Variations on Belyi's theorem

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Dedicated to the memory of José Manuel Souto who taught me the algebra I needed to write this article

Abstract

We establish a simple criterion for a variety to be defined over a number field, which allows us to extend to higher dimension several known results relative to the field of definition of complex curves as well as providing some new results in dimension one.

1 Introduction

Let C be a compact Riemann surface, that is a complex algebraic curve. The, by now well known, theorem of Belyi states that C can be defined over a number field if and only if there is a meromorphic function $f : C \to \mathbb{P}^1$ with three critical values, say $0, 1, \infty$ ([2]). Such functions (resp. Riemann surfaces) are often called *Belyi functions* (resp. *Belyi surfaces)*. Belyi's theorem has attracted much attention ever since Grothendieck noticed in his *Esquisse d'un Programme* (see [8]) that it implies amazing inter-relations between algebraic curves defined over number fields and a certain class of graphs embedded in a topological surface, which he named *dessins d'enfants*.

While the "only if" part of Belyi's theorem results from a surprisingly simple construction (in Grothendieck's words "*jamais sans doute un résultat profond et déroutant ne fut démontré en si peu de lignes!*", [8], $\frac{14}{15}$), for the "if part" Belyi invokes Weil's criterion in [26]. This criterion, however powerful, is not always easy to apply (at least by us, the non experts) since, as often occurs in practice, it is difficult to check if its hypotheses are satisfied in a given problem (cf. [27]). On the contrary, our criterion (Criterion 1) is easy to handle. Although it is much less ambitious, in that it only attempts to determine whether or not a given variety can be defined over a number field without specifying which one. It goes as follows

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1) *An irreducible complex projective variety X can be defined over a number field if and only if the family of all its conjugates X^σ, where σ is any field automorphism of $\mathbb{C}$, contains only finitely many isomorphism classes of complex projective varieties.*

The proof of Criterion 1 is based on the following key result (Theorem 12).

2) *If an irreducible complex projective variety X can be defined over two subfields of $\mathbb{C}$ algebraically disjoint over $\mathbb{Q}$ then X can in fact be defined over $\overline{\mathbb{Q}}$.*

A similar criterion holds for the field of definition of a morphism (Criterion 2).

Based on these two criteria we prove the following results.

3) *If C_1 and C_2 are curves defined over $\overline{\mathbb{Q}}$ and the genus of C_2 is greater or equal 2, then any morphism $f : C_1 \to C_2$ is necessarily defined over $\overline{\mathbb{Q}}$ too* (Proposition 20). The case in which $C_1 = C_2$, the automorphism case, is proved by different means in [15]. We also *extend this result to automorphisms of varieties of general type of arbitrary dimension* (Corollary 17).

4) *The holomorphic image of a Belyi surface is also a Belyi surface* (Theorem 21). Similarly, *if Y is a n-dimensional variety of general type which is the image of a n-dimensional variety X defined over $\overline{\mathbb{Q}}$ then Y can also be defined over $\overline{\mathbb{Q}}$* (Proposition 15).

5) *A Riemann surface C is a Belyi surface if and only if it can be uniformized by some Fuchsian group commensurable with the classical modular group $PSL_2(\mathbb{Z})$* (Theorem 25). The corresponding result for subgroups instead of commensurable groups appears first in the article [1] by Cohen, Itzykson and Wolfart (see also [10]), and perhaps goes back to Shabat-Voevodsky ([23]) and Grothendieck ([8]) himself.

We note that Criterion 1 can also be applied to give an elementary proof of the "if part" of Belyi's theorem (Theorem 19), different from those in [15] and [27], which requires only Hilbert's Nullstellensatz and some standard facts of algebraic field theory. In particular neither Grothendieck's notion of scheme nor Weil's concept of generic point is needed. Such a simple version of Belyi's theorem may be of independent interest in view of the growing attention being devoted to this subject by researchers from different areas of mathematics.

In conclusion we point out that, in the forthcoming paper ([7]) Criterion 1 will allow us to provide a Belyi type theorem for complex surfaces. It will turn out that in that case the role of Belyi functions is played by *composed Lefschetz pencils* with three critical values.

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2 A criterion for a variety to be defined over $\overline{\mathbb{Q}}$

Let $X \subset \mathbb{P}^n(\mathbb{C})$ be a projective variety. We shall say that X *is defined over a field* $k \subset \mathbb{C}$ if there is a finite collection of homogeneous polynomials with coefficients in k

$$\left\{ P_\alpha(X_0, ..., X_n) = \sum \alpha_\nu X_0^{\nu_0} ... X_n^{\nu_n} \right\}_\alpha$$

whose zero set $Z(P_\alpha)$ is precisely X.
We shall say that X *can be defined over* k if it is isomorphic to a variety defined over k. We are primarily interested in the question of whether a given variety X can be defined over a *number field*, or equivalently, over $\overline{\mathbb{Q}}$, the field of algebraic numbers.

For a subfield k of $\mathbb{C}$ we denote by $\overline{k}$ the algebraic closure of k in $\mathbb{C}$ and by Gal$(\mathbb{C}/k)$ the group of all field automorphisms of $\mathbb{C}$ which fix the elements in k. For $k = \mathbb{Q}$ we simply write Gal$(\mathbb{C}/\mathbb{Q})$= Gal$(\mathbb{C})$.

For given $\sigma \in$ Gal$(\mathbb{C})$ and $a \in \mathbb{C}$, we shall write a^σ instead of $\sigma(a)$. We shall employ the same rule to denote the obvious action induced by σ on the projective space $\mathbb{P}^n(\mathbb{C})$, the ring of polynomials $\mathbb{C}[X_0, .., X_n]$, etc. Namely, for a point $x = (x_0 : ... : x_n) \in \mathbb{P}^n(\mathbb{C})$ we put $x^\sigma = (x_0^\sigma : ... : x_n^\sigma)$, for a subset $U \subset \mathbb{P}^n(\mathbb{C})$ we write $U^\sigma = \{x^\sigma : x \in U\}$. For a polynomial $P = \sum a_\nu X_0^{\nu_0} ... X_n^{\nu_n}$ we put $P^\sigma = \sum a_\nu^\sigma X_0^{\nu_0} ... X_n^{\nu_n}$, etc. It follows that $X^\sigma = Z(P_\alpha^\sigma)$, hence if X is defined over K, then X^σ is defined over K^σ. We also see that if U is a Zariski open set of X, the set U^σ will be a Zariski open set of X^σ.

For a map $f : X \to Y$ from X to a second projective variety $Y \subset \mathbb{P}^r(\mathbb{C})$ we define $f^\sigma : X^\sigma \to Y^\sigma$ to be the map $f^\sigma = \sigma \circ f \circ \sigma^{-1}$. We see that if $f : X \to Y$ is a morphism of projective varieties with local expression

$$f_{|U} \equiv (F_0, ..., F_r)$$

for some homogeneous polynomials F_k, then $f^\sigma : X^\sigma \to Y^\sigma$ is a morphism locally defined by $f_{|U^\sigma} \equiv (F_0^\sigma, ..., F_r^\sigma)$ and that if f happens to be an isomorphism so will be f^σ.

We can now state our criterion for a complex variety to be defined over $\overline{k}$.

Criterion 1 *Let X be an irreducible complex projective variety and k a countable subfield of $\mathbb{C}$. The following conditions are equivalent*
i) X can be defined over $\overline{k}$.
ii) The family $\{X^\sigma\}_{\sigma \in Gal(\mathbb{C}/k)}$ contains only finitely many isomorphism classes of complex projective varieties.
iii) The family $\{X^\sigma\}_{\sigma \in Gal(\mathbb{C}/k)}$ contains only countably many isomorphism classes of complex projective varieties.

The proof of this criterion will be given in section 2.3.

