ON THE LEFSCHETZ NUMBER OF QUASICONFORMAL SELF-MAPPINGS OF COMPACT RIEMANN SURFACES.

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1. Introduction.

A well known theorem of Hurwitz states that if $\tau : S \to S$ is a conformal self-mapping of a compact Riemann surface of genus $g \geq 2$, then it has at most $2g + 2$ fixed points and that equality occurs if and only if τ is a hyperelliptic involution.

In this paper we consider this problem for a K-quasiconformal self-mapping $f : S \to S$. The result we obtain is that the number of fixed points (suitably counted) is bounded by $2 + g(K^{1/2} + K^{-1/2})$, and that this bound is sharp. We see that when $K = 1$, i.e. when f is conformal, our result agrees with the classical one.

To make matters more precise, we recall that the *Lefschetz number* of an orientation preserving homeomorphism $f : S \to S$ is defined by

$$L(f) = 2 - \text{trace } f^{\#}$$

where $f^{\#}$ is the induced linear automorphism of the first cohomology group $H^1(S, \mathbb{C})$.

The integer $L(f)$ counts the number of fixed points in the sense that, once f is perturbed so as to have finitely many fixed points,

$$L(f) = \sum_{f(x)=x} L_x(f)$$

where $L_x(f)$, *the local Lefschetz number* of f at the isolated point x, is defined to be the degree of the map $z \to \dfrac{f(z) - z}{|f(z) - z|}$ from the boundary of a small disc around x to the unit circle S^1. (See [G-P] and [B-T]).

With this notation we can describe our result as follows (Theorem 4.2.):

Let $f : S \to S$ be a K-quasiconformal self-mapping of a compact Riemann surface S of genus g; then we have

[1]Research of the second author supported by a grant of the CICYT. M.E.C. Spain.

a) *$L(f) \leq 2 + g(K^{1/2} + K^{-1/2})$. In other words, having a large number of fixed points (suitable counted) forces f to have big dilatation.*

b) *If $g \geq 2$ this bound is only attained by hyperelliptic involutions.*

c) *This is the best possible bound.*

These are, of course, statements about the relationship between dilatation and eigenvalues of a quasiconformal self-mapping.

A summary of the results we will have to prove previously includes (Theorem 3.3., Theorem 3.5. and Lemma 3.7.):

i) *For any eigenvalue λ, we have $|\lambda| \leq K^{1/2}$.*

ii) *f has an eigenvalue λ with $|\lambda| = K^{1/2}$ if and only if f is an absolutely extremal (or pseudo-Anosov) mapping whose associated quadratic differential equals the square of some abelian differential ω. Such an eigenvalue λ is unique, and $Re\omega$ is the unique eigenvector with eigenvalue λ.* (See §.2 for the definition of these concepts).

iii) *If $g \geq 2$, there is always an eigenvalue μ with $|\mu| < K^{1/2}$.*

In the last section we illustrate these results by working out an explicit example.

We are indebted to W. J. Harvey for drawing our attention to several articles related to this work.

2. Review of the Nielsen-Thurston-Bers classification of self-mappings of a surface.

Let S be a compact oriented surface of genus $g \geq 2$, and let $f : S \to S$ be an orientation preserving homeomorphism. If we endow S with a holomorphic atlas $\mathcal{A}$ to get a Riemann surface $(S, \mathcal{A})$, we can speak of its dilatation $K = K(f, \mathcal{A})$, which, we recall, is defined by

$$K = \begin{cases} \sup\limits_{z \in S} \dfrac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} & \text{a.e., if } f \text{ is quasiconformal} \\ \infty, & \text{otherwise} \end{cases}$$

We shall denote by $\bar{K} = \bar{K}(f)$ the infimum of $K(f', \mathcal{A})$, when $\mathcal{A}$ varies over all conformal structures on S, and f' over the isotopy class of f. We shall refer to this real number as the *dilatation of the mapping class* of f.

According to whether or not $\bar{K}$ is attained, Bers has introduced the following classification of mapping classes of S ([Be], see also [Ab]):

— f is called *elliptic* if $1 = \bar{K}(f) = K(f', \mathcal{A})$ for some conformal structure $\mathcal{A}$ and some mapping f' isotopic to f. In other words, if for a suitable conformal structure, f' can be realized as an automorphism of S.

— *parabolic* if $1 = \bar{K}(f) \neq K(f', \mathcal{A})$ for any conformal structure $\mathcal{A}$ on S and any mapping f' isotopic to f.

— *pseudo-hyperbolic* if $1 < \bar{K}(f) \neq K(f', \mathcal{A})$ for any conformal structure $\mathcal{A}$ on S and any f' isotopic to f.

— *hyperbolic* if $1 < \bar{K}(f) = K(f', \mathcal{A})$ for some conformal structure $\mathcal{A}$ on S and some mapping f' isotopic to f. These mapping classes are also called *irreducible* or *pseudo-Anosov* in the theory of Thurston; they are characterized by the fact that no set of (*admissible*) Jordan curves is preserved by f.

