The first integral cohomology of
pure mapping class groups

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It is a classical result of Powell that pure mapping class groups of connected, orientable surfaces of finite type and genus at least three are perfect. In stark contrast, we construct nontrivial homomorphisms from infinite-genus mapping class groups to the integers. Moreover, we compute the first integral cohomology group associated to the pure mapping class group of any connected orientable surface of genus at least 2 in terms of the surface's simplicial homology. In order to do this, we show that pure mapping class groups of infinite-genus surfaces split as a semi-direct product.

1 Introduction

Throughout this article, a surface refers to an orientable, connected, second-countable 2-manifold with (possibly empty) compact boundary. Associated to every such surface S is its mapping class group, written $\mathrm{Map}(S)$, which is the group of orientation-preserving homeomorphisms of S up to isotopy, where all homeomorphisms and isotopies fix the boundary of S pointwise. The *pure mapping class group*, written $\mathrm{PMap}(S)$, is the subgroup of $\mathrm{Map}(S)$ whose elements induce the trivial permutation on the set of topological ends of S.

The structure of $\mathrm{Map}(S)$ is generally well-understood when S is of finite type, that is, when $\pi_1(S)$ is finitely generated. In contrast, infinite-type mapping class groups have been less studied, although there has been a recent surge of activity in exploring their structure. The work to date has largely focused on highlighting the similarities between infinite-type mapping class groups and their finite-type counterparts [1, 3, 5, 7, 12, 18]. Nevertheless, the comparison between both has strong limitations [4, 18].

The goal of this article is to exhibit another fundamental difference between the finite-type and the infinite-type settings. Recall that a classical result of Powell [19, Theorem 1] states that if S is of finite type and has genus at least 3, then $\mathrm{PMap}(S)$ is perfect; in other words, it has trivial abelianization. It follows that every homomorphism to $\mathbb{Z}$ is

trivial. Our main result asserts that in most infinite-genus cases, there exist nontrivial homomorphisms to $\mathbb{Z}$; moreover, we construct all such homomorphisms.

More concretely, let $\hat{S}$ be the surface obtained from S by "filling in" the planar ends and capping off the boundary components of S (this is made more precise in Section 2.1). We let $H_1^{sep}(\hat{S}, \mathbb{Z})$ denote the subgroup of $H_1(\hat{S}, \mathbb{Z})$ generated by homology classes that can be represented by simple separating curves on the surface.

Equip the group of self-homeomorphisms of S with the compact-open topology and give $\mathrm{Map}(S)$ the corresponding quotient topology. Let $\mathrm{PMap}_c(S)$ be the subgroup of $\mathrm{PMap}(S)$ consisting of compactly supported mapping classes (see Section 2.2), or equivalently, the subgroup of $\mathrm{PMap}(S)$ generated by Dehn twists. Let $\overline{\mathrm{PMap}_c(S)}$ be the closure of $\mathrm{PMap}_c(S)$ and note that $\overline{\mathrm{PMap}_c(S)}$ is normal in $\mathrm{PMap}(S)$, which allows us to define the group $A_S = \mathrm{PMap}(S)/\overline{\mathrm{PMap}_c(S)}$. We will prove:

Theorem 1 *If S a surface of genus at least 2, then there exists a natural isomorphism between $H_1^{sep}(\hat{S}, \mathbb{Z})$ and $H^1(\mathrm{PMap}(S), \mathbb{Z})$. Moreover, every homomorphism from $\mathrm{PMap}(S)$ to $\mathbb{Z}$ factors through A_S.*

In Theorem 1 (and throughout the paper), $\mathbb{Z}$ is treated as the trivial $\mathrm{PMap}(S)$-module, which allows us to identify $H^1(\mathrm{PMap}(S), \mathbb{Z})$ with $\mathrm{Hom}(\mathrm{PMap}(S), \mathbb{Z})$.

The homomorphism in Theorem 1 is constructed in Proposition 4.4 and is shown to be an isomorphism in Theorem 4.6. Roughly speaking, if an element of $v \in H_1^{sep}(\hat{S}, \mathbb{Z})$ can be represented on S by a simple separating curve, then the corresponding element – given by Theorem 1 – in $H^1(\mathrm{PMap}(S), \mathbb{Z})$ applied to an element $f \in \mathrm{PMap}(S)$ measures how many handles f moves from the left side of v to the right side. This construction is extended to all elements of $H_1^{sep}(\hat{S}, \mathbb{Z})$ via linear combinations.

Intuitively, a topological end of a surface is *accumulated by genus* if there is a sequence of handles escaping to the end. Applying basic properties of homology, we obtain the following corollary to Theorem 1:

Corollary 2 *Let S be a surface of genus at least 2.*

(1) *S has at most one end accumulated by genus if and only if $H^1(\mathrm{PMap}(S), \mathbb{Z})$ is trivial.*

(2) *S has infinitely many ends accumulated by genus if and only if the rank of $H^1(\mathrm{PMap}(S), \mathbb{Z})$ is infinite.*

(3) *Let $n \in \mathbb{N}$[1]. The rank of $H^1(\mathrm{PMap}(S), \mathbb{Z})$ is n if and only if S has $n + 1$ ends accumulated by genus.*

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[1] For clarity, we set $\mathbb{N} = \{1, 2, 3, \ldots\}$; in particular, $0 \notin \mathbb{N}$.

A recent result of Bavard, Dowdall, and Rafi [3] proving a conjecture of the second and third author from [18] shows that the isomorphism type of either $\text{Map}(S)$ or $\text{PMap}(S)$ determines the homeomorphism type of S and vice versa. Corollary 2 serves as another example of the direct link between the algebra of $\text{PMap}(S)$ and the topology of S.

We note that the genus 2 restriction in Theorem 1 and Corollary 2 is necessary: pure mapping class groups of finite-type surfaces of genus less than 2 have infinite abelianization when there are at least 4 boundary components (see [8, §5.2.3 & 9.2]).

In order to prove Theorem 1, we first prove a structural result about $\text{PMap}(S)$. In [18, Theorem 4], the second and third author showed that $\overline{\text{PMap}_c(S)} = \text{PMap}(S)$ if and only if S has at most one end accumulated by genus. Moreover, when S has more than one end accumulated by genus, it was shown that $\text{PMap}_c(S)$ together with the set of handle shifts (see Section 2.3) topologically generate $\text{PMap}(S)$. The next result is a precise measure of the role of handle shifts in the failure of Dehn twists to topologically generate $\text{PMap}(S)$.

Let $H^1_{sep}(\hat{S}, \mathbb{Z})$ be the subgroup of $H^1(\hat{S}, \mathbb{Z})$ that is identified with $\text{Hom}(H^{sep}_1(\hat{S}, \mathbb{Z}), \mathbb{Z})$ under the natural isomorphism $H^1(\hat{S}, \mathbb{Z}) \to \text{Hom}(H_1(\hat{S}, \mathbb{Z}), \mathbb{Z})$ given by the universal coefficient theorem.