We will simultaneously prove a similar criterion for the field of definition of a morphism. Let $f \in$ Mor(X, Y) be a morphism between irreducible projective varieties X and Y both defined over k. We shall say that f *is defined over* k if it can be described by a finite collection of local expressions $f_{|U} \equiv (F_0, ..., F_r)$

with each F_k having coefficients in k, whereas we shall say that f *can be defined over* k if it is equivalent to a morphism $f_0 : X \to Y$ defined over k; that is, if there are automorphisms $h_1 : X \backsimeq X$, $h_2 : Y \backsimeq Y$ such that the following diagram commutes

$$\begin{array}{ccc} X & \stackrel{f}{\to} & Y \\ \downarrow h_1 & & \downarrow h_2 \\ X & \stackrel{f_0}{\to} & Y \end{array}$$

We have

Criterion 2 *Let $f : X \to Y$ be a morphism between irreducible projective varieties both defined over a countable subfield k of $\mathbb{C}$. The following conditions are equivalent*
i) f can be defined over $\overline{k}$.
ii) The family $\{f^\sigma\}_{\sigma \in Gal(\mathbb{C}/k)}$ contains only finitely many equivalence classes of morphisms.
iii) The family $\{f^\sigma\}_{\sigma \in Gal(\mathbb{C}/k)}$ contains only countably many equivalence classes of morphisms.

2.1 Specialization of k-algebras

We now recall some basic facts of the theory of transcendental field extensions that can be found e.g. in [3] or [16]. Let K be an extension field of a field k. A subset $\{\pi_1, ..., \pi_d\}$ of K is said to be *algebraically independent over* k if the *evaluation* map $k[X_1, .., X_d] \to K$ which sends a polynomial $a(X_1, ., ., X_d)$ to its value at $(\pi_1, .., \pi_d)$, $a(\pi_1, .., \pi_d)$ is injective; in other words, if it induces an isomorphism between $k[X_1, .., X_d]$ and its image, usually denoted by $k[\pi_1, ..., \pi_d]$. Because of this, given an arbitrary field extension $k \subset L$ and any d-tuple $(q_1, ..., q_d)$ of elements in L, the rule that sends $a(\pi_1, ..., \pi_d)$ to $a(q_1, ..., q_d)$ provides a well defined homomorphism of k-algebras $S : k[\pi_1, ..., \pi_d] \to L$. In fact, it is obvious that S extends to the subring $k[\pi_1, ..., \pi_d]_S$ of K consisting of all fractions $\frac{a_1(\pi_1,...,\pi_d)}{a_2(\pi_1,...,\pi_d)}$ with $a_2(q_1, ..., q_d) \neq 0$.
In what follows, for an element $a \in k[\pi_1, ..., \pi_d]_S$ we shall write $a^S = S(a) = a(q_1, ..., q_d)$ and so for a polynomial $q(T) := \sum q_l T^l \in k[\pi_1, ..., \pi_d]_S[T]$ we put $q^S(T) := \sum q_l^S T^l$.
An algebraically independent subset $\{\pi_1, .., \pi_d\} \subset K$ is called a *transcendence basis* of K over k if K is algebraic over $k(\pi_1, ..., \pi_d)$. Two extensions K_1 and K_2 of k, both assumed to be subfields of a common field, are said to be *algebraically disjoint* over k if for any pair of algebraically independent subsets over k, A_1 of K_1 and A_2 of K_2, $A_1 \cap A_2$ is empty and $A_1 \cup A_2$ is algebraically independent over k.
For the problem we are dealing with here, we can restrict ourselves to the case in which $L = \mathbb{C}$ and $k \subset K$ is a finitely generated extension of subfields of $\mathbb{C}$. In this situation, the Primitive Element Theorem implies that K is necessarily of the form $K = k(\pi_1, ..., \pi_d; u)$ where $u \in K$ is algebraic over $k(\pi_1, ..., \pi_d)$ and $\pi_1, ..., \pi_d$ are algebraically independent over k.

Definition 3 *By a standard set of generators of K over k we shall mean a set of generators $\{\pi_1, ..., \pi_d; u\}$ such that $\{\pi_1, ..., \pi_d\}$ is a transcendence basis.*
A specialization of $K; \pi_1, ..., \pi_d; u$ over k is a homomorphism of k-algebras $S : k[\pi_1, ..., \pi_d, u] \to \mathbb{C}$ such that if $m(T)$ is the irreducible polynomial of u over $k(\pi_1, ..., \pi_d)$ (with leading coefficient 1), then $m(T) \in k[\pi_1, ..., \pi_d]_S [T]$.
If $S(\pi_i) = q_i$ and $S(u) = b$, we denote S by $S^{(\pi_1, ..., \pi_d; u)}_{(q_1, ..., q_d; b)}$, or more simply by $S^{(\pi_i; u)}_{(q_i; b)}$. Similarly, for any elements $q_1, ..., q_d \in \mathbb{C}$ we write $S = S^{(\pi_i)}_{(q_i)}$ for the homomorphism S from $k[\pi_1, ..., \pi_d]$ to $\mathbb{C}$ given by $S(\pi_i) = q_i$.

Proposition 4 *Let $\{\pi_1, ..., \pi_d; u\}$ be a given standard set of generators of K over k, and $a = \sum a_i u^i$, with $a_i \in k(\pi_1, ..., \pi_d)$, a given element of K. Then there exists $\varepsilon > 0$ such that for any d-tuple of complex numbers $(q_i)_i$ with $|\pi_i - q_i| \leq \varepsilon$ the homomorphism $S = S^{(\pi_i)}_{(q_i)}$ satisfies the following two conditions*

i) The elements a_i as well as the coefficients of the irreducible polynomial $m(T)$ of Definition 3 lie all in $k[\pi_1, ..., \pi_d]_S$.

ii) For each root u_i of $m(T)$ there is a root b_i of $m^S(T)$ which is closer to u_i than any other root of $m^S(T)$.

Moreover, for such a homomorphism S the following property holds

iii) There is a bijective correspondence between roots b of $m^S(T)$ and specializations $S_b = S^{(\pi_i; u)}_{(q_i; b)}$ extending S to $k[\pi_1, ..., \pi_d, u, a]$.

Proof. i) Let $\{d_\beta(\pi_1, ..., \pi_d) \neq 0\}_\beta$, with $d_\beta(X_1, ..., X_d) \in k[X_1, ..., X_d]$ the finite set of denominators occurring in the elements a_i as well as in the coefficients of $m(T)$. Let (q_i) be a d-tuple of complex numbers and let $\varepsilon = \max|\pi_i - q_i|$. It is clear that if ε is sufficiently small we will still have $d_\beta(q_1, ..., q_d) \neq 0$ (in fact the set of d-tuples satisfying this property forms a Zariski open subset of $\mathbb{C}^d$ containing $(\pi_1, ..., \pi_d)$).
ii) This is a consequence of the continuous dependence of the zeros of a polynomial on its coefficients. Let $\{u_i\}$ and $\{b_i\}$ be the roots of $m(T)$ and $m^S(T)$ respectively. As ε tends to zero the polynomial $m^S(T)$ will come close to $m(T)$. It follows (see [16], 12.7) that if ε is sufficiently small, for each u_i there will be a root of $m^S(T)$, say b_i, such that the distance from u_i to b_i is as small as wanted. If, in particular, we require this distance to be less than $\frac{1}{2}\min|u_j - u_i|$ then the triangle inequality will imply that b_i is the only root of $m^S(T)$ satisfying this property.
iii) Since $m^S(T)$ is well defined, any extension of S to $k[\pi_1, ..., \pi_d, u]$ must, indeed, send u to one of its roots.
In order to show that for each root b of $m^S(T)$ one can define an extension S_b of S such that $S_b(u) = b$, let us begin by considering the following obvious isomorphisms produced by sending T to u

$$\frac{k(\pi_1, \ldots, \pi_d)[T]}{(m(T))} \simeq k(\pi_1, \ldots, \pi_d; u)$$

and

$$\frac{k\left[\pi_1,...,\pi_d\right]\left[T\right]}{(m(T))\cap k\left[\pi_1,...,\pi_d\right]\left[T\right]}\simeq k\left[\pi_1,...,\pi_d,u\right]$$