If f is hyperbolic and $(f, \mathcal{A})$ is an *absolutely extremal* pair, i.e. such that $\bar{K}(f) = K(f, \mathcal{A})$, then Bers ([Be]) has shown that on the *Riemann surface* $(S, \mathcal{A})$ there is a holomorphic quadratic differential Φ uniquely determined by f, except for a multiplicative positive constant, such that the following holds.

i) Let $P \in S$, and r the order of Φ at P, then there exists a local parameter z defined near and vanishing at P, such that

$$\Phi = \left(\frac{r+2}{2}\right)^2 z^r dz^2 \quad \text{near } P\ ;$$

if $r = 0$, then

$$\Phi = dz^2 \quad \text{near } P.$$

The parameter z is called a *natural parameter* belonging to Φ at P. It may be multiplied by an $(r+2)$-nd root of unity but is otherwise uniquely determined.

ii) The order r of Φ at $P \in S$ equals the order at $f(P)$.

iii) Let $z = x + iy$ be a natural parameter belonging to Φ at P, then for a suitable choice of one of the $(r+2)$ natural parameters belonging to Φ at $f(P)$ the mapping f is represented by

$$z \mapsto \zeta = \left(\frac{z^{r+2} + 2k|z|^{r+2} + k^2 \bar{z}^{r+2}}{1-k^2}\right)^{1/r+2}$$

with $\zeta > 0$ for $z > 0$, where $K = \dfrac{1+k}{1-k}$. In particular, if $r = 0$, then f is represented by

$$z \mapsto \zeta = K^{1/2} x + i\ K^{-1/2} y$$

3. Eigenvalues and dilatation.

We want to analyse the action of f on the de Rham model of $H^1(S, \mathbb{C})$. However, the fact that a quasiconformal mapping need not be differentiable possess a technical problem with regard to performing the pull-back operation of differentiable forms.

In order to overcome this difficulty we must bring in the Hilbert space $L^2(S)$ of (mesurable) square integrable 1-forms on the Riemann surface S. This space consists of 1-forms v that can be locally written as

$$v(z) = a(z)dz + b(z)d\bar{z}$$

where $a(z)$ and $b(z)$ are mesurable functions, and whose L^2-norm

$$\| v \|^2 = \int_S v \wedge^* \bar{v}$$

is finite. We recall that the $*$-operator is locally defined by

$$v^* = -ia(z)dz + ib(z)d\bar{z},$$

so that

$$v \wedge^* \bar{v} = i(|a(z)|^2 + |b(z)|^2)dz \wedge d\bar{z}$$

(concerning all this see [F-K] or [Sp]).

We observe that the usual pull-back operator f^* defined locally by

$$f^* v_{|U} = \left(a(f)\frac{\partial f}{\partial z} + b(f)\frac{\partial \bar{f}}{\partial z} \right) dz + \left(a(f)\frac{\partial f}{\partial \bar{z}} + b(f)\frac{\partial \bar{f}}{\partial \bar{z}} \right) d\bar{z},$$

transforms $L^2(S)$ into itself. This is so, because quasiconformal mappings enjoy the properties of preserving null-sets and having L^2-derivatives.

Besides the operator $f^* : L^2(S) \to L^2(S)$, f induces a linear automorphism of the first cohomology group which we shall denote by $f^\# : H^1(S, \mathbb{C}) \to H^1(S, \mathbb{C})$.

In practice, since we want to work with the de Rham model of $H^1(S, \mathbb{C})$, which consists of classes of C^∞ 1-forms on S, when computing $f^\#$, we shall replace f by a C^∞ mapping $f_1 : S \to S$ homotopic to f (see [K-L], p. 133, and [Mun]). As an alternative approach one could choose working with L^2-cohomology as in [D-S].

Warning: In this paper the term *eigenvalue of f* will always mean *eigenvalue of $f^\#$*.

Our basic result is the following:

Lemma 3.1. *Let f be a K-quasiconformal self-mapping of a Riemann surface S. Then for any $v \in L^2(S)$, we have $\| f^* v \|^2 \leq K \| v \|^2$.*

PROOF: We have

$$\| f^* v \|_U^2 := \int_U f^* v \wedge^* \overline{(f^* v)} = i \int_U \left(\left| a(f)\frac{\partial f}{\partial z} + b(f)\frac{\partial \bar{f}}{\partial z} \right|^2 + \left| a(f)\frac{\partial f}{\partial \bar{z}} + b(f)\frac{\partial \bar{f}}{\partial \bar{z}} \right|^2 \right) dz \wedge d\bar{z}$$

