. Since $\text{sst}(v) \subseteq \text{st}(v)$ we deduce that if D is the dominant component of $\Gamma - \text{st}(v)$, then $D \subseteq L_v$. Now, assume that D is shared. The proof of Proposition 4.1 implies that there exists $w \notin \text{sst}(v)$ which is linked to some vertex in C , and also to both v and u . As $u, w \in \Gamma - \text{sst}(v)$ and $C \subseteq \Gamma - \text{st}(v) \subseteq \Gamma - \text{sst}(v)$ we deduce $C \subseteq L_v$ too. \square

The graph $\hat{\Gamma}$. We are finally in a position to define the (simplicial) graph $\hat{\Gamma}$ defining the RAAG A which appears in the statement of Theorem 1.4. The vertices of $\hat{\Gamma}$ are in bijection with pairs (L, v) , where v is a vertex of Γ and L is a connected component $\Gamma - \text{sst}(v)$. Two such vertices (L, v) and (T, u) are linked in $\hat{\Gamma}$ unless $u \in L$ and $v \in T$ with $u \notin \text{lk}(v)$.

We need one more technical result before proving Theorem 1.4:

Lemma 4.5. *Let Γ be a connected graph, v a vertex of Γ and $L \subseteq \Gamma - \text{sst}(v)$ a connected component. Then $L \not\subseteq \text{st}(v)$.*

Proof. Let $w \in L$. First, if $w \notin \text{st}(v)$ we are done. Hence assume that $w \in \text{st}(v)$. Since $w \notin \text{sst}(v)$, we know that w is not v -SIL irrelevant. In other words, there exists a 3-claw embedded as a full subgraph of Γ , with center w and leaves v, v_2 and v_3 , which fails to satisfy condition of (i) in Proposition 4.1. In particular, $v_2 \notin \text{st}(v)$ thus $v_2 \notin \text{sst}(v)$ also but as v_2 and $w \in L$ are linked, $v_2 \in L \cap (\Gamma - \text{st}(v))$, as desired. \square

We are finally in a position to prove Theorem 1.4:

Proof of Theorem 1.4. We first define φ on the vertices of $\hat{\Gamma}$ by the formula

$$\varphi(L, v) = c_{L, v}$$

where recall that $c_{L, v}$ is the partial conjugation of L by v . We first check that φ is well-defined. To this end, it suffices to see that φ preserves the standard relators of $A_{\hat{\Gamma}}$. Let (L, v) and (T, u) be two vertices. If either $u = v$ or $u \in \text{lk}_\Gamma(v)$, then the partial conjugations $c_{L, v}$ and $c_{T, u}$ commute. So we may assume that v and u are not linked in Γ and, say, $v \notin T$. Note that $T \cap (\Gamma - \text{st}(u)) = T - (T \cap \text{st}(v))$ is a union of connected components of $\Gamma - \text{st}(v)$, denoted D_1, \dots, D_r , so that

$$c_{T, u} = c_{D_1, u} \cdots c_{D_r, u}$$

and that none of them is the dominant component of the SIL as $v \notin D_1 \cup \dots \cup D_r$. Now we distinguish two cases:

- (i) Suppose first that L is the connected component of $\Gamma - \text{sst}(v)$ that contains u . Then, Proposition 4.4 yields that L contains the dominant and all the shared components of $\Gamma - \text{st}(v)$ with respect to u . Now, Theorem 2.4 implies that $[c_{L, v}, c_{D_i, u}] = 1$, for $i = 1, \dots, r$. Hence $[c_{L, v}, c_{T, u}] = 1$, as desired.
- (ii) Otherwise, L must be a union of shared components, and we are done again by Theorem 2.4.

At this point, we know that the map φ is well-defined. Next, let $\Lambda \subseteq \Gamma$ be the full subgraph whose vertices are linked to every vertex of Γ other than themselves. By a result of Servatius [21], the subgroup A_Λ is precisely the center of A_Γ . Moreover, if $\Gamma_0 = \Gamma - \Lambda$ then $A_{\Gamma_0} \cong A_\Gamma / A_\Lambda = \text{Inn}(A_\Gamma)$. Thus we have a map

$$\iota : A_{\Gamma_0} \rightarrow A_{\hat{\Gamma}}$$

given by the rule

$$v \mapsto \prod \{(L, v) \mid L \text{ connected component of } \Gamma - \text{sst}(v)\},$$

which is obviously well-defined. To see that its image is normal in $A_{\hat{\Gamma}}$ we proceed as follows. Let u, v be vertices of Γ , and T any connected component of $\gamma - \text{sst}(u)$. If $v \notin T$, then

$$\iota(v)^{(T, u)} = \iota(v).$$

Otherwise, if $u \in L$,

$$(L, v)^{(T, u)} = (L, v)^{\iota(v)},$$

while if $u \notin L$

$$(L, v)^{(T, u)} = (L, v).$$

In particular,

$$\iota(v)^{(T,u)} = \iota(v)^{\iota(u)},$$

as desired.