Now we claim that if $b\in\mathbb{C}$ is any root of $m^S(T)$, the extension of $S:k\left[\pi_1,...,\pi_d\right]\to\mathbb{C}$ to $k\left[\pi_1,...,\pi_d\right]\left[T\right]$ obtained by sending T to b factorizes through $k\left[\pi_1,...,\pi_d,u\right]$, thereby providing the desired extension S_b. This is so because its kernel contains the ideal $(m(T))\cap k\left[\pi_1,...,\pi_d\right]\left[T\right]$. (Here we use Gauss lemma, see e.g. [16], to ensure that if $p(T)\in k(\pi_1,..,\pi_d)[T]$ is such that $q(T)=p(T)m(T)\in k\left[\pi_1,...,\pi_d\right]\left[T\right]$ then $p(T)$ lies, in fact, in $k\left[\pi_1,...,\pi_d\right]_S\left[T\right]$ and so $q^S(b)=p^S(b)m^S(b)=0$).
Finally, we note that the requirements in i) relative to the element $a\in K$ mean that $a\in k\left[\pi_1,...,\pi_d\right]_S\left[u\right]$, thus $S_b(a)=\sum a_i^S b^i$ is well defined. ■

The root b_i of $m^S(T)$ corresponding to the root u_i of $m(T)$ in part ii) of the previous proposition is sometimes referred to as *the root that belongs to* u_i ([16], 12.7). We are, of course, interested in the root that belongs to u. This one we shall denote by $b(u)$. Maintaining the rest of our notation, we make the following

Definition 5 *Let $\{\pi_1,...,\pi_d;u\}$ be a standard set of generators of K over k and Σ a finite subset of K. We shall say that $\varepsilon>0$ is good for $\{\pi_1,...,\pi_d;u\}$ and Σ if for each $a\in\Sigma$ the positive number ε satisfies the three conditions in Proposition 4. If Σ is the empty set we will simply say that ε is good for $\{\pi_1,...,\pi_d;u\}$.
Let $F=\sum a_\nu X_0^{\nu_0}...X_n^{\nu_n}$ be a polynomial with coefficients in K. We shall say that ε is good for $\{\pi_1,...,\pi_d;u\}$ and F if it is good for $\{\pi_1,...,\pi_d;u\}$ and the set of the coefficients of F.*

In what follows, given a specialization $S=S_{(q_i;b)}^{(\pi_i;u)}$ of $K;\pi_1,...,\pi_d;u$ over k, we will write

$$\delta_S=\max_i|\pi_i-q_i|$$

Thus, if ε is good for $\{\pi_1,...,\pi_d;u\}$ and a set $\Sigma=\{a_\alpha\}$, any specialization $S=S_{(q_i;b(u))}^{(\pi_i;u)}$ with $\delta_S<\varepsilon$ extends to the ring $k\left[\pi_1,...,\pi_d,u,\{a_\alpha\}\right]$. In particular, if ε is good for $\{\pi_1,...,\pi_d;u\}$ and a polynomial $F=\sum a_\nu X_0^{\nu_0}...X_n^{\nu_n}$ with coefficients in K, we are allowed to write $F^S=\sum a_\nu^S X_0^{\nu_0}...X_n^{\nu_n}$.

Lemma 6 *Let K_3 be a finitely generated extension field of k, and K_1 and K_2 two subextensions algebraically disjoint over k. Let us choose a standard set of generators of K_1 over k*

$$\pi_1,...,\pi_d;u_1$$

and a standard set of generators of K_3 over k

$$\pi_1,\ldots\pi_d,\pi_{d+1},\ldots,\pi_s,\pi_{s+1},\ldots,\pi_l;u_3$$

where $\{\pi_{d+1},...,\pi_s\}$ is a transcendence basis of K_2 over k. Finally, let Σ_3 be a finite subset of K_3. Then there exists $\varepsilon>0$ good for $\{\pi_1,...,\pi_d;u_1\}$ and *at the same time good for $\{\pi_1, ..., \pi_l; u_3\}$ and Σ_3, such that any specialization $S_1 = S^{(\pi_i;u_1)}_{(q_i;b(u_1))}$ of $K_1; \pi_1, ..., \pi_d; u_1$ over k with $\delta_{S_1} \leq \varepsilon$ extends to a specialization S_3 of $K_3; \pi_1, ..., \pi_l; u_3$ over k with $\delta_{S_3} \leq \varepsilon$ such that $S_3(K_2 \cap \Sigma_3) \subset \overline{k}$.*

Proof. Let $u_2 \in K_2$ such that $K_2 = k(\pi_{d+1}, ..., \pi_s; u_2)$ and let $\varepsilon_2 > 0$ be good for $\{\pi_{d+1}, ..., \pi_s; u_2\}$. Let $\varepsilon_1 > 0$ be good for $\{\pi_1, ..., \pi_d; u_1\}$ and $\varepsilon_3 > 0$ be good for $\{\pi_1, ..., \pi_l; u_3\}$ and the set $\Sigma_3 \cup \{u_1, u_2\}$. Set $\varepsilon' = \min \varepsilon_i, i = 1, 2, 3$ and assume that $\delta_{S_1} \leq \varepsilon'$. We are aiming for an extension of S_1.

Well, let $q_{d+1}, ..., q_l \in \overline{k}$, with $|\pi_i - q_i| < \varepsilon'$. Then $S_2 = S^{(\pi_i;u_2)}_{(q_i;b(u_2))}$ and $S_3 = S^{(\pi_i;u_3)}_{(q_i;b(u_3))}$ are specializations of $(K_2; \pi_{d+1}, ..., \pi_s; u_2)$ and $(K_3; \pi_1, ..., \pi_l; u_3)$ over k such that $\delta_{S_1}, \delta_{S_3} \leq \varepsilon'$. It is clear that the restriction of S_3 to $k[\pi_1, ..., \pi_d]$ (resp. $k[\pi_{d+1}, ..., \pi_s]$) agrees with that of S_1 (resp. S_2). Therefore, if we denote by $m_1(T)$ and $m_2(T)$ the irreducible polynomials of u_1 over $k(\pi_1, ..., \pi_d)$ and u_2 over $k(\pi_{d+1}, ..., \pi_s)$, we have that $S_3(u_k)$ must be a root of $m_k^{S_3}(T) = m_k^{S_k}(T)$ for $k = 1, 2$. Let us now write $u_1 = \sum c_i(\pi_1, ..., \pi_l) u_3^i$. It is clear that by taking ε' is sufficiently small, $S_3(c_i)$ (resp. $S_3(u_3) = b(u_3)$) can be made to be as close to c_i (resp. u_3) as wanted, hence $S_3(u_1)$ can be made to be as close to u_1 as wanted. In particular, closer to u_1 than any other root of $m_1^{S_1}(T)$ (see part ii of Proposition 4), hence $S_3(u_1) = S_1(u_1) = b(u_1)$. This proves the existence of ε such that if $\delta_{S_3} \leq \varepsilon$ then S_3 is an extension of S_1.

Now, by construction, we obviously have $S_3(K_2 \cap \Sigma_3) \subset \overline{k}(S_3(u_2))$, so it only remains to be seen that $S_3(u_2)$ is algebraic over k. But this is clear since, again by construction, $m_2^{S_2}(T)$ is a polynomial over $\overline{k}$. ■

2.2 Specialization of morphisms of projective varieties

In this section $X = Z(P_\alpha)$ and $Y = Z(Q_\beta)$ will be irreducible projective subvarieties of $\mathbb{P}^n(\mathbb{C})$ and $\mathbb{P}^r(\mathbb{C})$ respectively. We would like to have a purely algebraic formulation of the concept of *morphism* or *regular mapping* between X and Y.