From here, using the identity $\overline{(\partial f/\partial z)} = \partial \bar{f}/\partial \bar{z}$ and the inequality $2|a|\ |b| \leq |a|^2 + |b|^2$, we deduce that

$$\begin{aligned}
\int_U f^* v \wedge^* \overline{(f^* v)} &\leq i \int_U \left\{ \left(|a(f)|^2 + |b(f)|^2\right)\left(|f_z|^2 + |f_{\bar{z}}|^2\right) + 4|a(f)|\ |b(f)|\ |f_z|\ |f_{\bar{z}}| \right\} dz \wedge d\bar{z} \\
&\leq i \int_U \left(|a(f)|^2 + |b(f)|^2\right)\left(|f_z|^2 + |f_{\bar{z}}|^2 + 2|f_z|\ |f_{\bar{z}}|\right) dz \wedge d\bar{z} \\
&= i \int_U \left(|a(f)|^2 + |b(f)|^2\right) \frac{(|f_z| + |f_{\bar{z}}|)^2}{|f_z|^2 - |f_{\bar{z}}|^2} \left(|f_z|^2 - |f_{\bar{z}}|^2\right) dz \wedge d\bar{z} \\
&= i \int_U \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \left(|a(f)|^2 + |b(f)|^2\right) \det(Df) dz \wedge d\bar{z} \\
&\leq K \int_{f(U)} v \wedge^* \bar{v} = K \| v \|_{f(U)}^2
\end{aligned}$$

as wanted.

In a somewhat more sophisticated language this result can be rephrased as follows.

Lemma 3.1.A. *Let f be a K-quasiconformal self-mapping. Then, the pull-back operation of 1-forms induces a bounded linear operator f^* of the Hilbert space $L^2(S)$, whose norm is bounded by $K^{1/2}$.*

Remark 3.2. In fact, we have $K^{-1/2} \leq\| f^* \|\leq K^{1/2}$; the lower bound being obtained by reversing the inequalities in the proof of Lemma 3.1.

Theorem 3.3. *Let λ be an eigenvalue of a K-quasiconformal mapping f, then $|\lambda| \leq K^{1/2}$.*

PROOF: Let us approximate f by a sequence of C^∞ mappings $f_n : S \to S$, such that

i) f_n converges to f in the C^0-topology; and

ii) the partial derivatives of f_n converge to those of f in the L^2-topology.

That this can be done for a local representation $f : U \to V$ of f in terms of local charts U of x and V of $f(x)$, is a basic result in Real Analysis. Passage from local approximation to global can be performed in the standard gluing fashion (e.g. mimic the proof of Theorem 4.2. in [Mun]).

Thus, by taking n sufficiently large, we can achieve two goals: on the one hand i) implies that f_n is going to be homotopic to f (see e.g. [Mun]), and on the other hand, by applying i) and ii) simultaneously, it is clear from the definitions at the begining of this section that for any $v \in L^2(S)$, the sequence of 1-forms f_n^*v converges to f^*v in the L^2-topology.

Now, λ being an eigenvalue of $f^\#$ means that $f_n^*v = \lambda v + d\varphi_n$, for some C^∞ function φ_n and some non-exact closed 1-form v. Moreover, if we assume that v is harmonic, as we shall do, then v and $d\varphi_n$, viewed as elements of $L^2(S)$, will be orthogonal (see [F-K] or [Sp]).

Thus, we have $\| \lambda v \|^2\leq\| f_n^*v \|^2$ which tends to $\| f^*v \|^2\leq K \| v \|^2$, as n tends to infinity. This ends the proof.

A similar argument allows us to make the following observation:

Lemma 3.4. *i) Let f be a K-quasiconformal self-mapping, and let v be a harmonic 1-form which is an eigenvector of the linear operator $f^* : L^2(S) \to L^2(S)$ with eigenvalue λ. Then v is also an eigenvector of the linear automorphism $f^\# : H^1(S, \mathbb{C}) \to H^1(S, \mathbb{C})$, with same eigenvalue λ .*

ii) *The converse holds when $|\lambda| = K^{1/2}$.*

PROOF: i) With the notation as in the proof of the theorem above, we see that the sequence $f_n^*v = \eta + d\varphi_n$, with η harmonic, converges in the L^2-topology to $f^*v = \lambda v$. Thus, we find that the harmonic 1-form $\lambda v - \eta$ lies in the L^2-closure of the vector space of exact 1-forms. From here we deduce that $\lambda v - \eta = 0$ as desired (see [F-K] or [Sp]).

ii) We see that, on the one hand, $f_n^* v = \lambda v + d\varphi_n$ converges to $f^* v$; and on the other hand, $\| f^* v \| \leq K^{1/2} \| v \|$. Combining these two facts we find that, when $|\lambda| = K^{1/2}$, $d\varphi_n$ converges to zero, which is the same as saying that $f^* v = \lambda v$.

Theorem 3.5. *Let f be a K-quasiconformal self-mapping of a compact Riemann surface S. Then, the following statements are equivalent:*

i) f has an eigenvalue λ with $|\lambda| = K^{1/2}$.

ii) f is an absolutely extremal mapping whose associated quadratic differential Φ equals the square of an abelian holomorphic differential ω .