Theorem 3 *Let S be a surface with at least two ends accumulated by genus. There is an injection $\kappa\colon H^1_{sep}(\hat{S}, \mathbb{Z}) \to \text{PMap}(S)$ such that $\pi \circ \kappa\colon H^1_{sep}(\hat{S}, \mathbb{Z}) \to A_S$ is an isomorphism, where $\pi\colon \text{PMap}(S) \to A_S$ is the natural projection. Moreover, there exist pairwise commuting handle shifts $\{h_i\}_{i=1}^r$ such that*

$$\kappa(H^1_{sep}(\hat{S}, \mathbb{Z})) = \prod_{i=1}^r \langle h_i \rangle,$$

where $r \in \mathbb{N} \cup \{\infty\}$ denotes the rank of $H^{sep}_1(\hat{S}, \mathbb{Z})$.

Note that in Theorem 3 the rank of $H^{sep}_1(\hat{S}, \mathbb{Z})$ can be infinite, so it is necessary to use the direct product. As an immediate corollary of Theorem 3, we obtain:

Corollary 4 $\text{PMap}(S) = \overline{\text{PMap}_c(S)} \rtimes \kappa(H^1_{sep}(\hat{S}, \mathbb{Z}))$ *for any surface S.*

Note that:

- if S has at most one end accumulated by genus, then Corollary 4 is a restatement of [18, Theorem 4].

- if S has n ends accumulated by genus with $2 \leq n < \infty$, then $\kappa(H^1_{sep}(\hat{S}, \mathbb{Z})) \cong \mathbb{Z}^{n-1}$.

- if S has infinitely many ends accumulated by genus, then $\kappa(H^1_{sep}(\hat{S}, \mathbb{Z})) \cong \mathbb{Z}^{\mathbb{N}}$, where $\mathbb{Z}^{\mathbb{N}}$ is the Baer-Specker group.

To go from Corollary 4 to Theorem 1 is, to the authors' knowledge, the first application of descriptive set theory to the study of mapping class groups. We note that descriptive set theory has been previously used in the study of homomorphisms of homeomorphism groups of compact manifolds, for instance see the work of Rosendal-Solecki [22], Rosendal [21], and Mann [17].

It follows from recent work of Hernandez–Morales–Valdez [11] and Bavard–Dowdall–Rafi [3] on the automorphism group of the curve graph that $\mathrm{PMap}(S)$ is Polish, that is, it is separable and completely metrizable as a topological space (see Section 2.4 for more details). The key tool used to go from Corollary 4 to Theorem 1 is a theorem of Dudley [6, Theorem 1] showing that every homomorphism from a Polish group to $\mathbb{Z}$ is continuous.

As mentioned above, every finite-type pure mapping class group of genus at least 3 is perfect [19, Theorem 1], and hence so is $\mathrm{PMap}_c(S)$, being a direct limit of perfect groups. Applying Dudley's theorem to $\overline{\mathrm{PMap}_c(S)}$, which is Polish by Corollary 2.5, we see that $H^1(\overline{\mathrm{PMap}_c(S)}, \mathbb{Z})$ is trivial. Given this discussion and Theorem 1, the natural generalization of the perfectness of pure mapping class groups in the finite-type setting would appear to be the following conjecture:

Conjecture 5 *If S a surface of genus at least 3, then $\overline{\mathrm{PMap}_c(S)}$ is perfect.*

As a direct consequence of Corollary 4, we have

Corollary 6 *For any surface S, the group $A_S = \mathrm{PMap}(S)/\overline{\mathrm{PMap}_c(S)}$ is abelian and hence*

$$[\mathrm{PMap}(S), \mathrm{PMap}(S)] \leq \overline{\mathrm{PMap}_c(S)},$$

where $[\mathrm{PMap}(S), \mathrm{PMap}(S)]$ denotes the commutator subgroup of $\mathrm{PMap}(S)$.

Given Theorem 1 and Corollary 6, we are led to the following conjecture, which would also follow from Conjecture 5.

Conjecture 7 *For any surface S of genus at least 3, the abelianization of $\mathrm{PMap}(S)$ is the group A_S; in particular, $[\mathrm{PMap}(S), \mathrm{PMap}(S)] = \overline{\mathrm{PMap}_c(S)}$.*

We finish the introduction with two additional applications of the work above. In the setting of pure mapping class groups of finite-type surfaces, the first author and Souto [2] showed that there are many restrictions on the existence of homomorphisms between pure mapping class groups. In stark contrast, we have:

Corollary 8 *Let S be a surface with at least 2 ends accumulated by genus. For every surface S' of negative Euler characteristic not homeomorphic to a thrice-punctured sphere there exists a nontrivial homomorphism $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$.*

Proof To see this, simply take a nontrivial homomorphism $\mathrm{PMap}(S) \to \mathbb{Z}$ and then any nontrivial homomorphism $\mathbb{Z} \to \mathrm{PMap}(S')$. $\qquad\square$

Relating to Kazhdan's property (T), it is a famous open question whether mapping class groups of finite-type surfaces virtually surject onto $\mathbb{Z}$. By restricting the number of ends accumulated by genus, we see that this can be done for a large class of big mapping class groups:

Corollary 9 *If the number of ends of a surface S accumulated by genus is finite and at least 2, then $\mathrm{Map}(S)$ virtually surjects onto $\mathbb{Z}$.*

Acknowledgements

2 Preliminaries

All surfaces in this article are assumed to be connected, orientable, second-countable, and to have (possibly empty) compact boundary. The notion of a simple closed curve is critical to the study of mapping class groups, so for clarity, a simple closed curve on a surface S is an embedding of the circle $\mathbb{S}^1$ in S. A simple closed curve is *trivial* if it is homotopic to a point; it is *peripheral* if it is either homotopic to a boundary component or bounds a once-punctured disk; it is *separating* if its complement is disconnected.

We will routinely abuse notation and conflate a mapping class with a representative homeomorphism and a simple closed curve with its isotopy class.

2.1 Topological ends and the classification of surfaces

The classification of surfaces of infinite type relies on the notion of a topological end – first introduced by Freudenthal. For a reference on the following material see [20].

Definition 2.1 The *space of ends* of a surface S, denoted $\mathcal{E}(S)$, is the set

$$\mathcal{E}(S) = \operatorname*{proj\,lim}_{K}(\pi_0(S \smallsetminus K)),$$

where the projective limit is taken over the set of all compact subsets of S, which is directed with respect to inclusion. Further, by equipping $\pi_0(S \smallsetminus K)$ with the discrete topology for each compact set K, the set $\mathcal{E}(S)$ can be given the limit topology, that is, the coarsest topology such that the projection maps $\varphi_K \colon \mathcal{E}(S) \to \pi_0(K)$ given by the universal property of projective limits are continuous.