By a *homogeneous* $(r+1)$-tuple of polynomials we shall mean a $(r+1)$-tuple of homogeneous polynomials $(F_0, ..., F_r)$ all of which have the same degree. Thus, a homogeneous $(r+1)$-tuple defines a map from a Zariski open set U of X to $\mathbb{P}^r(\mathbb{C})$. In fact we can take $U = \cup_k D(F_k)$ with $D(F_k) = \{x \in X : F_k(x) \neq 0\}$.

Proposition 7 *Defining a morphism $f : X \to Y$ is equivalent to specifying a finite collection of homogeneous $(r+1)$-tuples of polynomials $\{(F_{k,0}, ..., F_{k,r})\}_k$ satisfying the following three conditions*
i)

$$\cup_{j,k} D(F_{k,j}) = X \tag{1}$$

ii) For each of these $(r+1)$-tuples $(F_{k,0}, ..., F_{k,r})$ there are polynomials $W_{\alpha,\beta} \in \mathbb{C}[X_0, .., X_n]$ and a positive integer q such that the following identity holds

$$(Q_\beta (F_{k,0}(X_0, .., X_n), ..., F_{k,r}(X_0, .., X_n))^q = \sum W_{\alpha,\beta} P_\alpha \tag{2}$$

iii) Any pair of these $(r+1)$-tuples $(F_{k,0},...,F_{k,r})$ and $(F_{l,0},...,F_{l,r})$ is compatible in the sense that for each pair of indices $0 \le i,j \le r$ there are polynomials T_α and a positive integer p such that the following identity holds

$$(F_{k,i}F_{l,j} - F_{k,j}F_{l,i})^p = \sum T_\alpha P_\alpha \tag{3}$$

The morphism $f : X \to Y$ is then defined by

$$f(x) = (F_{k,0}(x),...,F_{k,r}(x)), \text{ if } x \in \bigcup_j D(F_{k,j}) \tag{4}$$

Furthermore, two such collections $\{(F_{k,0},...,F_{k,r})\}_k$ and $\{(G_{l,0},...,G_{l,r})\}_l$ define the same morphism if its union defines a morphism.

Proof. Clearly, the map given by (4) is a regular mapping. Identity (1) precisely means that f is defined in the whole set X, identity (2) expresses the fact that the image $f(\cup_j D(F_{k,j})) \subset \mathbb{P}^r(\mathbb{C})$ lies in fact in Y, and identity (3) shows that f is coherently defined in the intersection of $\cup_j D(F_{k,j})$ and $\cup_j D(F_{l,j})$.

Conversely, recall (see e.g. [21], I.4.2) that a map $f : X \to Y \subset \mathbb{P}^r(\mathbb{C})$ is called a morphism if there is an open cover $\{U_k\}$ of X such that

$$f_{|U_k} \equiv (F_{k,0},...,F_{k,r})$$

for some of homogeneous $(r+1)$-tuple of polynomials $F_{k,j}$. By the *Identity theorem for holomorphic functions* f is given by this expression on the whole Zariski open set $\cup_j D(F_{k,j})$. As $U_k \subset \cup_j D(F_{k,j})$ our collection of $(r+1)$-tuples satisfy condition i). Now, since the target variety Y is defined by the polynomials $\{Q_\beta\}$, we must have $Q_\beta(F_0(x),...,F_r(x)) = 0$ for all x in $\cup_j D(F_{k,j})$, hence for all $x \in X$. By Hilbert's Nullstellensatz this occurs if and only if the polynomial $Q_\beta(F_{k,0}(X_0,..,X_n),...,F_{k,r}(X_0,..,X_n))$ belongs to the *radical* of the *ideal* generated by $\{P_\alpha\}$; this is what condition ii) asserts. Moreover, at any point $x \in U_k \cap U_l$ we must have $(F_{k,0}(x),...,F_{k,r}(x)) = (F_{l,0}(x),...,F_{l,r}(x))$ which in turn implies that for any pair of indices i,j the identity $F_{k,j}(x)/F_{k,i}(x) = F_{l,j}(x)/F_{l,i}(x)$ holds on $D(F_{k,i}) \cap D(F_{l,i})$, hence $F_{k,i}F_{l,j} - F_{k,j}F_{l,i}$ vanishes on the whole X. But again by Hilbert's Nullstellensatz, this is the same as identity (3). Finally, it is clear that two morphisms f and g coincide if their corresponding collections of $(r+1)$-tuples are compatible with each other, that is if the larger collection of $(r+1)$-tuples resulting as the union of these two collections satisfy condition iii). ■

We shall write $f \equiv \{(F_{k,0},...,F_{k,r})\}$. We observe that even if X or Y are not irreducible, a collection of $(r+1)$-tuples enjoying the three properties above will still define a morphism. Each of the defining $(r+1)$-tuples will be called a *local expression for f*. The sets $\cup_j D(F_{k,j})$ and $f(\cup_j D(F_{k,j}))$ are called, respectively, *the domain* of definition and the *image* of the local expression $(F_{k,0},...,F_{k,r})$.

Corollary 8 *Let $f : X \to Y$ be as before and $g : Y \to Z$ and $h : X \to Z$ be morphisms to a third irreducible variety $Z \subset \mathbb{P}^m(\mathbb{C})$. Then $h = g \circ f$ if* *and only if for any triple of local expressions* $(F_0,...,F_r)$ *for* f*,* $(G_0,...,G_m)$ *for* g *and* $(H_0,...,H_m)$ *for* h *such that the image of the first one has non-empty intersection with the domain of the second one and for each pair of indices* $0 \le i,j \le r$ *there exist a positive integer* l *and polynomials* D_α *such that the following identity holds*

$$(H_j G_i(F_0,...,F_r) - H_i G_j(F_0,...,F_r))^l = \sum D_\alpha P_\alpha,\ 0 \le i,j \le m \qquad (5)$$

Proof. Clearly the morphism $g \circ f$ is defined by the collection of local expressions of the form $(G_0(F_0,...,F_r),...,G_m(F_0,...,F_r))$ where $(F_0,...,F_r)$ and $(G_0,...,G_m)$ are as in the statement. Now observe that identity (5) is a particular case of identity (3); it merely expresses the compatibility of this collection and that defining h. ■

Corollary 9 *A morphism* $f \equiv \{(F_{k,0},...,F_{k,r})\}_k$ *between two irreducible varieties* X *and* Y *is an isomorphism if and only if there is a collection of homogeneous* $(n+1)$*-tuples* $\{(G_{s,0},...,G_{s,n})\}_s$*, with* $G_{s,u} \in \mathbb{C}[Y_0,..,Y_r]$*, defining a morphism* $g : Y \to X$ *(which is going to be its inverse), so that for any indices* k,s *and any pair of indices* $0 \le i,j \le r$ *and* $0 \le u,v \le n$*, there are positive integer* s,t *and polynomials* M_α *and* M_β *such that the following identities hold*

$$(X_j G_{s,u}(F_{k,0},...,F_{k,r}) - X_i G_{s,v}(F_{k,0},...,F_{k,r}))^s = \sum M_\alpha P_\alpha \qquad (6)$$

and

$$(Y_j F_{k,i}(G_{s,0},...,G_{s,n}) - Y_i F_{k,j}(G_{s,0},...,G_{s,n}))^t = \sum M_\beta Q_\beta \qquad (7)$$

Proof. Relations (6) and (7) are a particular case of relation (5) in which we have made $H_j = X_j$ and $H_j = Y_j$ respectively. Thus, they express the identities $id = g \circ f$ and $id = f \circ g$ respectively. ■

We are now in position of specializing morphisms

Proposition 10 *Let* $k \subset K$ *be a finitely generated extension of subfields of* $\mathbb{C}$ *with standard generators* $\{\pi_1,...,\pi_d;u\}$*.*