When this occurs λ is the unique eigenvalue with the property that $|\lambda| = K^{1/2}$, and $Re\omega$ is (up to multiplicative constant) the unique eigenvector whose eigenvalue is λ.

Proof:

i) $\Rightarrow$ ii): The fact that λ is an eigenvalue of f is independent of the isotopy class of f as well as of the Riemann surface structure on S. Since, by Theorem 3.3., we must always have $|\lambda| \leq K^{1/2}$, we see that when the identity holds f has to be absolutely extremal.

Moreover, we see from Lemma 3.4.ii) that if this occurs all the inequalities that intervene in the proof of Lemma 3.1. have to be, in fact, equalities. Thus, if the local expression of the eigenvector v, which we can assume to be harmonic, is $v = a(z)dz + \bar{b}(z)d\bar{z}$, with $a(z)$ and $b(z)$ holomorphic, we must have

a) $\left| a(f)\dfrac{\partial f}{\partial z} + \bar{b}(f)\dfrac{\partial \bar{f}}{\partial z} \right| = \left| a(f)\dfrac{\partial f}{\partial z} \right| + \left| \bar{b}(f)\dfrac{\partial \bar{f}}{\partial z} \right|$ and

b) $2|a||b| = |a^2| + |b^2|$, i.e. $|a| = |b|$

Now, f being absolutely extremal, we know that it can be written, in terms of the *natural parameters* $z = \int \sqrt{\Phi}$, as (see §.2)

$$f(z) = \frac{z + k\bar{z}}{\sqrt{1-k^2}}$$

The identity a) means now that the function $a(z)$ is a positive multiple of the function $\bar{b}(z)$, whereas b) implies that this multiple is 1. In other words, we have $a(z) = \bar{b}(z)$ with $a(z)$ and $b(z)$ holomorphic funtions of z, which implies that $a(z)$ and $\bar{b}(z)$ must be identically equal to a certain constant C, and that, in terms of these natural parameters, $v = C\ dz + Cd\bar{z} = 2CRe(dz)$.

In summary, we have shown that, except for a finite set of points of S - the zeros of Φ - the harmonic 1-form v agrees (locally) with the real part of (a branch of) $\sqrt{\Phi}$, therefore the holomorphic 1-form $\omega = v + i^* v$ (see [F-K] or [Sp]) coincides (locally) with $\sqrt{\Phi}$, and hence we have $\Phi = \omega^2$.

ii) $\Rightarrow$ i): Let f be an absolutely extremal mapping, and let $\Phi = \omega^2$ be its associated quadratic differential. Around each point $\mathrm{P} \in S$ which is not a zero of ω , we fix the local natural parameter $z = \int_{\mathrm{P}} \omega$.

By the results quoted in §.2 we know that $f(P)$ is not a zero of ω either; there are therefore precisely two natural parameters around $f(P) = Q$, namely

$$z = \int_Q \omega \quad \text{and} \quad -z = \int_Q (-\omega).$$

By making the *correct* choice of sign, f can be expressed as $f(z) = \dfrac{z + k\bar{z}}{\sqrt{1-k^2}}$, or in real coordinates

$$f(x+iy) = K^{1/2}x + iK^{-1/2}y$$

Since the set of points Q at which the correct choice is, say z, is both open and closed, we see that the sign is constant all over S.

Now from this expression we deduce that

$$f^*\left(dx_{|Q}\right) = K^{1/2}dx_{|P}\ , \text{ or in other words}$$
$$f^*\left(\pm Re\omega_{|Q}\right) = K^{1/2}Re\omega_{|P}\ .$$

This means that $Re\omega$ is an eigenvector of the operator f^* and, by Lemma 3.4.i), also of the linear isomorphism $f^{\#}$, with eigenvalue $\pm K^{1/2}$ (the sign being the same as the correct sign of the natural parameter around Q). This completes the proof of the theorem.

Remark 3.6. i) The proof of the above theorem shows that in fact λ is equal to either $K^{1/2}$ or $-K^{1/2}$ (both may occur: see §.5). In any case the corresponding eigenspace is the 1-dimensional vector space generated by $Re\sqrt{\Phi}$. A similar statement holds for $|\lambda| = K^{-1/2}$ and $Im\sqrt{\Phi}$ (see Remark 3.2.).

ii) These results imply the existence of absolutely extremal (or pseudo-Anosov) mappings f whose associated quadratic differential is not of the form $\Phi = \omega^2$ (although it is readily seen that it will be of this form if we take a suitable lifting of f, see [Kra]). To see this, let f be a parabolic mapping class, then by Theorem 3.3. (see also Remark 3.2.) it only has unitary eigenvalues. Now, according to Papadopoulos ([Pa]), there exists a pseudo-Anosov diffeomorphism with $K > 1$ such that its action on the first homology group (and hence its eigenvalues) is the same as that of f. Now apply Theorem 3.5.