The *Freudenthal compactification* of S, denoted $\bar{S}$, is the set $\bar{S} = S \sqcup \mathcal{E}(S)$ equipped with the topology whose open sets are the open subsets of S together with sets of the form $U \cup U^*$, where U is an open connected set of S with compact boundary and $U^* = \varphi_{\partial U}^{-1}([U])$.

An end e of S is *accumulated by genus* if every neighborhood of e in $\bar{S}$ has infinite genus. The set of all ends accumulated by genus is a closed subset of $\mathcal{E}(S)$ and is denoted $\mathcal{E}_\infty(S)$. Each end $e \in \mathcal{E}(S) \smallsetminus \mathcal{E}_\infty(S)$ is *planar*, that is, e has a neighborhood in $\bar{S}$ homeomorphic to a disk. The surface

$$\hat{S} = \bar{S} \smallsetminus \mathcal{E}_\infty(S)$$

is a surface in which every end is accumulated by genus and can be thought of as being obtained from S by "forgetting" or "filling in" the planar ends. (If S has finite genus, then $\bar{S} = \hat{S}$.)

The classification of surfaces states that a surface S is determined up to homeomorphism by the quadruple $(g, b, \mathcal{E}(S), \mathcal{E}_\infty(S))$, where g is the genus of S and b is the number of boundary components of S.

2.2 Mapping class groups

Let $\operatorname{Homeo}_\partial(S)$ denote the group of self-homeomorphisms of S that fix the boundary of S pointwise. The *extended mapping class group* of S, denoted $\operatorname{Map}^\pm(S)$, is defined to be

$$\operatorname{Map}^\pm(S) = \operatorname{Homeo}_\partial(S)/\sim,$$

where $f \sim g$ if they are isotopic through homeomorphisms fixing ∂S pointwise. If ∂S is empty, then the *mapping class group*, denoted $\mathrm{Map}(S)$, is the index two subgroup of orientation-preserving mapping classes; otherwise, every element of $\mathrm{Homeo}_\partial(S)$ is orientation-preserving and we identify $\mathrm{Map}(S)$ with $\mathrm{Map}^\pm(S)$. The *pure mapping class group* is the kernel of the action of $\mathrm{Map}(S)$ on $\mathcal{E}(S)$.

Given $f \in \mathrm{Homeo}_\partial(S)$, we say f is *compactly supported* if the closure of the set $\{x \in S : f(x) \neq x\}$, called the *support of f*, is compact, or equivalently, if f is the identity outside of a compact set. We say a mapping class is *compactly supported* if it has a compactly supported representative.

The *compactly-supported pure mapping class group* of S, denoted $\mathrm{PMap}_c(S)$, is the subgroup of $\mathrm{Map}(S)$ consisting of compactly-supported mapping classes.

Equipped with the compact-open topology, $\mathrm{Homeo}_\partial(S)$ becomes a topological group. The *compact-open topology* on $\mathrm{Map}^\pm(S)$ is defined to be the quotient topology induced by the epimorphism

$$\mathrm{Homeo}_\partial(S) \to \mathrm{Map}^\pm(S).$$

We will view any subgroup of $\mathrm{Map}^\pm(S)$ as a topological group by equipping it with the subspace topology, which will also be referred to as the compact-open topology.

2.3 Handle shifts

The abelian group in the decomposition of $\mathrm{PMap}(S)$ presented in Corollary 4 is generated by handle shifts, which were introduced in [18]. We recall their definition now: Let Σ be the surface obtained by gluing handles onto $\mathbb{R} \times [-1, 1]$ periodically with respect to the transformation $(x, y) \mapsto (x + 1, y)$. Let $\sigma \colon \Sigma \to \Sigma$ be the homeomorphism up to isotopy determined by requiring

(1) $\sigma(x, y) = (x + 1, y)$ for $(x, y) \in \mathbb{R} \times [-1 + \epsilon, 1 - \epsilon]$ for some $\epsilon > 0$ and

(2) $\sigma(x, y) = (x, y)$ for $(x, y) \in \mathbb{R} \times \{-1, 1\}$.

We say a homeomorphism $h \colon S \to S$ is a *handle shift* if there exists a proper embedding $\iota \colon \Sigma \to S$ such that

$$h(x) = \begin{cases} (\iota \circ \sigma \circ \iota^{-1})(x) & \text{if } x \in \iota(\Sigma) \\ x & \text{otherwise} \end{cases}$$

We will say that h is *supported* on $\iota(\Sigma)$, or simply on Σ, to relax notation. Since the embedding ι is required to be proper, it induces a map $\iota_\infty \colon \mathcal{E}(\Sigma) \to \mathcal{E}_\infty(S)$. We
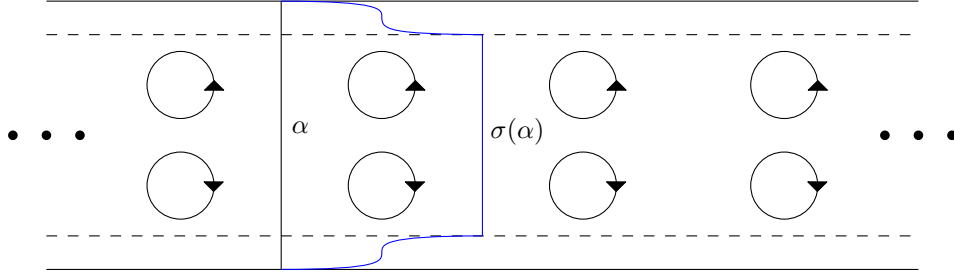
Figure 1: The circles are identified vertically to obtain Σ.

therefore see that a handle shift h has an attracting and repelling end, which we denote h^+ and h^-, respectively. That is, for any x in the interior of $\iota(\Sigma)$ we have

$$\lim_{n\to\infty} h^n(x) = h^+ \text{ and } \lim_{n\to-\infty} h^n(x) = h^-,$$

where the limit is taken in the Freudenthal compactification $\bar{S}$ of S. Note that if h_1 and h_2 are homotopic handle shifts, then $h_1^+ = h_2^+$ and $h_1^- = h_2^-$; therefore, there are well-defined notions of attracting and repelling ends for the mapping class associated to a handle shift.