Now, the maps ι and φ fit in a commutative diagram similar to the one in [6, Page 9]:

$$\begin{array}{ccccccc} 1 & \longrightarrow & A_{\Gamma_0} & \xrightarrow{\iota} & A_{\hat{\Gamma}} & \longrightarrow & Q \longrightarrow 1 \\ & & \cong \downarrow & & \varphi \downarrow & & \bar{\varphi} \downarrow \\ 1 & \longrightarrow & \text{Inn}(A_{\Gamma}) & \longrightarrow & \text{Im}(\varphi) & \longrightarrow & \hat{Q} \longrightarrow 1 \end{array}$$

where the injectivity of ι follows from the commutativity of the diagram, Q and \hat{Q} are the corresponding quotient groups and $\bar{\varphi}$ is the induced map.

We now check that $\text{Im}(\varphi)$ is normal in $\text{PAut}(A_{\hat{\Gamma}})$. Let u, v be vertices of Γ , L a connected component of $\Gamma - \text{sst}(v)$, and C a connected component of $\Gamma - \text{st}(u)$. The same argument used above to show that φ is well-defined shows that $[c_{L,v}, c_{C,u}] = 1$, except possibly in the case when C is the dominant component, i.e., when $v \in C$. In this case, let T be the connected component of $\Gamma - \text{sst}(u)$ that contains C . Then $T - (T \cap \text{st}(u)) = C \cup C_1 \cup \dots \cup C_s$ for some (non-dominant) components C_1, \dots, C_s and we get

$$[c_{L,v}, c_{C_i,u}] = 1$$

for $i = 1, \dots, s$. As a consequence,

$$c_{L,v}^{c_{C,u}} = c_{L,v}^{c_{C,u}} \prod_{i=1}^s c_{C_i,u} = c_{L,v}^{c_{T,u}} \in \text{Im}(\varphi).$$

Finally, we claim that the induced map $\bar{\varphi} : Q \rightarrow \hat{Q}$ is injective. We follow the strategy of [6, Theorem 3.6], and first show that Q is abelian. To this end, let (L, v) and (T, u) be two vertices of $\hat{\Gamma}$, which are necessarily linked unless $v \in T$. In this case, if T_1, \dots, T_r are the remaining connected components of $\Gamma - \text{sst}(u)$, we have $[(L, v), (T_i, u)] = 1$ for $i = 1, \dots, r$, and thus

$$[(L, v), (T, u)] = [(L, v), (T, u)] \prod_{i=1}^r [(L, v), (T_i, u)] \in A_{\Gamma_0},$$

as desired. In fact, it is easy to give an explicit presentation of Q , through which one sees that Q is the free abelian group generated by the cosets of the (L, v) 's, after removing exactly one of them for each v (in an arbitrary manner).

To see that $\bar{\varphi}$ is injective, observe that if we had a linear combination \bar{z} of elements such that $\bar{\varphi}(\bar{z}) = 1$ then any preimage $z \in A_{\hat{\Gamma}}$ would be such that $\varphi(z) \in \text{Inn}(A_{\Gamma})$. Let (L, v) be some vertex whose coset appears in \bar{z} . As $L \not\subseteq \text{st}(v)$ by Lemma 4.5, $\varphi(z)$ conjugates any $w \in L$ by (at least) v . However, it does not conjugate any $u \in T_v$ and, using again that $T_v \not\subseteq \text{st}(v)$, we deduce that $\varphi(z)$ cannot be inner. \square

Remark 4.6. Observe that if Γ has no SIL, then $\text{sst}(v) = \text{st}(v)$ for any v thus $\hat{\Gamma}$ has a vertex for each connected component of $\Gamma - \text{st}(v)$ and the group $A_{\hat{\Gamma}}$ is precisely $\text{PAut}(A_{\Gamma})$, as shown already in [6].