Let $f : X = Z(P_\alpha) \to Y = Z(Q_\beta)$ *with* $f \equiv \{(F_{k,0},...,F_{k,r})\}_k$ *be a morphism between irreducible projective varieties. Assume that* $P_\alpha, Q_\beta, F_{k,i}$ *are all defined over* K*. Let* $\varepsilon > 0$ *be good for the given generators and all these polynomials, and let* S *be a specialization of* $K;\pi_1,...,\pi_d;u$ *over* k *with* $\delta_S < \varepsilon$*. Let us consider the varieties* $X^S = Z(P_\alpha^S)$*,* $Y^S = Z(Q_\beta^S)$ *and the collection of homogeneous* $(r+1)$*-tuples* $\{(F_{k,0}^S,...,F_{k,r}^S)\}_k$*. Then, if* ε *is sufficiently small, this collection defines a morphism* $f^S : X^S \to Y^S$*. Moreover, if* $f : X \to Y$ *is an isomorphism,* ε *can be chosen so that* f^S *is an isomorphism too.*

Proof. In order to prove that $f^S = \{(F_{k,0}^S,...,F_{k,r}^S)\}_k$ defines a morphism between X^S and Y^S it is enough to check conditions i), ii) and iii) of Proposition 7. To deal with ii) (resp. iii)) we first consider the larger field K' obtained

by adding to K the set Σ of coefficients of the polynomials $W_{\alpha\beta}$ (resp. T_α) occurring in the identities of type (2) (resp. (3)) existing among the various local expressions for f. Let us now apply Lemma 6 with $K_1 = K$, $K_2 = k$, $K_3 = K'$, $\Sigma_3 = \Sigma$ and generators of K_3 and K_1 over k as required in the statement of this lemma. We then infer that if δ_S is sufficiently small there is a specialization S_3 extending S which can be applied to identities (2) and (3) to show that conditions ii) and iii) are satisfied. As for the equality $\cup_{j,k} D(F^S_{k,j}) = Z(P^S_\alpha)$ of which condition i) consists of, it will also be fulfilled provided δ_S is sufficiently small. Indeed, suppose there is a sequence of parameters $(q_{1,m}, ..., q_{d,m})$ converging to $(\pi_1, ..., \pi_d)$ and points $x_m \in Z(P^{S_m}_\alpha) \subset \mathbb{P}^n$ at which all polynomials $F^{S_m}_{k,j}$ vanish (Here $S_m = S^{(\pi_i;u)}_{(q_{i,m};b(u))}$). Then, by continuity, any accumulation point x would lie in X but, at the same time, it would be a zero of all polynomials $F_{k,j}$ contradicting the fact that $\cup_{j,k} D(F_{k,j}) = X$. Thus, f^S *is* a morphism.
If, moreover, f is an isomorphism, we take as K_3 the field obtained by adding to K' the coefficients of the polynomials defining f^{-1} as well as those occurring in the corresponding identities (6) and (7). Still the same argument proves that if δ_S is sufficiently small f^S must be an isomorphism too. ∎

Obviously, the varieties X^S, Y^S and the morphism f^S introduced in Proposition 10 depend on the choice of the defining polynomials. However, we have the following uniqueness result

Proposition 11 *Let the notation be as in Proposition 10, then the following properties hold*

a) If X is given by another collection of polynomials P'_α defined over K, then ε can be chosen sufficiently small so that $Z(P^S_\alpha) = Z(P'^S_\alpha)$.

b) If $g = \{(G_{k,0}, ..., G_{k,r})\}_k$ is another collection of $(r+1)$-tuples with coefficients in K defining f, then ε can be chosen sufficiently small so that $f^S = g^S$.

Proof. a) By hypothesis the radicals of the ideals generated by both sets of polynomials coincide. This means that we have polynomial identities of the form

$$\left(P'_\beta\right)^p = \sum W_{\alpha,\beta} P_\alpha \text{ and } P^q_\beta = \sum W'_{\alpha,\beta} P'^S_\alpha$$

and hence it is enough to require the positive number ε in Proposition 10 to be good also for the polynomials $W_{\alpha,\beta}$, $W'_{\alpha,\beta}$ and P'_α.

b) Proposition 10 applied to the defining collection for f obtained as the union of both collections of $(r+1)$-tuples yields the equality $f^S = g^S$ (see last statement in Proposition 7). ∎

Theorem 12 *1) If an irreducible projective complex variety X can be defined over two subfields of $\mathbb{C}$ algebraically disjoint over k then X can in fact be defined over $\overline{k}$.*

2) Let X and Y be irreducible projective complex varieties defined over k. If a morphism $f \in Mor(X, Y)$ can be defined over two subfields of $\mathbb{C}$ algebraically disjoint over k then f can in fact be defined over $\overline{k}$.

Proof. 1) Let $h : X \to Y$ be an isomorphism between irreducible projective varieties $X \subset \mathbb{P}^n(\mathbb{C})$ and $Y \subset \mathbb{P}^r(\mathbb{C})$. Write $X = Z(P_\alpha)$, $Y = Z(Q_\beta)$ and $h \equiv \{(F_{k,0}, ..., F_{k,r})\}$. Let Σ_1 (resp. Σ_2, resp. Σ) be the set of coefficients of the polynomials P_α (resp. Q_β, resp. $F_{k,i}$). Denote by K_1 (resp. K_2, resp. K_3) the field generated over k by Σ_1 (resp. Σ_2, resp. $\Sigma_3 = \Sigma_1 \cup \Sigma_2 \cup \Sigma$). We have to show that if K_1 and K_2 are algebraically disjoint over k then X can be defined over $\overline{k}$. In order to do that let us choose generators of K_3 and K_1 over k as in Lemma 6. Then this lemma tells us that there is a specialization $S = S_3$ of K_3 and these generators over k, with δ_S as small as wanted, which extends the identity specialization of K_1 and such that for any $a \in \Sigma_2$ one has $S(a) \in \overline{k}$. By Proposition 10, if δ_S is sufficiently small, we can specialize $h : X \to Y$ to obtain a new isomorphism $h^S : X^S \to Y^S$. Now, by construction, $X^S = X$ and Y^S is defined over $\overline{k}$ as desired.

2) This is similar to part 1). Suppose there is a commutative diagram

$$\begin{array}{ccc} X & \stackrel{f}{\to} & Y \\ \downarrow h_1 & & \downarrow h_2 \\ X & \stackrel{g}{\to} & Y \end{array}$$

where the vertical arrows are isomorphisms. Let us choose sets of defining polynomials for f, g, h_1 and h_2. Let us denote by Σ_1 and Σ_2 be the set of coefficients of the local expressions defining f and g respectively. Let Σ be the set of coefficients of the local expressions defining h_1 and h_2 as well as those of the polynomials intervening in the identities of type (3) expressing the compatibility of the local expressions defining $g \circ h_1$ with those defining $h_2 \circ f$. Denote by K_1 (resp. K_2, resp. K_3) the field generated over k by Σ_1 (resp. Σ_2, resp. $\Sigma_3 = \Sigma_1 \cup \Sigma_2 \cup \Sigma$). We now have to show that if K_1 and K_2 are algebraically disjoint over k then f can be defined over $\overline{k}$. In order to do that we apply Lemma 6 in the same way as before to conclude that there is a specialization $S = S_3$ of K_3 and a suitable set of generators over k, with δ_S as small as wanted, extending the identity specialization of K_1 and such that for any $a \in \Sigma_2$ we have $S(a) \in \overline{k}$. If we now apply S to our commutative diagram for the chosen sets of defining polynomials we obtain $g^S \circ h_1^S = h_2^S \circ f^S$. On the other hand, by the construction of S, we see that g^S is defined over $\overline{k}$ and that $X^S = X$, $Y^S = Y$ and $f^S = f$. Moreover, by Proposition 10, if δ_S is sufficiently small, then h_1^S and h_2^S are again isomorphisms, thus f is equivalent to g^S. This concludes the proof. ■

2.3 Proof of Criteria 1 and 2

Theorem 12 has the following implication

Corollary 13 *Criteria 1 and 2 hold.*

Proof. We will prove the equivalence of the three statements in criteria 1 and 2 simultaneously.