2.4 Polish groups

A topological group is *Polish* if its underlying topological space is separable and completely metrizable. (See [14] for a reference to the theory of Polish spaces/groups and for the following material.) Let Γ be a graph with a countable vertex set. Given any subset A of Γ, define

$$U(A) = \{g \in \mathrm{Aut}(\Gamma)\colon g(a) = a \text{ for all } a \in A\}.$$

We can then equip $\mathrm{Aut}(\Gamma)$ with the *permutation topology*, which is generated by all $\mathrm{Aut}(\Gamma)$-translates of the pointwise stabilizers $U(A)$, where A is a finite subset of Γ. This makes $\mathrm{Aut}(\Gamma)$ into a topological group; furthermore, it is clear that $\mathrm{Aut}(\Gamma)$ is separable (in fact, it is second-countable).

Lemma 2.2 $\mathrm{Aut}(\Gamma)$ *with the permutation topology is Polish.*

Proof As Γ is countable, choose an enumeration $\{x_i\}_{i\in\mathbb{N}}$ of Γ. Let $d\colon \Gamma \times \Gamma \to \mathbb{R}$ be given by

$$d(g, g') = \inf_{n\in\mathbb{N}\cup\{0\}} \{2^{-n}\colon g(x_i) = g'(x_i) \text{ for all } i < n\}.$$

It turns out that d is not complete; however, we leave as an exercise to check that $\rho(g, g') = d(g, g') + d(g^{-1}, (g')^{-1})$ is a complete metric and the corresponding metric topology is the permutation topology. $\qquad\square$

Let us turn back to mapping class groups. Let $\mathcal{C}(S)$ denote the curve graph of S, that is, the graph whose vertices consist of nontrivial non-peripheral isotopy classes of simple closed curves on S and where two vertices are adjacent if they have disjoint representatives. The following recent theorem is a generalization of a result of Ivanov [13] (see also [16]) in the finite-type setting:

Theorem 2.3 (Hernandez–Morales–Valdez [11], Bavard–Dowdall–Rafi [3]) *If S is of infinite type, then* $\mathrm{Aut}(\mathcal{C}(S)) = \mathrm{Map}^{\pm}(S)$.

A straightforward application of the Alexander Method [8, Proposition 2.8] yields:

Proposition 2.4 *The compact-open topology and the permutation topology are the same topologies on* $\mathrm{Map}^{\pm}(S)$.

As every closed subspace of a Polish space is Polish we obtain:

Corollary 2.5 *Every closed subgroup of* $\mathrm{Map}^{\pm}(S)$ *is Polish. In particular,* $\mathrm{Map}(S)$, $\mathrm{PMap}(S)$, *and* $\overline{\mathrm{PMap}_c(S)}$ *are Polish.*

Proof If S is borderless, then the result is an immediate consequence of Lemma 2.2, Theorem 2.3, and Proposition 2.4. If S has boundary, then it is straightforward to embed S in a larger borderless surface S' so that $\mathrm{Map}(S)$ is a closed subgroup of $\mathrm{Map}(S')$. $\qquad\square$

3 Colorings and homomorphisms

In this section, we will use colorings of subgraphs of the curve graph to build nontrivial homomorphisms of pure mapping class groups of infinite-genus surfaces to $\mathbb{Z}$. The constructions given here are directly motivated by the colorings presented in [10, Section 5] and their relations to the Chillingworth homomorphism described in [10, Section 6].

Let S be any surface. For a homology class $v \in H_1(S, \mathbb{Z})$, we will regularly attribute properties of a representative curve to v; for instance, v is *simple* if it there exists a simple closed curve on S representing v. An element of $H_1(S, \mathbb{Z})$ is *primitive* if it cannot be written as a positive multiple of another element. If a surface is compact with

at most one boundary component or non-compact, borderless, and with at most 1 end, then every primitive homology class is simple (see [8, Proposition 6.2]); however, in all other cases there are primitive homology classes that fail to be simple.

Let γ be an oriented simple closed curve on S. The homology class of γ is determined, up to orientation, by its partition of $\mathcal{E}(S) \cup \partial S$; in particular, each simple element $v \in H_1^{sep}(S, \mathbb{Z})$ determines a partition of $\mathcal{E}(S)$ into two disjoint sets v^- and v^+, where v^- consists of the ends to the left of v and v^+ those to the right. Note that if v is simple, then $v = 0$ if and only if one of v^- or v^+ is empty.

For a simple nonzero element $v \in H_1(S, \mathbb{Z})$, let $\mathcal{C}_v(S)$ denote the subgraph of $\mathcal{C}(S)$ induced on the set of vertices that can be oriented to be represent v. A *proper coloring* of a graph is a labelling of the vertices such that no two adjacent vertices share a label.

Now, let S be an infinite-genus surface. Fix a simple nonzero homology class $v \in H_1^{sep}(S, \mathbb{Z})$ and let γ be an oriented simple closed curve on S representing v. Choose an end $e_v \in v^-$. We will now give a proper $\mathbb{Z}$-coloring $\phi_\gamma \colon \mathcal{C}_v(S) \to \mathbb{Z}$.

Let $c \in \mathcal{C}_v(S)$. As $c \cup \gamma$ is compact, there exists a connected compact surface R such that each component of $\partial R \smallsetminus \partial S$ is separating and $c \cup \gamma \subset \mathring{R}$, where $\mathring{R}$ denotes the interior of R. Let ∂_0 be the boundary component of R such that e_v and $\mathring{R}$ are on opposite sides of ∂_0. Let $\mathfrak{g}_R(c)$ and $\mathfrak{g}_R(\gamma)$ be the genera of the component of $R \smallsetminus c$ and $R \smallsetminus \gamma$ containing ∂_0. Define

$$\phi_\gamma(c) = \mathfrak{g}_R(c) - \mathfrak{g}_R(\gamma).$$

It is straightforward to check that the definition of ϕ_γ is independent of the choice of R.

Lemma 3.1 ϕ_γ *is a proper $\mathbb{Z}$-coloring of $\mathcal{C}_v(S)$.*

Proof If $b, c \in \mathcal{C}_v(S)$ are adjacent, then they cobound a compact surface F of positive genus satisfying

$$0 < \mathrm{genus}(F) = |\phi_\gamma(c) - \phi_\gamma(d)|.$$

It follows that ϕ_γ is a proper coloring. $\qquad\square$

Since a nonzero, non-peripheral, simple, separating homology class v is determined, up to orientation, by its partition of $\mathcal{E}(S) \cup \partial S$, the action of $\mathrm{PMap}(S)$ on $\mathcal{C}(S)$ restricts to an action on $\mathcal{C}_v(S)$.