iii) The theorem also shows that the dilatation (or *expansion factor*) K of a pseudo-Anosov diffeomorphism with the property that $\Phi = \omega^2$ is an algebraic integer, for either $K^{1/2}$ or $K^{-1/2}$ is an eigenvalue of a matrix with integer coefficients, namely the matrix of $f^{\#} : H^1(S, \mathbf{Z}) \to H^1(S, \mathbf{Z})$. In fact this is true for any pseudo-Anosov diffeomorphism, for, as it has been noted above, if f does not have this property a suitable lifting of it will do (see [Kra]). This result was first proved by Thurston (see [As] p. 167).

Lemma 3.7. *Let f be an absolutely extremal self-mapping of dilatation $K > 1$, on a Riemann surface of genus $g \geq 2$. Then f has an eigenvalue μ with $|\mu| < K^{1/2}$.*

PROOF: Let us consider in $H^1(S,\mathbb{C})$ the intersection product

$$(v|w)=\int_S v\wedge w=-\langle v,^* \bar{w}\rangle$$

(see [F-K] or [Sp]). (Observe that unlike the Hermitian product we have been considering so far, the intersection product is well defined over de Rham classes).

We can assume that f is in the situation described in Theorem 3.5. for otherwise there will be nothing to be proved. With the notation used there, let us denote by V the subspace of $H^1(S,\mathbb{C})$ generated by $Re\sqrt{\Phi}$ and $Im\sqrt{\Phi}$, and by $V^\perp$ its orthogonal space with respect to the bilinear form just defined.

We claim that $V^\perp$ is $f^\#$-invariant.

Let $\eta\in V^\perp$, i.e. $(\eta|Re\sqrt{\Phi})=(\eta|Im\sqrt{\Phi})=0$. Then, denoting by f_1 a diffeomorphism homotopic to f, we have e.g.

$$\begin{aligned}(f^\#\eta|Re\sqrt{\Phi})&=\int f_1^*\eta\wedge Re\sqrt{\Phi}=\int\eta\wedge(f_1^{-1})^*(Re\sqrt{\Phi})=(\eta|(f^{-1})^\# Re\sqrt{\Phi})\\&=\pm K^{-1/2}\int\eta\wedge Re\sqrt{\Phi}=0,\end{aligned}$$

where in the last identity we have used the fact that, by Remark 3.5., $(f^{-1})^\#(Re\sqrt{\Phi})=\pm K^{-1/2}Re\sqrt{\Phi}$. This shows that $V^\perp$ is indeed an invariant subspace.

Now we are almost done. Let η be an eigenvector of $f^\#$ in $V^\perp$ and μ its corresponding eigenvalue. Then $\eta\neq Re\sqrt{\Phi}$, for we have $(Re\sqrt{\Phi}|Im\sqrt{\Phi})>0$ which implies that $Re\sqrt{\Phi}\notin V^\perp$; so, by Theorem 3.3 and Remark 3.6.i), we see that $|\mu|<K^{1/2}$.

We now collect what we have gathered, so far, about the eigenvalues of a quasiconformal self-mapping.

Theorem 3.8. *Let f be a K-quasiconformal self-mapping of a Riemann surface S of genus g, with $K>1$. Then we have*

i) If λ is an eigenvalue of f, then $|\lambda|\leq K^{1/2}$.

ii) If $g\geq 2$, then f always possesses an eigenvalue μ with $|\mu|<K^{1/2}$.

PROOF: Statement i) is precisely Theorem 3.3. As for ii) we easily see it to be a consequence of Theorem 3.5. and Lemma 3.7.

4. The Lefschetz number of f.

Lemma 4.1. *Let M be a symplectic matrix, and let $\lambda\in\mathbb{C}$ be an eigenvalue of it. Then λ^{-1} is also an eigenvalue of M, with the same multiplicity as λ .*

PROOF: We recall that a symplectic matrix satisfies the identity $MJ({}^tM) = J$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is the intersection matrix of a canonical basis (see [F-K]). In particular, we see that $J({}^tM)J^{-1} = M^{-1}$, which shows that the eigenvalues of M (and their multiplicities) are the same as those of M^{-1} .

It reminds to be shown that for a non-singular matrix A, λ being an eigenvalue of A is equivalent to λ^{-1} being an eigenvalue of A^{-1} (with same multiplicity). But this follows from the following identities

$$(A - \lambda I)^p = [\lambda A(\lambda^{-1}I - A^{-1})]^p = [(\lambda^{-1}I - A^{-1})\lambda A]^p = \lambda^p A^p(\lambda^{-1}I - A^{-1})^p$$

that cleary imply that

$$\ker(A - \lambda I)^p = \ker(A^{-1} - \lambda^{-1}I)^p \ .$$

Theorem 4.2. *Let f be a K-quasiconformal self-mapping of a compact Riemann surface of genus g; then we have*

a) $L(f) \leq 2 + g(K^{1/2} + K^{-1/2})$

b) Assume $g \geq 2$. Then this bound is attained if and only if S is hyperelliptic and f is the hyperelliptic involution.

c) The bound is sharp in the sense that for any genus $g \geq 1$, there is a mapping class f, with dilatation $\bar{K}$, such that the identity $L(f) = 2 + g(\bar{K}^{1/2} + \bar{K}^{-1/2})$ holds.