Let us denote by K the field generated over k by the coefficients of the polynomials defining X (resp. the local expressions for f).

Assume that $K \subset \overline{k}$, then K is clearly a finite extension of k and therefore the number of distinct restrictions of $\mathrm{Gal}(\mathbb{C}/k)$ to K is finite, hence the family $\{X^\sigma\}$ (resp. $\{f^\sigma\}$) can only contain finitely many distinct projective varieties (resp. morphisms). In other words, i) implies ii). On the other hand it is obvious that ii) implies iii).
It remains to prove that iii) implies i). Let $\{\pi_1, ..., \pi_d\}$ be a transcendence basis of K over k. We may assume that $d \geq 1$ for otherwise there would be nothing to prove. Since K is a countable field we can construct an uncountable family of field automorphisms $\sigma_s \in \mathrm{Gal}(\mathbb{C}/k)$ by first sending $\{\pi_1, ..., \pi_d\}$ bijectively to pairwise algebraically independent sets over k, $\{\sigma_s(\pi_1), ..., \sigma_s(\pi_d)\} \subset \mathbb{C}$, and then extending that to an automorphism of $\mathbb{C}$ (see e.g. [3] chap. 5.14.4). Now assumption iii) implies that in this family there are plenty of distinct elements $\sigma_1, \sigma_2 \in \mathrm{Gal}(\mathbb{C}/k)$ such that X^{σ_1} and X^{σ_2} (resp. f^{σ_1} and f^{σ_2}) are equivalent and hence so must be X and $X^{\sigma_1^{-1}\sigma_2}$ (resp. f and $f^{\sigma_1^{-1}\sigma_2}$). Thus X (resp. f) can be defined over finite extensions K_1 of $k(\pi_1, ..., \pi_d)$ and K_2 of $k(\sigma_1^{-1}\sigma_2(\pi_1), ..., \sigma_1^{-1}\sigma_2(\pi_d))$. From here, we only have to invoke Theorem 12. ■

3 Some applications in higher dimensions

Before we concentrate on the Riemann surface case we give some applications in higher dimensions.

Proposition 14 *Let S be a non singular projective surface defined over a number field and $C \subset S$ an irreducible curve with negative self-intersection $C^2 < 0$. Then C is also defined over a number field.*

Proof. Suppose $C = Z(Q_\alpha)$ for certain polynomials Q_α such that the field K generated over $\overline{\mathbb{Q}}$ by their coefficients is transcendental. Choose a standard set of generators $\pi_1, ..., \pi_d; u$ of K over $\overline{\mathbb{Q}}$. Let ε be good for $\pi_1, ..., \pi_d; u$ and the polynomials Q_α, and let S_m be a sequence of specializations of $K; \pi_1, ..., \pi_d; u$ over $\overline{\mathbb{Q}}$ with $\delta_{S_m} < \varepsilon$ tending to zero and $S_m(\pi_i) \in \overline{\mathbb{Q}}$. Then we can apply S_m to the inclusion morphism $C \subset S$ to obtain a corresponding sequence of curves C^{S_m} defined over $\overline{\mathbb{Q}}$ inside the surface $S^{S_m} = S$. When C^{S_m} gets sufficiently close to C, the intersection numbers $C^{S_m} \cdot C$ equal $C \cdot C < 0$. This inequality implies that $C^{S_m} = C$ as wanted. Indeed, let

$$C^{S_m} = C_1 \cup \cdots \cup C_d$$

be the expression of C^{S_m} as the finite union of its irreducible components, then $C^{S_m} \cdot C = \sum C_i \cdot C$. Now, if $C_i \neq C$ then, by definition, $C_i \cdot C \geq 0$, therefore for at least one of the components, say C_1, we must have $C_1 = C$. What remains to be observed is that because C^{S_m} is defined over $\overline{\mathbb{Q}}$ each of its irreducible components can be defined over $\overline{\mathbb{Q}}$ too. This can be easily seen as follows. First let the group $\mathrm{Gal}(\mathbb{C})$ act on both sides of the expression for C^{S_m} above to conclude that there are only finitely many isomorphism classes of Galois conjugates of C_1 and then apply Criterion 1. ■

Proposition 15 *If a smooth n-dimensional projective variety of general type Y occurs as image of a $n-$dimensional projective variety X defined over a number field, then Y can be defined over a number field too.*

Proof. For any $\sigma \in \mathrm{Gal}(\mathbb{C})$ we have a morphism $f^\sigma : X^\sigma \to Y^\sigma$. Since X is defined over a finite extension of $\mathbb{Q}$, in the collection of varieties $\{X^\sigma\}_{\sigma \in Gal(\mathbb{C})}$ we only have finitely many isomorphism classes. Now the result of Tsai ([25]) that a given n-dimensional variety can be surjectively mapped to only finitely many smooth n-dimensional varieties of general type implies that the same must be true for the family $\{Y^\sigma\}_{\sigma \in Gal(\mathbb{C})}$. Thus Criterion 1 applied to Y gives the desired conclusion. ■

Actually I don't know if this result is completely general, that is, whether it holds true if one makes no assumptions on the variety Y at all. An affirmative answer to this question would follow from an affirmative answer to the following, apparently unsolved, problem

Are there only finitely many smooth n-dimensional varieties Y to which a given n-dimensional variety X can be surjectively mapped with given degree d?

De Franchis-Severi theorem, which will be formulated in detail in the next section, implies that this is always the case when $n = 1$.

Now following [11] we denote by $\mathrm{Hol}_k(X, Y)$ the set of holomorphic maps f with $rank(f) \geq k$ from a complex analytic space X into a complex manifold Y. Here $rank(f)$ denotes the maximum rank of f on the regular points of X; thus $f(X)$ is is an analytic space and $dim(f(X)) = rank(f)$. In particular $\mathrm{Hol}_1(X, Y)$ is the subset of all non constant morphisms and, if $m = \dim Y$, $\mathrm{Hol}_m(X, Y)$ is the subset of the surjective ones. As usual, $\mathrm{Isom}(X, Y)$ will stand for the subset consisting of the isomorphisms.

Proposition 16 *Let X and Y be irreducible projective varieties defined over $\overline{\mathbb{Q}}$ such that Y is non singular. Assume that $Hol_k(X, Y)$ (resp. $Isom(X, Y)$) is finite. Then any f in $Hol_k(X, Y)$ (resp. $Isom(X, Y)$) is defined over a number field.*

Proof. Let K be the field generated over $\overline{\mathbb{Q}}$ by the set Σ of the coefficients of the local expressions defining f. Choose a standard set of generators $\pi_1, ..., \pi_d; u$ of K over $\overline{\mathbb{Q}}$ and let ε be good for $\pi_1, ..., \pi_d; u$ and the set Σ. Let S_n be a sequence of specializations of $K; \pi_1, ..., \pi_d; u$ over $\overline{\mathbb{Q}}$ with $\delta_{S_n} < \varepsilon$ tending to zero such that $S_n(\pi_i) \in \overline{\mathbb{Q}}$. Then we can apply S_n to the morphism $f : X \to Y$ to obtain a sequence of morphisms $f^{S_n} : X \to Y$ all defined over $\overline{\mathbb{Q}}$. Moreover, since the requirement $rank(f) \geq k$ is an open condition on the coefficients of f, we infer that if f^{S_n} is sufficiently close to f then f^{S_n} lies in $\mathrm{Hol}_k(X, Y)$ (resp. $\mathrm{Isom}(X, Y)$). Thus, for n sufficiently large f^{S_n} must equal f. ■

Due to work by Kobayashi and Ochiai ([14]) the typical case in which Proposition 16 applies occurs when Y is of general type and $k = \dim Y$. In particular we have

Corollary 17 *The automorphisms of a non singular variety of general type which is defined over $\overline{\mathbb{Q}}$ are all defined over $\overline{\mathbb{Q}}$.*

But Proposition 16 can be applied in many other situations such as the various cases discussed by Kalka, Shiffman and Wong in [11]. One can show, for instance, that the above statement holds also for any $K3$ surface Y. This is because on the one hand Corollary 6 in that paper states that $\mathrm{Hol}_2(Y,Y)$ is discrete, and on the other it is clear that the proof of Proposition 16 still works if instead of finiteness one only requires discreteness.