Lemma 3.2 *If $v \in H_1^{sep}(S, \mathbb{Z})$ is simple and nonzero, γ and δ are oriented simple closed curves representing v, and $f \in \mathrm{PMap}(S)$, then the following equalities hold:*

$$(3\text{--}1) \qquad\qquad \phi_\gamma \circ f - \phi_\gamma = \phi_\gamma(f(\gamma)).$$

$$(3\text{--}2) \qquad\qquad \phi_\gamma - \phi_\delta = \phi_\gamma(\delta) = -\phi_\delta(\gamma).$$

$$(3\text{--}3) \qquad\qquad \phi_\gamma \circ f - \phi_\gamma = \phi_\gamma - \phi_{f(\gamma)}.$$

Proof Let $c \in \mathcal{C}_v(S)$. Choose a compact surface R containing $c, f(c), \gamma, f(\gamma)$ and such that each component of $\partial R \smallsetminus \partial S$ is separating. To see (3–1), a direct computation yields:

$$\begin{aligned}
\phi_\gamma(f(c)) &= \mathfrak{g}_R(f(c)) - \mathfrak{g}_R(\gamma) \\
&= \mathfrak{g}_R(f(c)) - \mathfrak{g}_R(f(\gamma)) + \mathfrak{g}_R(f(\gamma)) - \mathfrak{g}_R(\gamma) \\
&= \phi_\gamma(c) + \mathfrak{g}_R(f(\gamma)) - \mathfrak{g}_R(\gamma) \\
&= \phi_\gamma(c) + \phi_\gamma(f(\gamma))
\end{aligned}$$

(3–2) follows from a near identical argument. Combining (3–1) and (3–2) with $\delta = f(\gamma)$ yields (3–3). $\qquad\square$

Given a handle shift h and a simple element $v \in H_1^{sep}(S, \mathbb{Z})$, we say v *cuts* h if h^+ and h^- are on opposite sides of v. In light of Lemma 3.2, we are now in a position to define a homomorphism $\varphi_v \colon \mathrm{PMap}(S) \to \mathbb{Z}$ for $v \in H_1^{sep}(S, \mathbb{Z})$ simple.

Proposition 3.3 (Definition of φ_v) *Let S be an infinite-genus surface. Let $v \in H_1^{sep}(S, \mathbb{Z})$ be simple and nonzero. If γ is any oriented simple closed curve representing v, then*

$$\varphi_v \colon \mathrm{PMap}(S) \to \mathbb{Z}$$

given by

$$\varphi_v(f) = \phi_\gamma \circ f - \phi_\gamma,$$

is a well-defined homomorphism. In addition, φ_v satisfies:

(1) $\varphi_v\left(\overline{\mathrm{PMap}_c(S)}\right) = 0$.

(2) $\varphi_{-v} = -\varphi_v$.

(3) *If $h \in \mathrm{PMap}(S)$ is a handle shift, then $\varphi_v(h) \neq 0$ if and only if v cuts h.*

(4) $\varphi_v = 0$ *if and only if $\mathcal{E}_\infty(S) \subset v^+$ or $\mathcal{E}_\infty(S) \subset v^-$.*

Proof Fix $v \in H_1^{sep}(S, \mathbb{Z})$. We first need to show that ϕ_v is well-defined. Let γ and δ be oriented simple closed curves homologous to v. Using Lemma 3.2, we have

$$\phi_\gamma \circ f - \phi_\gamma = \phi_\gamma(f(\gamma))$$
$$= \phi_\delta(f(\gamma)) - \phi_\delta(\gamma)$$
$$= \phi_\delta \circ f - \phi_\delta.$$

Hence, φ_v is well defined.

If $f, g \in \mathrm{PMap}(S)$, then

$$\varphi_v(fg) = \phi_\gamma \circ (fg) - \phi_\gamma \text{ (by definition)}$$
$$= \phi_\gamma - \phi_{fg(\gamma)} \text{ (by (3–3))}$$
$$= \phi_\gamma - \phi_{g(\gamma)} + \phi_{g(\gamma)} - \phi_{fg(\gamma)}$$
$$= \phi_\gamma \circ g - \phi_\gamma + \phi_{g(\gamma)} \circ f - \phi_{g(\gamma)} \text{ (by (3–3))}$$
$$= \varphi_v(g) + \varphi_v(f) \text{ (by definition and well-definedness)}.$$

This establishes $\varphi_v \colon \mathrm{PMap}(S) \to \mathbb{Z}$ as a homomorphism.

To see (1), let $\bar{f} \in \overline{\mathrm{PMap}_c(S)}$, let $c \in \mathcal{C}_v(S)$, and let $f \in \mathrm{PMap}_c(S)$ such that $f(c) = \bar{f}(c)$. Choose a compact surface R containing c and the support of f such that each component of $\partial R \smallsetminus \partial S$ is separating. It must be that $\mathfrak{g}_R(f(c)) = \mathfrak{g}_R(c)$. In particular, if we orient c to represent v, then

$$0 = \phi_c(f(c)) = \phi_c(\bar{f}(c)) = \varphi_v(\bar{f}).$$

(2) can be seen from the definitions but is also a direct consequence of (3–2) and (3–3).

To see (3), first let $h \in \mathrm{PMap}(S)$ be a handle shift cut by v. Let h be supported in the subsurface $\Sigma \subset S$. We can then choose an oriented representative γ of v so that $|\gamma \cap \partial\Sigma| = 2$. In this case $h(\gamma)$ is disjoint from γ and hence they cobound a positive genus subsurface yielding $\varphi_v(h) \neq 0$. Conversely, if v does not cut a handle shift h, then there is a representative γ of v disjoint from the support of h. It follows that $\varphi_v(h) = 0$.

Now to (4): If $v \in H_1^{sep}(S, \mathbb{Z})$ satisfies either $\mathcal{E}_\infty(S) \subset v^+$ or $\mathcal{E}_\infty(S) \subset v^-$, then for every curve $c \in \mathcal{C}_v(S)$ a component of $S \smallsetminus c$ has finite genus, say g_c. Therefore, for every $f \in \mathrm{PMap}(S)$, we have that $S \smallsetminus f(c)$ also has a genus-g_c component. Tracing through the definitions, we see that $\varphi_v(f) = 0$; as f was arbitrary, $\varphi_v = 0$.

Conversely, if v partitions $\mathcal{E}_\infty(S)$ nontrivially, then there exists a handle shift $h \in \mathrm{PMap}(S)$ cut by v; in particular, by (3), $\varphi_v(h) \neq 0$. $\qquad\square$
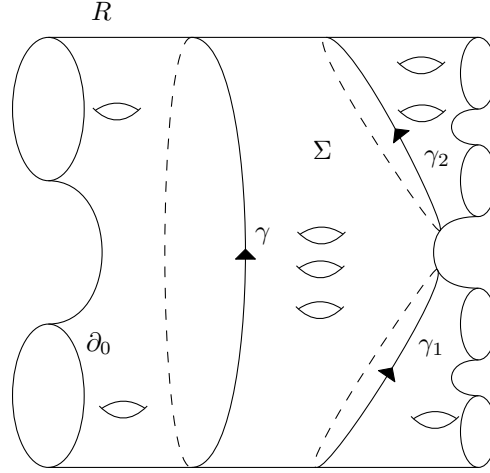
Figure 2: $R, \Sigma, \gamma, \gamma_1, \gamma_2, \partial_0$ as in the proof of Lemma 4.1.