PROOF:

Part a) follows from Theorem 3.8.i) and Lemma 4.1. along with the observation that the real function $t \mapsto t + 1/t$ is an increasing function for $t \geq 1$.

On the other hand, Theorem 3.8.ii) implies that, if $g \geq 2$, the bound can never be reached by a K-quasiconformal mapping f, if $K > 1$, i.e. if f is not conformal. Thus, part b) is now a consequence of the classical theorem of Hurwitz about the number of fixed points of a holomorphic automorphism, quoted in the introduction.

As for part c), it follows from the following gluing construction due to Bers ([Be]):

Let E be a complex torus and let $f : E \to E$ be an absolutely extremal self-mapping of dilatation K; this is induced by a linear mapping $A \in SL_2(\mathbf{Z})$ that satisfies trace$(A) = K^{1/2} + K^{-1/2}$ (see e.g. [Na] p.156). Then, for any $\varepsilon > 0$ there is a quasiconformal mapping $f_\varepsilon : E \to E$, isotopic to f and satisfying

$$K(f_\varepsilon) < K + \varepsilon$$
$$f_\varepsilon(\bar{0}) = \bar{0}$$

and there is a Jordan arc α emanating from $\bar{0}$ such that $f_{\varepsilon|\alpha} \equiv$ identity ([Be]), Lemma 7).

Now one can construct a Riemann surface S, by taking two copies E_1, E_2 of E and gluing them along the arc α . Thus, S will be a Riemann surface of genus 2, in which

α gives rise to a Jordan curve which is homology (but not homotopy) trivial, and that divides S into two homeomorphic holed tori.

Next, we can define a quasiconformal mapping $F_\varepsilon : S \to S$ by declaring that $F_{\varepsilon|E_i} = f_\varepsilon (i = 1, 2)$ ([Be], p.94).

We see that trace $\left(F_\varepsilon^\#\right) = 2$ trace $(f^\#)$, and that $K(F_\varepsilon)$ tends to K. This completes the proof of part c) for $g = 2$.

In order to prove it for genus $g = 3$, we take 3 copies of E and we, first, glue E_1 to E_2 around $\bar{0}$ and then E_2 to E_3 around a second fixed point of f, $P \neq \bar{0}$. It is clear that this construction can be carried on as long as we choose f with sufficiently many fixed points.

Corollary 4.3. *For elliptic or parabolic self-mappings we have $L(f) \leq 2 + 2g$.*

Example 4.4. (cases $g = 0, 1$)

i) When $g = 0$, it is clear that we always have $L(f) = 2$, as the first cohomology group is trivial in this case.
ii) When $g = 1$, it is well known that the bound is reached by those hyperbolic elements of $SL_2(\mathbf{Z})$ whose trace is negative (see [Nag] p. 156).

5. An example.

Probably the easiest way to show the existence of pseudo-Anosov diffeomorphisms on a compact surface of genus $g \geq 2$ is to take an Anosov diffeomorphism on the torus $f : E \to E$ and lift it to a suitable ramified cover $\pi : S \to E$ to obtain $\tilde{f} : S \to S$ satisfying $\pi \circ \tilde{f} = f \circ \pi$ (see [As] p. 245). As noted in this reference, the problem with this method is that it is difficult to keep track of the topological nature of $\tilde{f}$, e.g. of its action on homology.

The results of this paper show that there is a close relationship between dilatation, spectrum and number of fixed points of $\tilde{f}$. This interaction implies that, unless theoretically, knowledge of some of these data will provide information about the others. Below we work out a (very) special example in which the spectrum of $\tilde{f}$ can be completely determined.

Our diffeomorphism f is going to be given by the matrix $A = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix} \in SL_2(\mathbb{Z})$. It is well known that, since $|\text{trace}A| > 2, f$ is Anosov and, in fact, a complex structure that minimises its dilatation – and makes of f an absolutely extremal mapping – is given by $E = \dfrac{\mathbb{C}}{\mathbb{Z} \oplus \mathbb{Z}i}$. (This is seen, see [Nag] p. 156, by first finding out the two real points left fixed by the action of A as a Möbius transformation, which turn out to be $\dfrac{1+\sqrt{5}}{2}$ and $\dfrac{1-\sqrt{5}}{2}$; and then checking that the complex number i lies on the hyperbolic line through them).