Remark 4.7. In [5], in the case when Γ is connected and has no triangles, the authors construct a map from a finite index subgroup of $\text{Out}(A_\Gamma)$ to a product of groups of the form $\text{Out}(F)$, with F free. The kernel K turns out to be abelian, and is generated by the cosets in $\text{Out}(A_\Gamma)$ of those partial conjugations which are of the form $c_{L,v}$ where L is a connected component of $\Gamma - \{v\}$. This construction was subsequently generalized [7] to arbitrary graphs Γ .

In general, if we consider the subgroup N of $\text{PAut}(A_\Gamma)$ generated by partial conjugations $c_{L,v}$, with L a connected component of $\Gamma - \{v\}$, one can proceed along similar lines as above to show that N is a RAAG containing $\text{Inn}(A_\Gamma)$ and that it is normal in $\text{PAut}(A_\Gamma)$ and the corresponding quotient is abelian. Moreover, N embeds in $A_{\hat{\Gamma}}$, but in general as a proper subgroup. For example, this is what happens for the graphs of Example 4.3.

We finish this section with a few words about a possible obstruction to the linearity of $\text{PAut}(A_\Gamma)$ in the case when Γ has a SIL. Consider the group G defined as

$$G = (F_2 \times F_2) *_t$$

where the stable group is the diagonal group $\Delta(F_2 \times F_2)$ and t acts as the identity on $\Delta(F_2 \times F_2)$, i.e., $(g, g)^t = (g, g)$. G admits the following explicit presentation:

$$G = \langle x, a, y, b, t \mid [x, y] = [x, b] = [a, y] = [a, b] = [xy, t] = [ab, t] = 1 \rangle.$$

Metaftsis and Prassidis [19] have announced a proof that this group is linear. In fact, if G is not linear then the next lemma would imply that $\text{PAut}(A_\Gamma)$ cannot be linear either, provided Γ has a SIL:

Lemma 4.8. *If Γ has a SIL, then $\text{PAut}(A_\Gamma)$ has a subgroup isomorphic to $(F_2 \times F_2) *_t$ where $(g, g)^t = (g, g)$.*

Proof. Observe that we may express the defining relators of G as

$$G = \langle x, a, y, b, t \mid [x, y] = [x, b] = [a, y] = [a, b] = [xy, t] = [ab, t] = 1 \rangle \\ \langle x, a, y, b, t \mid a^y = a^b = a, x^y = x^b = x, t^y = t^{x^{-1}}, t^b = t^{a^{-1}} \rangle,$$

and that this means that G is the semidirect product of the (normal) subgroup K generated by x, a and t , which is free of rank 3, and a free group of rank 2 generated by y and b , where the action is given by partial conjugations.

Now, assume that Γ has a SIL pair v and u , and let C be a shared component. Choose some vertex $w \in C$ and consider the inner automorphisms c_u, c_v and c_w that conjugate by u, v and w respectively. Consider also the partial conjugations $c_{C,v^{-1}}$ and $c_{C,u^{-1}}$, and let H be the subgroup of $\text{PAut}(A_\Gamma)$ generated by these five elements. We claim that the map $f : G \rightarrow H$ induced by

$$\begin{aligned} x &\mapsto c_u, \\ a &\mapsto c_v, \\ t &\mapsto c_w, \\ y &\mapsto c_{C,u^{-1}}, \\ b &\mapsto c_{C,v^{-1}} \end{aligned}$$

is an isomorphism. To see this, observe that as the vertices v , u and w are pairwise not linked, the subgroup of A_Γ that they generate is free of rank 3, and thus the same holds for the conjugations c_u , c_v , c_w . Thus the restriction of f yields an isomorphism from K to the subgroup generated by c_u , c_v and c_w . The same happens for the restriction of f to the subgroup of G generated by y and b and the subgroup of H generated by $c_{C,u^{-1}}$ and $c_{C,v^{-1}}$. Taking into account the structure of G we deduce that f is in fact an isomorphism. \square

Remark 4.9. It is not difficult to construct explicit linear representations of G . Let A, B, D, M be matrices so that $[A, D] = 1 = [B, D]$ and A, B generate a free group (we might choose $D = I$). Represent the generators of G as follows:

$$\begin{aligned} x &\mapsto \text{diag}(A, I) \\ y &\mapsto \text{diag}(I, A) \\ a &\mapsto \text{diag}(B, I) \\ b &\mapsto \text{diag}(I, B) \\ t &\mapsto M \otimes D \end{aligned}$$

where \otimes is the Kronecker product. Note that $xy \mapsto \text{diag}(A, A) = I \otimes A$ which commutes with $M \otimes D$.