Remark. If S has two ends accumulated by genus and $v \in H_1^{sep}(S, \mathbb{Z})$ is simple and nonzero, then the function $d \colon \mathcal{C}_v(S) \to \mathbb{R}$ defined by $d(a, b) = |\varphi_v(a) - \varphi_v(b)|$ is a metric. Moreover, $\mathcal{C}_v(S)$ with this metric agrees with the metric on the graph $\mathcal{G}(S)$ defined in [7, Section 9]. In [7], this metric space and the corresponding action of $\mathrm{Map}(S)$ is used to show that $\mathrm{Map}(S)$ admits a left-invariant infinite-diameter pseudo-metric.

4   Proofs of Theorems 1 and 3

In Section 3, we constructed an element of $H^1(\mathrm{PMap}(S), \mathbb{Z})$ for each nonzero simple element of $H_1^{sep}(S, \mathbb{Z})$. The remainder of the article is dedicated to showing that these homomorphisms span all of $H^1(\mathrm{PMap}(S), \mathbb{Z})$. To this end, the goal of the next sequence of lemmas is to extend the definition of φ_v to an arbitrary element $v \in H_1^{sep}(S, \mathbb{Z})$ by linearity.

Lemma 4.1 *Let $v_1, \ldots, v_n$ be simple homology classes in $H_1^{sep}(S, \mathbb{Z})$ such that there exist pairwise disjoint oriented simple closed curves $\gamma_1, \ldots, \gamma_n$ so that γ_i represents v_i. If $v = \sum v_i$ is simple, then*

$$\varphi_v = \sum_{i=1}^{n} \varphi_{v_i}.$$

Proof   Let γ be an oriented simple closed curve representing v. By the hypothesis of the lemma, γ can be chosen to be disjoint from γ_i for all $i \in \{1, \ldots, n\}$; furthermore,

$\gamma_1, \ldots, \gamma_n$, and $-\gamma$ cobound a compact surface F. By part (2) of Proposition 3.3, we can assume without loss of generality that F is to the right of γ as in Figure 2.

Now fix an element $f \in \mathrm{PMap}(S)$ and choose a compact surface R such that each component of ∂R is separating and $F \cup f(F) \subset \mathring{R}$. By our choice of orientation, we can choose $e_{v_1}, \ldots, e_{v_n}, e_v \in v^-$ so that there exists a component ∂_0 of ∂R such that ∂_0 separates e_{v_i} from γ_i and e_v from γ for each $i \in \{1, \ldots, n\}$.

Observe that

$$\mathfrak{g}_R(\gamma) = \left(\sum_{i=1}^{n} \mathfrak{g}_R(\gamma_i) \right) - (n-1) \operatorname{genus}(R) - \operatorname{genus}(F)$$

and, similarly,

$$\mathfrak{g}_R(f(\gamma)) = \left(\sum_{i=1}^{n} \mathfrak{g}_R(f(\gamma_i)) \right) - (n-1) \operatorname{genus}(R) - \operatorname{genus}(f(F)).$$

Since $\operatorname{genus}(F) = \operatorname{genus}(f(F))$, we see that

$$\begin{aligned} \varphi_v(f) &= \mathfrak{g}_R(f(\gamma)) - \mathfrak{g}_R(\gamma) \\ &= \left(\sum_{i=1}^{n} \mathfrak{g}_R(f(\gamma_i)) \right) - \left(\sum_{i=1}^{n} \mathfrak{g}_R(\gamma_i) \right) \\ &= \sum_{i=1}^{n} \left(\mathfrak{g}_R(f(\gamma_i)) - \mathfrak{g}_R(\gamma_i) \right) \\ &= \sum_{i=1}^{n} \varphi_{v_i}(f). \end{aligned}$$

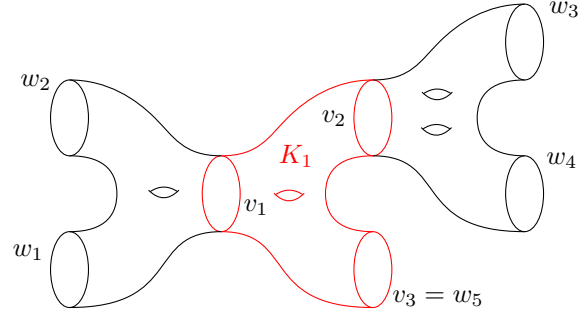As $f \in \mathrm{PMap}(S)$ was arbitrary, the result follows. $\qquad \square$

A *principal exhaustion* of S is a collection of finite-type surfaces $\{K_i\}_{i \in \mathbb{N}}$ satisfying:

(1) $K_i \subset K_j$ whenever $i < j$,

(2) each component of ∂K_i is separating, and

(3) each component of $S \smallsetminus K_i$ is of infinite type.

We record a lemma regarding the structure of $H_1^{sep}(S, \mathbb{Z})$:

Lemma 4.2 *If $\{K_i\}_{i \in \mathbb{N}}$ is a principal exhaustion of S, then*

$$H_1^{sep}(S, \mathbb{Z}) = \varinjlim H_1^{sep}(K_n, \mathbb{Z}).$$

Figure 3: An example of K_1 (red) sitting in K_2 and the corresponding basis elements.

In particular, there exists $n, m \in \mathbb{N}$ such that every nonzero element $v \in H_1^{sep}(S, \mathbb{Z})$ can be written as

$$v = \sum_{k=1}^{m} a_k v_k,$$

where $a_k \in \mathbb{Z}$ and v_k can be represented by a peripheral curve on K_n.

Proof The first statement follows directly from the fact that $S = \varinjlim K_i$. The second follows from the first and basic facts about the homology of finite-type surfaces. □

Now fix a principal exhaustion $\{K_i\}_{i \in \mathbb{N}}$ of S and let $v \in H_1^{sep}(S, \mathbb{Z})$ be arbitrary. Write $v = \sum a_k v_k$ as in Lemma 4.2 and define $\varphi_v \colon \operatorname{PMap}(S) \to \mathbb{Z}$ by

$$\varphi_v = \sum_{k=1}^{m} a_k \varphi_{v_k}.$$

Lemma 4.3 φ_v *is well defined.*

Proof There are two stages to check: (1) φ_v is well defined with respect to a fixed principal exhaustion and (2) φ_v does not depend on the choice of principal exhaustion.