We see that $L(f) = 2 - \text{trace} f^\# = 5$, and accordingly f has five fixed points, namely $0 = 0 + i0, e = \frac{1}{5} + i\frac{2}{5}, e_1 = \frac{2}{5} + i\frac{4}{5}, e_2 = \frac{3}{5} + i\frac{1}{5}, e_3 = \frac{4}{5} + i\frac{3}{5}$; all of which have Lefschetz number equal to $+1$. (We recall that the local Lefschetz number of a differentiable mapping

f at an isolated fixed point x equals the sign of the determinant of $(Df_x - Id)$, whenever this is non-zero, see [G-P]).

We also see that the dilatation (or expansion factor) of f is $K = \left(\frac{1+\sqrt{5}}{2}\right)^2$; this can be computed either directly or by arguing in a more geometric fashion as in [Nag] p. 156. Its eigenvalues are $-K^{1/2}$, $-K^{-1/2}$ and the corresponding eigenvectors are $v = \frac{1+\sqrt{5}}{2}\,dx - dy$, $\eta = dx + \frac{1+\sqrt{5}}{2}\,dy$.

As for S we shall take a fixed 2 to 1 cover of E ramified precisely over $0, e$. In other words S is going to be a compact Riemann surface of genus 2 admitting an involution J with precisely two fixed points $\tilde{0}$, $\tilde{e}$ so that the quotient surface $\frac{S}{\langle J \rangle}$ is isomorphic to E in such a way that the points $\tilde{0}$, $\tilde{e}$ map into the points $0, e$ respectively. We shall denote by $\pi : S \to E$ the corresponding projection, and by $\tilde{f}$, and $J \circ \tilde{f}$ the two lifts of f to S, i.e. $\tilde{f}$ is a selfmapping of S that satisfies $\pi \circ \tilde{f} = f \circ \pi$, or equivalently $J \circ \tilde{f} = \tilde{f} \circ J$. The existence of the cover S and the lift $\tilde{f}$ is justified in Lemma 5.2 below.

It is clear that $\tilde{f}$ is again an absolutely extremal selfmapping with same dilatation K, and so is that $\pi^* v$ and $\pi^* \eta$ are eigenvectors of $\tilde{f}$ with eigenvalues $-K^{1/2}$ and $-K^{-1/2}$ respectively. According to our previous results the associated quadratic differential must be $\Phi = \omega^2$, with

$$\omega = \pi^* v + i \pi^* \eta = \left(\frac{1+\sqrt{5}}{2} + i\right) \pi^* dz$$

Its Lefschetz number will be

$$L(\tilde{f}) = 2 + K^{1/2} + K^{-1/2} - (\mu + \mu^{-1}) = 5 - (\mu + \mu^{-1})$$

where μ and μ^{-1} are the two remaining eigenvalues of $\tilde{f}$ which by our results must satisfy $\mu + \mu^{-1} < K^{1/2} + K^{-1/2} = 3$; thus we have the following list of possibilities

$$\mu + \mu^{-1} = \begin{cases} 0 \\ \pm 1 \\ \pm 2 \end{cases}$$

Proposition 5.1. *Depending on which of the two liftings of f we are considering, we have either $\mu + \mu^{-1} = 1$ or $\mu + \mu^{-1} = -1$ $\left(\text{i.e. either } \mu = \frac{1+i\sqrt{3}}{2} \text{ or } \mu = -\frac{1+i\sqrt{3}}{2}\right)$.*

Accordingly its characteristic polynomial is either $P(\lambda) = \lambda^4 + 2\lambda^3 - \lambda^2 + 2\lambda + 1$ or $P(\lambda) = \lambda^4 + 4\lambda^3 + 5\lambda^2 + 4\lambda + 1$.

Of course we could prove this if we knew the number of fixed points of $\tilde{f}$ (and their local Lefschetz numbers). However this method presents the problem that, while it is clear that the only candidates to be fixed points of $\tilde{f}$ are the π-fibers of the fixed points of f, there is no way to decide whether the points in each fibre are actually fixed or interchanged by $\tilde{f}$. (In fact the behaviour of the liftings $\tilde{f}$ and $J \circ \tilde{f}$ will be opposite to each other).

This difficulty can be avoided if, instead of $\tilde{f}$, we consider the self-mapping $\tilde{f}^2 := f \circ f$.

Indeed the fixes set of $\tilde{f}^2$ consists of the set of $2+6$ points $\pi^{-1}(\{0, e, e_1 , e_2 , e_3\})$. This is because, on the one hand, it is clear that $\tilde{f}^2$ fixed each fibre pointwise and, on the other hand, it cannot fix more than that since $\tilde{f}^2$ is a lift of f^2 which has the same fixed point set as f, although each fixed point has now local Lefschetz number equal to -1. (Note that $L(f^2) = 2 - \mathrm{trace} f^2 = 2 - 7 = -5$).