4 Applications in the Riemann surface case

4.1 Riemann surfaces

In one direction the proof of Belyi's theorem consists of a very elementary (and clever) algorithm to construct a Belyi function on a given Riemann surface defined over $\overline{\mathbb{Q}}$. For the converse statement he invokes a criterion of rationality due to Weil ([26]). We begin this section by applying Criteria 1 and 2 to produce an elementary proof of this part of Belyi's theorem as well as of the supplementary fact that the Belyi function can be defined over $\overline{\mathbb{Q}}$.

Theorem 18 *Let $f : C_1 \to C_2$ be a surjective morphism between algebraic curves. If C_2 and the branch values of f, $y_1, ...y_r \in C_2$ are defined over $\overline{\mathbb{Q}}$, then C_1 and f can also be defined over $\overline{\mathbb{Q}}$.*

Proof. For any $\sigma \in \mathrm{Gal}(\mathbb{C})$ the morphism $f^\sigma : C_1^\sigma \to C_2^\sigma$ has the same degree as $f : C_1 \to C_2$. By hypothesis, in the family $\{(C_2^\sigma, y_1^\sigma, ..., y_r^\sigma)\}_{\sigma \in Gal(\mathbb{C})}$ there are only finitely many distinct pointed curves, therefore standard monodromy theory (see e.g. [18]) puts us in position to apply Criterion 1 to C_1 and Criterion 2 to f. $\blacksquare$

This obviously implies the Weil part in Belyi's theorem. Namely

Corollary 19 *Let C be a complex algebraic curve admitting a morphism $f : C \to \mathbb{P}^1$ ramified over three values. Then C can be defined over a number field. Moreover, the morphism f can be defined over a number field too.*

Our next results depend on the de Franchis-Severi theorem ([5], see also [6], [9], [12] or [17]). This theorem has two parts:

1) If C_1 and C_2 are compact Riemann surfaces and the genus of C_2 is at least two, then $Mor(C_1, C_2)$ contains only finitely many non constant elements.

2) The number of isomorphism classes of Riemann surfaces of genus $g \geq 2$ (resp. of genus 1) to which a given one C can be surjectively mapped (resp. mapped with fixed degree) is finite.

Proposition 20 *If C_1 and C_2 are defined over $\overline{\mathbb{Q}}$ and the genus of C_2 is at least two any non constant morphism $f : C_1 \to C_2$ is necessarily defined over $\overline{\mathbb{Q}}$ too.*

Proof. In view of the first part of the de Franchis-Severi theorem, this is a consequence of Proposition 16. ■

Theorem 21 *Let $f : C_1 \to C_2$ be a surjective morphism between compact Riemann surfaces. Suppose C_1 is a Belyi surface, then so must be C_2.*

Proof. For any $\sigma \in \mathrm{Gal}(\mathbb{C})$ we have a morphism $f^\sigma : C_1^\sigma \to C_2^\sigma$ with the same degree as $f : C_1 \to C_2$. Since, by hypothesis, C_1 can be assumed to be defined over a finite extension of $\mathbb{Q}$, in the collection of curves $\{C_1^\sigma\}_{\sigma \in Gal(\mathbb{C})}$ we only have finitely many isomorphism classes. By the second part of the de Franchis-Severi theorem, the same is true for the family $\{C_2^\sigma\}_{\sigma \in Gal(\mathbb{C})}$, provided the genus of C_1 is ≥ 1. In this circumstance Criterion 1 applied to C_2 gives the desired conclusion. If the genus of C_1 is 0, then C_1 is isomorphic to $\mathbb{P}^1$ and so the theorem is trivial in this case. ■

As the referee has pointed out to me there is an alternative proof of Theorem 21 in the case f is a Galois cover and the genus of C_1 is at least 2 which avoids using de Franchis-Severi theorem. In this situation C_2 is the quotient of C_1 by the corresponding Galois group. As the automorphism group of C_1 is finite, Proposition 16 implies that all automorphisms of C_1 are defined over $\overline{\mathbb{Q}}$. Hence, the quotient C_2 is defined over $\overline{\mathbb{Q}}$ as well.

Combining Theorem 18 and Theorem 21 we get

Corollary 22 *Let $f : C_1 \to C_2$ be an unramified covering of compact Riemann surfaces. Then, C_2 is a Belyi surface if and only if C_1 is.*

We observe that when the genus is 1, the above Corollary 22 gives, as a particular case, a classical result in Number Theory, namely that if τ is a complex number with positive imaginary part, $\alpha \in GL_2(\mathbb{Q})^+$ and j is the Jacobi modular function, then $j(\tau)$ *is an algebraic number if and only if* $j(\alpha(\tau))$ *is.* This is because, on the one hand, $\mathbb{Q}(j(\tau))$ is the minimum field of definition for the complex torus $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$ and, on the other, E_τ is always isogenous to $E_{\alpha(\tau)}$ ([22], [19]).
In turn the theory of elliptic curves shows that in general $\mathbb{Q}(j(\alpha(\tau)))$ is different from $\mathbb{Q}(j(\tau))$. In other words, in Corollary 21, one cannot expect C_2 to be defined over each field of definition of C_1.

4.2 Fuchsian groups

In this section the reader is assumed to have some familiarity with the most elementary aspects of the theory of *Fuchsian groups*, that is discrete subgroups of $PSL(2,\mathbb{R})$, and uniformization of Riemann surfaces. The *triangle Fuchsian (resp. Euclidean) group* of signature (p, q, r) is the index 2 orientation preserving subgroup of the group generated by reflections in the sides of a hyperbolic (resp. Euclidean) triangle in the upper-half plane $\mathbb{H}$ (resp. the complex plane $\mathbb{C}$) whose angles are $\pi/p, \pi/q$ and π/r. What is of interest here is that in both cases taking the quotient by a triangle group gives $\mathbb{P}^1$ with three branching values of order p, q and r corresponding to the vertices of the triangle in question. We are thus tacitly assuming that the angles $\pi/p, \pi/q$ and π/r are nonzero.

Let C be a compact Riemann surface isomorphic to a quotient space $\mathbb{U}/\Gamma$, with $\mathbb{U} = \mathbb{H}$ or $\mathbb{C}$, and Γ a finite index subgroup of triangle subgroup Δ. Then the canonical map $\mathbb{U}/\Gamma \to \mathbb{U}/\Delta$ affords a Belyi map from $C = \mathbb{U}/\Gamma$ to $\mathbb{P}^1 = \mathbb{U}/\Delta$. In fact the converse is also true. We have ([1], see also [10])

Proposition 23 *A compact Riemann surface of genus $g \geq 1$ is a Belyi surface if and only if C is isomorphic to a quotient space $\mathbb{U}/\Gamma$, with $\mathbb{U} = \mathbb{H}$ or $\mathbb{C}$, and Γ a finite index subgroup of a triangle group.*

Unfortunately the group Γ in Proposition 23 may not be a *surface group,* that is, a co-compact group which acts freely on $\mathbb{U}$, hence isomorphic to the fundamental group of C. This question in the genus 1 case has been dealt with in [24].