In light of the above remark, a natural question is:

Question 4.10. *Is there is some choice of A , B , D and M so that this representation is faithful?*

REFERENCES

- [1] J. Aramayona, C. Martínez-Pérez, On the first cohomology of automorphism groups of graph groups. *J. Algebra* 452 (2016)
- [2] N. Brady, J. McCammond, J. Meier, A. Miller, The Pure Symmetric Automorphisms of a Free Group Form a Duality Group. *J. of Algebra* 246 (2) (2001).
- [3] M. R. Bridson, The rhombic dodecahedron and semisimple actions of $\text{Aut}(F_n)$ on CAT(0) spaces, *Fund. Math.* 214 (2011).
- [4] M. R. Bridson, J. Howie, C. F Miller III, H. Short, Subgroups of direct products of limit groups. *Ann. of Math.* (2) 170 (2009)
- [5] R. Charney, J. Crisp, K. Vogtmann, Automorphisms of 2-dimensional right-angled Artin groups. *Geom. Topol.* 11 (2007), 2227-2264.
- [6] R. Charney, K. Ruane, N. Stambaugh, A. Vijayan, The automorphism group of a graph product with no SIL. *Illinois J. Math.* 54 (2010),
- [7] R. Charney, K. Vogtmann, Finiteness properties of automorphism groups of right-angled Artin groups. *Bull. Lond. Math. Soc.* 41 (2009).
- [8] M. Day, Peak reduction and finite presentations for automorphism groups of right-angled Artin groups, *Geometry & Topology* 13 (2009).
- [9] M. Day, R. Wade, Subspace arrangements, BMS invariants and pure symmetric outer automorphisms of right-angled Artin groups. To appear in *Group, Geometry and Dynamics*.
- [10] Michael W. Davis, Tadeus Januszkiewicz. Right-angled Artin groups are commensurable with right-angled Coxeter groups. *Journal of Pure and Applied Algebra* 153 (2000) 229-235.
- [11] E. Formanek, C. Procesi, The automorphism group of a free group is not linear. *J. Algebra* 149 (1992).
- [12] D. L. Goldsmith, The theory of motion groups. *Michigan Math. J.* 28 (1981)

- [13] Mauricio Gutierrez, Adam Piggott and Kim Ruane. On the Automorphisms of a Graph Product of Abelian Groups. *Groups Geom. Dyn.* 6 (2012), no. 1, 125-153.
- [14] S. P. Humphries, On weakly distinguished bases and free generating sets of free groups. *Quart. J. Math. Oxford Ser. (2)* 36 (1985),
- [15] N. Koban, A. Piggott, The Bieri-Neumann-Strebel invariant of the pure symmetric automorphisms of a right-angled Artin group. *Illinois J. Math.* 58 (2014),
- [16] M. R. Laurence. *Automorphisms of graph products of groups*. PhD thesis, Queen Mary College, University of London, 1993.
- [17] M. R. Laurence. A generating set for the automorphism group of a graph group. *J. London Math. Soc. (2)*, 52(2): 318-334, 1995.
- [18] J. McCool, On basis-conjugating automorphisms of free groups. *Canad. J. Math.* 38 (1986).
- [19] V. Metaftsis, S. Prassidis, *personal communication*.
- [20] B. Mühlherr. Automorphisms of graph-universal Coxeter groups. *J. Algebra* 200(2) 1998.
- [21] H. Servatius, Automorphisms of graph groups, *J. Algebra* 126 (1989).
- [22] Jaques Tits, Sur le groupe des automorphismes de certains groupes de Coxeter, *Journal of Algebra*, 113, no. 2 (1988).
- [23] E. Toinet, A finitely presented subgroup of the automorphism group of a right-angled Artin group. *J. Group Theory* 15 (2012)

JAVIER ARAMAYONA: DEPARTMENT OF MATHEMATICS, UNIVERSIDAD AUTÓNOMA DE MADRID, 28049 MADRID, SPAIN; AND INSTITUTO DE CIENCIAS MATEMÁTICAS (ICMAT).
E-mail address: javier.aramayona@uam.es

CONCHITA MARTÍNEZ-PÉREZ: DEPARTMENT OF MATHEMATICS, UNIVERSIDAD DE ZARAGOZA, PEDRO CERBUNA S/N, 50009 ZARAGOZA, SPAIN.
E-mail address: conmar@unizar.es