Now we have

$$\begin{aligned} L(\tilde{f}^2) &= 2 - \{(K + K^{-1}) + (\mu^2 + \mu^{-2})\} = 2 - 7 - (\mu^2 + \mu^{-2}) \\ &= L_{\tilde{0}}(\tilde{f}^2) + L_{\tilde{e}}(\tilde{f}^2) - 6 \ ; \end{aligned}$$

the first identity because the eigenvalues of $\tilde{f}^2$ are the square of those of $\tilde{f}$, and the last one, because at the six points of $\pi^{-1}(\{e_1 , e_2 , e_3\})$ π is a local homeomorphism and therefore the local Lefschetz numbers of $\tilde{f}^2$ coincide with those of f^2 , of which it is a lift.

Similarly we have

$$\begin{aligned} L(J \circ \tilde{f}^2) &= 2 - \{(K + K^{-1}) - (\mu^2 + \mu^{-2})\} = -5 + (\mu^2 + \mu^{-2}) \\ &= L_{\tilde{0}}(J \circ \tilde{f}^2) + L_{\tilde{e}}(J \circ \tilde{f}^2). \end{aligned}$$

Now, to compute the trace of $J \circ \tilde{f}^2$, we have used the fact that the eigenvalues of J are $\lambda = 1$ (with eigenspace generated be $Re\omega$ and $Im\omega$), and $\lambda = -1$; whereas the last identity follows from the observation that the involution J permutes the fibers of π .

From here we rule out the possibilities $\mu + \mu^{-1} = 0, \pm 2$, and hence prove Proposition 5.1, by inserting into the expressions for $L(\tilde{f}^2)$ and $L(J \circ \tilde{f}^2)$ the following two ingredients:

(1) $$L_{\tilde{0}}(\tilde{f}^2) + L_{\tilde{0}}(J \circ \tilde{f}^2) = L_{\tilde{e}}(\tilde{f}^2) + L_{\tilde{e}}(J \circ \tilde{f}^2) = -2$$

This is because, putting $h = f^2$ and $\tilde{h} = \tilde{f}^2$, we can (locally) write:

$$\begin{aligned} 2\deg\left(\frac{h-z}{|h-z|}\right) &= \deg\left(\frac{h-z}{|h-z|} \circ \pi\right) = \deg\left(\frac{h\circ\pi - \pi(z)}{|h\circ\pi - \pi(z)|}\right) = \deg\left(\frac{\pi\tilde{h} - \pi(z)}{|\pi\circ\tilde{h} - \pi(z)|}\right) \\ &= \deg\left(\frac{(\tilde{h})^2 - z^2}{|(\tilde{h})^2 - z^2|}\right) = \deg\left(\frac{\tilde{h}-z}{|\tilde{h}-z|}\right) + \deg\left(\frac{\tilde{h}+z}{|\tilde{h}+z|}\right) \\ &= \deg\left(\frac{\tilde{h}-z}{|\tilde{h}-z|}\right) + \deg\left(\frac{\tilde{h}+z}{|\tilde{h}+z|} \circ J\right) = \deg\left(\frac{\tilde{h}-z}{\tilde{h}-z}\right) + \deg\left(\frac{J\circ\tilde{h}-z}{J\circ\tilde{h}-z}\right) \end{aligned}$$

(2) $$L_{\tilde{0}}(\tilde{f}^2) \text{ is equal to either } L_{\tilde{e}}(\tilde{f}^2) \text{ or } L_{\tilde{e}}(J \circ \tilde{f}^2).$$

The reason for this is that, with the notation as above, the local expressions of h and π at the points 0 and e coincide (the last one being $\pi(z) = z^2$); hence the same statement must hold for $\tilde{h}$, except, perhaps, for composition with $J(z) = -z$.

It only remains to be proved the following

Lemma 5.2. *The covering S and the mapping $\tilde{f}$ do exist.*

This is best done within the language of Füchsian groups (see [Har] and [Gon] p. 484).

Let Γ be a Füchsian group given by

$$\begin{aligned} &\text{Generators} : a, b, x_1 , x_2 \; ; \text{ and} \\ &\text{Relations } : aba^{-1}b^{-1} = x_1^2 = x_2^2 = 1 \end{aligned}$$

Then any group epimorphism $\phi : \Gamma \to \{\pm 1\}$ with $\phi(x_1) = \phi(x_2) = -1$ produces as kernel a free acting Füchsian group K. If the generators are suitably chosen then the natural covering $\mathbb{H}/K \to \mathbb{H}/\Gamma$ solves our problem. (Here, of course, $\mathbb{H}$ denotes the upper half plane).

The mapping f induces a natural group isomorphism $f_* : \Gamma \to \Gamma$ and a lift $\tilde{f}$ exists if and only if $\phi \circ f_* = \phi$. It is easily seen that for our $f = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$, and indeed for any $A \in SL_2(\mathbb{Z})$ whose trace is odd, a suitable homomorphism ϕ can be found.

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