(1) Let $\{K_i\}_{i \in \mathbb{N}}$ be a principal exhaustion of S and fix $n, m \in \mathbb{N}$ such that $m > n$. Let k_n and k_m be the number of boundary components of K_n and K_m, respectively. Choose $v_1, \ldots, v_{k_n-1}$ to be a maximal linearly independent set of simple peripheral elements in $H_1(K_n, \mathbb{Z})$ and let $v_{k_n} = -\sum_{i=1}^{k_n-1} v_i$. There then exist integers $n_i \in \{0, \ldots, k_m\}$ for $i \in \{0, 1, \ldots, k_n\}$ with $n_0 = 0$ and simple peripheral elements $w_1, \ldots, w_{k_m}$ of $H_1(K_m, \mathbb{Z})$ such that

$$v_i = \sum_{j=n_{i-1}+1}^{n_i} w_j.$$

(See Figure 3 where v_1 would be a sum of w_1 and w_2.) In this setup we can apply Lemma 4.1 to see that

$$(4\text{–}1) \qquad\qquad \varphi_{v_i} = \sum_{j=n_{i-1}+1}^{n_i} \varphi_{w_j}.$$

Now if $v \in H_1^{sep}(S, \mathbb{Z})$ such that

$$v = \sum_{i=1}^{k_n} a_i v_i \quad \text{and} \quad v = \sum_{j=1}^{k_m} b_j w_j,$$

then, by applying (4–1) to each basis element,

$$\sum_{i=1}^{k_n} a_i \varphi_{v_i} = \sum_{j=1}^{k_m} b_j \varphi_{w_j}.$$

with $b_j = a_i$ whenever $n_{i-1} < j \le n_i$. This completes (1).

Now let $\{K_i\}_{i \in \mathbb{N}}$ and $\{K'_j\}_{j \in \mathbb{N}}$ be two principal exhaustions of S. Suppose $v \in H_1^{sep}(S, \mathbb{Z})$ such that v can be written in both $H_1^{sep}(K_n, \mathbb{Z})$ and $H_1^{sep}(K'_m, \mathbb{Z})$. Choose $N \in \mathbb{N}$ such that $K'_m, K_n \subset K_N$, then a very similar argument to that of (1) yields the result. $\qquad\square$

Neither the integral homology of S nor the curve graph $\mathcal{C}(S)$ can distinguish between an isolated planar end (i.e. puncture) of S and a boundary component. Furthermore, the subgroup of $\mathrm{PMap}(S)$ generated by Dehn twists about boundary components is contained in $\mathrm{PMap}_c(S)$. Therefore, for notational simplicity, we will assume that S is borderless for the remainder of the section. Only very slight modifications to the remaining proofs are needed to deal with the case in which S has compact boundary.

For an infinite-genus surface S, let $\hat{S}$ be the surface obtained by "forgetting" the planar ends of S, that is,

$$\hat{S} = \bar{S} \smallsetminus \mathcal{E}_\infty(S).$$

Let $\iota \colon S \hookrightarrow \hat{S}$ denote the inclusion and $\iota_* \colon H_1^{sep}(S, \mathbb{Z}) \to H_1^{sep}(\hat{S}, \mathbb{Z})$ the induced homomorphism.

Proposition 4.4 (Definition of $\hat{\Phi}$)   *Let S be an infinite-genus surface. The map*

$$\Phi \colon H_1^{sep}(S, \mathbb{Z}) \to H^1(\mathrm{PMap}(S), \mathbb{Z})$$

given by $\Phi(v) = \varphi_v$ is a homomorphism. Furthermore, Φ factors through the homomorphism $\iota_* \colon H_1^{sep}(S, \mathbb{Z}) \to H_1^{sep}(\hat{S}, \mathbb{Z})$ yielding an monomorphism

$$\hat{\Phi} \colon H_1^{sep}(\hat{S}, \mathbb{Z}) \to H^1(\mathrm{PMap}(S), \mathbb{Z}).$$

Proof Linearity is implicit in the definition of φ_v; the fact that Φ is a homomorphism then follows immediately from Lemma 4.3. By property (4) of Proposition 3.3, $\ker \Phi = \ker \iota_*$; hence, $\hat{\Phi}$ is a well-defined injective homomorphism. $\square$

The remainder of this section is dedicated to proving that $\hat{\Phi}$ is an isomorphism. To start, the combination of the definition of φ_v, Property (1) from Proposition 3.3, and the proof of [18, Theorem 4] yield the following:

Theorem 4.5 *Let S be any surface. If $f \in \mathrm{PMap}(S)$, then $\varphi_v(f) = 0$ for all $v \in H_1^{sep}(S, \mathbb{Z})$ if and only if $f \in \overline{\mathrm{PMap}_c(S)}$.*

We can now prove Theorem 3.

Proof of Theorem 3 Let $A_S = \mathrm{PMap}(S)/\overline{\mathrm{PMap}_c(S)}$ and let $H_S = H_{sep}^1(\hat{S}, \mathbb{Z})$. Let us start with assuming S has no planar ends, i.e. $S = \hat{S}$. Let $r \in \mathbb{N} \cup \{0, \infty\}$ denote the rank of $H_1^{sep}(S, \mathbb{Z})$. Combining Lemma 4.2 with the fact $H_1^{sep}(S, \mathbb{Z})$ is a free abelian group, we see there exists a collection of simple separating homology classes $\{v_i\}_{i=1}^r$ such that

(1) $H_1^{sep}(S, \mathbb{Z}) = \bigoplus_{i=1}^r \langle v_i \rangle$ and

(2)   there exists a pairwise disjoint collection of oriented simple closed curves $\{\gamma_i\}_{i=1}^r$ on S such that γ_i represents v_i for each $i \in \mathbb{N}$.

It follows that

$$H_S = \prod_{i=1}^r \langle \mathrm{pr}_i \rangle,$$

where $\mathrm{pr}_i \colon H_1^{sep}(S, \mathbb{Z}) \to \mathbb{Z}$ is the projection onto the ith basis element, that is,

$$\mathrm{pr}_i(v_j) = \left\{ \begin{array}{ll} 1 & j = i \\ 0 & j \neq i \end{array} \right.$$

We will construct a handle shift h_i that is naturally associated to the projection pr_i. Let S' be the surface obtained from S by removing pairwise disjoint regular neighborhoods of each γ_i. Observe that each separating curve in S' bounds a compact surface: if not, then this would contradict $\{v_i\}_{i=1}^r$ being a basis. It follows that each component of S' is one-ended and has infinite-genus. Every such surface is classified up to homeomorphism by the number of boundary components; for $n \in \mathbb{N} \cup \{\infty\}$, let Z_n be the one-ended infinite-genus surface with n boundary components. We can obtain Z_n by removing n
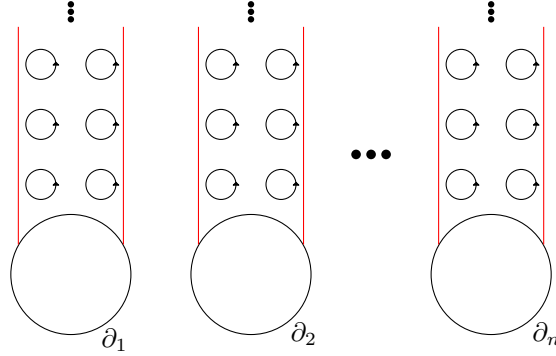
Figure 4: The surface Z_n.

open discs centered along the horizontal axis in $\mathbb{R}^2$ and attaching handles periodically and vertically above each removed disk as shown in Figure 4.