Let us assume for the rest of this section that $g \geq 2$. In this case, which corresponds to $\mathbb{U} = \mathbb{H}$, surface groups are Fuchsian groups uniquely determined by the isomorphism class of the Riemann surface they uniformize (up to conjugation in $PSL(2,\mathbb{R})$). If a Riemann surface C is simultaneously obtained as quotient by a surface group G and by a not necessarily torsion free Fuchsian group Γ, then there is an isomorphism $H : C = \mathbb{H}/\Gamma \to \mathbb{H}/G$ which, by covering space theory, lifts to a holomorphic map $h : \mathbb{H} \to \mathbb{H}$. This map induces a group homomorphism $\phi : \Gamma \to G$ characterized by

$$h \circ \gamma = \phi(\gamma) \circ h \quad \text{for all} \quad \gamma \in \Gamma \tag{8}$$

It is not hard to show (see [4], 4.1) that both the map h and the homomorphism ϕ are surjective. Let us say that a group homomorphism $\phi : \Gamma \to G$ between Fuchsian groups Γ and G is *holomorphically induced* if there is a surjective holomorphic map $h : \mathbb{H} \to \mathbb{H}$ with respect to which ϕ satisfies identity (8). We can then characterize the surface groups that uniformise Belyi surfaces as follows.

Proposition 24 *Let G be a surface group of genus $g \geq 2$. Then $C = \mathbb{H}/G$ is a Belyi surface if and only if G is the image of a holomorphically induced epimorphism $\phi : \Gamma \twoheadrightarrow G$ where Γ is a finite index subgroup of some co-compact triangle group.*

Proof. The "only if" part results from Proposition 23 together with the comments that follow it. Conversely, if the epimorphism $\phi : \Gamma \twoheadrightarrow G$ is induced by a holomorphic surjection $h : \mathbb{H} \to \mathbb{H}$ satisfying (8), then it is clear that we have an induced morphism of compact Riemann surfaces $H : \mathbb{H}/\Gamma \to \mathbb{H}/G$. As Γ is contained in a triangle group we see that $\mathbb{H}/\Gamma$ is a Belyi surface. Now we apply Theorem 21. ■

As J. Wolfart has pointed out to me, this idea of comparing the various groups inducing the same quotient Riemann surface was already considered, although in a different language, by Klein in [13], p.301, for the case in which the quotient is $\mathbb{P}^1$ with three distinguished points.

We can also characterize Belyi surfaces via uniformization by finite volume Fuchsian groups. Recall that two Fuchsian groups Γ_1 and Γ_2 are called *commensurable* if Γ_1 has a subgroup of finite index which is conjugate in $\mathbb{PSL}_2(\mathbb{R})$ to a finite index subgroup of Γ_2. We have

Theorem 25 *A compact Riemann surface C is a Belyi surface if and only if there is a finite set $\Sigma \subset C$ such that $C \setminus \Sigma$ is isomorphic to a quotient of the form $\mathbb{H}/\Gamma$ for some Fuchsian group commensurable with the classical modular group $\mathbb{PSL}_2(\mathbb{Z})$. Furthermore, Γ can be chosen to be torsion free.*

Proof. If $f : C \to \mathbb{P}^1$ is a Belyi function, then the restriction $C \setminus \Sigma \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$, with $\Sigma = f^{-1}(\{0, 1, \infty\})$, is a smooth cover isomorphic to one of the form $\mathbb{H}/\Gamma \to \mathbb{H}/\Gamma(2)$ for some finite index subgroup of the level 2 principal congruence subgroup $\Gamma(2)$ which itself has finite index in $\mathbb{PSL}_2(\mathbb{Z})$ and is torsion free.

Conversely, assume that there is a finite set $\Sigma \subset C$ such that $C \setminus \Sigma$ is isomorphic to $\mathbb{H}/\Gamma$ for some Fuchsian group commensurable with $\mathbb{PSL}_2(\mathbb{Z})$. Let H be a finite index subgroup of Γ and α a real Möbius transformation such that $\alpha H \alpha^{-1}$ has finite index in $\mathbb{PSL}_2(\mathbb{Z})$, then the natural projection $\mathbb{H}/\alpha H \alpha^{-1} \to \mathbb{H}/\mathbb{PSL}_2(\mathbb{Z})$ yields a Belyi function on the Riemann surface C' obtained by compactifying $\mathbb{H}/\alpha H \alpha^{-1} \simeq \mathbb{H}/H$. Similarly, the projection $\mathbb{H}/H \to \mathbb{H}/\Gamma$ uniquely determines a non constant morphism from C' to C. Now we only have to invoke Theorem 21. ■

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References

[1] P. Beazley Cohen, C. Itzykson, J. Wolfart. Fuchsian triangle groups and Grothendieck dessins. Variations on a theme of Belyi. *Comm. Math. Phys.* **163** (1994),3, 605-627.

[2] G. Belyi. On Galois extensions of a maximal cyclotomic field, *Math, USSR Izv.* **14**, No.2 (1980), 247-256.

[3] N. Bourbaki. Elements of Mathematics. Algebra II. Springer (1990)

[4] C. Earle and I. Kra. On sections of some holomorphic families of closed Riemann surfaces. *Acta Math.* **137** (1976), 49-79

[5] M. de Franchis. Un teorema sulle involutioni irrazionali, *Rend. Circ. Mat. Palermo.* **36** (1913)

[6] Y. Fuertes and G. González-Diez. On the number of coincidences of morphisms between closed Riemann surfaces. *Publ. Mat.* **37**, 2, (1993) 339-353.

[7] G. González-Diez. Belyi's theorem for complex surfaces (preprint)

[8] A. Grothendieck. "Esquisse d'un Programme". *London. Math. Soc, L.N.Series* **242**. (L.Schneps and P. Lochak, editors)

[9] A. Howard and A. J. Sommese. On the theorem of de Franchis. *Ann. Scuola Norm. Pisa* Cl. Sci.(4) **10** (1983), 429–436

[10] G. Jones and D. Singerman. Belyi functions, hypermaps and Galois groups. *Bull. London Math. Soc.* **28** (1996), no. 6, 561–590.

[11] M. Kalka, B. Shiffman and B. Wong. Finiteness and rigidity theorems for holomorphic mappings. *Michigan Math. J.* **28** (1981), 289-295

[12] E. Kani. Bounds on the number of non-rational subfields of a function field. *Invent. Math.* **85 (**1986**),** 185-198

[13] F. Klein. Vorlesungen über die hypergeometrische Funktion, Springer Grundlehren **39** (1933).

[14] S. Kobayashi and T. Ochiai. Meromorphic mappings onto compact complex spaces of general type. *Invent. Math.***31** (1975), 7-16

[15] B. Köck. The moduli field of a curve revisited. *Beiträge Algebra Geom.* **45** (2004), 253-265

[16] S. Lang. Algebra. Second Edition. Addison-Wesley (1984)

[17] H. H. Martens. Observations on morphisms of closed Riemann surfaces. II. *Bull. London Math. Soc.* **20** (1988), 253–254.

[18] R. Miranda., Algebraic curves and Riemann surfaces, Graduate Studies in Mathematics Vol. **5**. Amer. Math. Soc. (1995)

[19] A. Robert. Elliptic curves. *Lecture Notes in Mathematics.* **326.** Springer-Verlag (1986)

[20] L. Schneps. The Grothendieck theory of Dessins d'Enfants. *London. Math. Soc, L.N.Series* **200**. Cambridge Univ. Press (1994)

[21] I.R. Shafarevich. Basic Algebraic Geometry. Springer Verlag (1977)

[22] G. Shimura. Introduction to the arithmetic theory of automorphic functions, Princeton University Press (1971)

[23] G. B. Shabat and V.A. Voevodsky. Equilateral triangulations of Riemann surfaces, and curves over algebraic number fields. *Soviet Math. Dokl.* **39**, 1 (1989), 38-41.

[24] D. Singerman and R.I. Sydall. Belyi uniformization of elliptic curves. Bull. London Math. Soc. **29** (1997), 443–451

[25] I. Tsai. Dominating the varieties of general type. *J. Reine Angew. Math.* **483** (1997), 197-219

[26] A. Weil. The field of definition of a variety, *Amer. J. Math.* **78** (1956), 509-524.

[27] J. Wolfart. The "Obvious" Part of Belyi's Theorem and Riemann surfaces with many automorphisms. *London. Math. Soc, L.N.Series* **242**. (L. Schneps and P. Lochak, editors)