Let Y be the surface obtained from $[0, 1] \times [0, \infty) \subset \mathbb{R}^2$ by attaching handles periodically with respect to the map $Y \hookrightarrow Y$ given by $(x, y) \mapsto (x, y + 1)$. In our construction of Z_n, there are n disjoint embeddings of Y into Z_n shown in Figure 4. In each embedding, there is a unique boundary component of Z_n containing the image of $[0, 1] \times \{0\} \subset Y$.

Fix $i \in \mathbb{N}$ and let X_1 and X_2 be the two components of S' containing boundary components b_1 and b_2, respectively, homotopic to γ_i in S. Let Y_i be the image of Y in X_i intersecting b_i. As b_1 and b_2 bound an annulus in S, we can connect the intervals $Y_1 \cap b_1$ and $Y_2 \cap b_2$ with a strip $T \cong [0, 1] \times [0, 1]$ in the annulus. Now the connected subsurface $\Sigma_i = Y_1 \cup T \cup Y_2$ gives a proper embedding of Σ (see Section 2.3) into S. Let h_i be a handle shift supported in Σ_i such that $\varphi_{v_i}(h_i) = 1$. Observe that γ_j is disjoint from Σ_i whenever $i \neq j$; hence,

(i) $h_i h_j = h_j h_i$

(ii) $\varphi_{v_j}(h_i) = 0$ whenever $i \neq j$.

It follows from (i) that $\prod_{i=1}^{r} \langle h_i \rangle < \mathrm{PMap}(S)$. We can therefore define the monomorphism

$$\kappa \colon H_S \to \mathrm{PMap}(S)$$

by setting $\kappa(\mathrm{pr}_i) = h_i$.

Now if $S \neq \hat{S}$, then we can repeat the above with $\hat{S}$ and by choosing commuting preimages of the h_i in $\mathrm{PMap}(S)$ under the forgetful map $\mathrm{PMap}(S) \to \mathrm{PMap}(\hat{S})$. This is always possible as one can choose the Σ_i above to avoid the planar ends of S.

By (1) of Proposition 3.3 and Proposition 4.4, every homomorphism $\hat{\Phi}(v)$ for $v \in H_1^{sep}(\hat{S}, \mathbb{Z})$ factors through A_S yielding a homomorphism $\bar{\Phi}(v) \colon A_S \to \mathbb{Z}$. Let $h \in \kappa(H_S)$, so we can write $h = \prod h_i^{m_i}$. If h is not the identity, then there exists $i \in \mathbb{N}$ such that $m_i \neq 0$. It follows that $\hat{\Phi}(v_i)(h) = m_i$ and thus $\bar{\Phi}(v_i)(\pi(h)) = m_i$. This implies that $\pi(h) \neq 0$ and as h was arbitrary we can conclude that $\pi \circ \kappa$ is injective.

Given $a \in A_s$, define $a^* \in H_S$ by $a^*(v) = \bar{\Phi}(v)(a)$. It is easy to check that the map $\Psi \colon A_S \to H_S$ given by $\Psi(a) = a^*$ is a homomorphism. Furthermore, if $a^* = 0$, then $\bar{\Phi}(v)(a) = 0$ for all $v \in H_1^{sep}(\hat{S}, \mathbb{Z})$; it follows from Theorem 4.5 that $a = 0$ and Ψ is injective. Now by the definition of h_i and property (2) above, we see that $\pi(h_i)^* = \mathrm{pr}_i$. The injectivity of $\pi \circ \kappa$ now implies that the composition $\pi \circ \kappa \circ \Psi$ is the identity. We can therefore conclude that $\pi \circ \kappa$ is an isomorphism whose inverse is Ψ. □

We can now show that $\hat{\Phi}$ is an isomorphism.

Theorem 4.6 *For any surface S of genus at least 2, the map $\hat{\Phi} \colon H_1^{sep}(\hat{S}, \mathbb{Z}) \to H^1(\mathrm{PMap}(S), \mathbb{Z})$ is an isomorphism.*

Proof By Proposition 3.3, we already know that $\hat{\Phi}$ is injective. It is left to show surjectivity. Let $\varphi \in H^1(\mathrm{PMap}(S), \mathbb{Z})$. If the genus of S is at least 3, then by [19, Theorem 1], $\mathrm{PMap}_c(S)$ is perfect as it is the direct limit of perfect groups; hence, $\varphi(\mathrm{PMap}_c(S)) = 0$. If the genus is 2, then $\overline{\mathrm{PMap}_c(S)}$ is the direct limit of groups whose abelianization is finite and the same argument applies (see [8, Section 5.1.2] or [15, Theorem 5.1]). By Corollary 2.5, we can apply [6, Theorem 1] to see that φ is continuous, which allows us to conclude that $\varphi(\overline{\mathrm{PMap}_c(S)}) = 0$. Therefore, by Corollary 4, we can identify $H^1(\mathrm{PMap}(S), \mathbb{Z})$ with $\mathrm{Hom}(\kappa(H_S), \mathbb{Z})$.

Using the notation from Theorem 3, we write

$$\kappa(H_S) = \prod_{i=1}^{r} \langle h_i \rangle,$$

where r is the rank of $H_1^{sep}(\hat{S}, \mathbb{Z})$ and the h_i are handle shifts. Recall from the proof of Theorem 3 that there exists a basis $\{v_i\}_{i=1}^{r}$ for $H_1^{sep}(\hat{S}, \mathbb{Z})$ such that

$$\hat{\Phi}(v_i)(h_j) = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

Applying standard properties of Hom in the finite-rank case or an old result of Specker [23] (see also [9, Lemma 94.1]) in the case of infinite rank, we see that

$$\mathrm{Hom}(\kappa(H_S), \mathbb{Z}) = \bigoplus_{i=1}^{r} \langle \hat{\Phi}(v_i) \rangle.$$

This implies that $\hat{\Phi}$ is surjective and hence an isomorphism. □

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