On the structure of Whittaker sublocus of moduli space of algebraic curves*

Ernesto Girondo
Departamento de Matemáticas. Universidad Autónoma de Madrid
28049 Madrid, Spain
E-mail: ernesto.girondo@uam.es

Gabino González-Diez
Departamento de Matemáticas. Universidad Autónoma de Madrid
28049 Madrid, Spain
E-mail: gabino.gonzalez@uam.es

(MS received ;)

We prove the compactness of Whittaker sublocus of moduli space of Riemann surfaces (complex algebraic curves). This is the subset of points representing hyperelliptic curves which satisfy Whittaker's conjecture on the uniformization of hyperelliptic curves via monodromy of Fuchsian differential equations. In the last part of the article we drive our attention to the statement made by R.A. Rankin more than forty years ago to the effect that the conjecture "has not been proved for any algebraic equation containing irremovable arbitrary constants". We combine our compactness result with other facts coming from Teichmüller theory to show that in the most natural interpretations of this sentence we can think of, this is, in fact, impossible.

1. Introduction

The fact that a compact Riemann surface of genus $g > 1$ can be obtained either as quotient space of the unit disc $\mathbb{D}$ under the action of a Fuchsian group or as an algebraic curve is one of the corner stones of the theory of Riemann surfaces. Unfortunately, this correspondence can be made explicit only in few special cases (see e. g. [1], [2]).

Genus g hyperelliptic Riemann surfaces are simply double covers of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ ramified over $2g+2$ points. As algebraic curves, they are given by equations of the form $y^2 = f(x)$, where $f(x)$ is a polynomial in x with $2g+2$ different roots (or $2g+1$ if ∞ is a branch value).

There is a classical approach to the problem of finding the Fuchsian group uniformizing a given genus g hyperelliptic curve C of equation $y^2 = f(x)$. The strategy, which is based on ideas of H. Poincaré that were later retaken by E.T. Whittaker [15], consists of exploiting a known fact: if K uniformizes C, then K is an index two normal subgroup of a Fuchsian group Γ with signature $(0; 2, \ldots, 2)$. The determination of K is equivalent to that of Γ. But Γ turns out to be the monodromy

*Work partially supported by the MCyT research project BFM2003-04964.

group of a second order differential equation of the form

$$y'' + \frac{1}{2}S(z)y = 0,$$

where $S(z)$ is a certain rational function with double poles at the roots of f. Theoretically this equation is fully determined in terms of the algebraic curve, that is in terms of (the roots of) f, but the complete determination of this differential equation has remained an elusive problem for decades.

In the thirties, E.T. Whittaker [16], building on previous work of Burnside [2] that ultimately goes back to Fricke, Klein and Poincaré, proposed the following conjectural expression for the differential equation:

$$y'' + \frac{1}{2}W(z)y = 0,$$

with

$$W(z) := \frac{3}{8}\left(\left(\frac{f'(z)}{f(z)}\right)^2 - \frac{2g+2}{2g+1}\frac{f''(z)}{f(z)}\right).$$

The validity of this conjecture would have led to the solution of the uniformization problem for hyperelliptic curves.

As far as we know, the history of Whittaker conjecture goes roughly as follows:

It was early noticed that the analogous statement for genus one surfaces holds, as can be seen after some manipulation with Weierstrass $\wp$ function (see [18], p.439). Already in his 1929 paper [16] E.T. Whittaker checked the conjecture for the genus two curve $y^2 = x^5 + 1$. This result was extended one year later by M. Mursi [12] to the curve $y^2 = x^7 + 1$, of genus three, and to all curves of the form $y^2 = x^{2g+1} + 1$ by S.C. Dahr [5] in 1935. Other works on Whittaker's conjecture carried out about the same time are due to D.P. Dalzell [4], J. Hodgkinson [10] and Whittaker's son J.M. Whittaker [17]. It was not until some twenty years later that R.A. Rankin [13] enlarged substantially the list of known Whittaker surfaces, as they should be called in this paper, by proving the conjecture for a collection of curves "whose branch points" (the roots w_i of f and, possibly, the point at infinity) "form sets possessing certain symmetrical properties" in the author's own words. The conjecture was taken up by D.V. Chudnovsky and G.V Chudnovsky in their 1990 paper [3]. There, they performed a number of numerical experiments to find that a randomly chosen surface is most likely to be non-Whittaker. In fact, using tools from Teichmüller theory, I. Kra was able to prove in [11] that the coefficients in the rational function $S(z)$ depend only real-analytically (but not holomorphically) on the values w_i. As the coefficients in $W(z)$ clearly depend holomorphically on these values, it can be deduced that Whittaker's conjecture is not true in general.

So it was natural to believe (see e.g. [3]) that the only Whittaker surfaces were those encountered by Rankin (hyperelliptic surfaces *with many automorphisms*). Nevertheless, in [7] we showed that this prediction is not correct, as the existence of Whittaker surfaces without many automorphisms was proved.

In this article we study the subset of the moduli space of algebraic curves of given genus consisting of points representing hyperelliptic curves which satisfy Whittaker's conjecture. This subset shall be referred as the *Whittaker locus* and will be denoted by $\mathcal{W}$. Our main result is (see Theorem 3.1) that $\mathcal{W}$ is a compact real analytic space.

The motivation for this paper is the work of R.A. Rankin on the conjecture and, in particular, his 1958 article [13] (see also the nicely written survey article [14]). In its introduction he writes: *In particular, it* (Whittaker's conjecture) *has not been proved for any algebraic equation containing irremovable arbitrary constants.* In Theorem 4.1 we show that in the most natural interpretations of this sentence we can think of, this is, in fact, impossible.

2. Whittaker's conjecture on the uniformization of hyperelliptic surfaces via differential equations

Any hyperelliptic Riemann surface arises from a set $\{z_1, \ldots, z_{2g+2}\}$ of $(2g+2)$ points in the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, as a hyperelliptic algebraic curve. Its equation is given by $y^2 = \prod_{i=1}^{2g+2}(x - z_i)$ if $\{z_i, 1 \le i \le 2g+2\} \subset \mathbb{C}$ or $y^2 = \prod_{i=1}^{2g+1}(x - z_i)$ if, say, $z_{2g+2} = \infty$.

As any set of three points of the sphere can be always mapped, via a Möbius transformation, to any other set of three points, we shall normalize three of them to be $\{0, 1, \infty\}$ as usual.

Definition 2.1. We shall denote by W^{2g-1} the complement in $\hat{\mathbb{C}}^{2g-1}$ of the normalized diagonal subset $\Delta := \{w_j = 0, 1, \infty \text{ for some } j\} \cup \{w_j = w_i, i \neq j\}$.

Thus, any hyperelliptic curve is isomorphic to one of the form

$$y^2 = x(x-1)(x-w_4)\cdot \cdots \cdot (x - w_{2g+2}), \tag{2.1}$$

where $w = (w_4, \ldots, w_{2g+2}) \in W^{2g-1}$.

In fact, it can be shown that the moduli space of hyperelliptic curves of genus g is obtained as a quotient space of W^{2g-1} modulo the action of the symmetric group Σ_{2g+2} which acts on it in a natural Möbius fashion (see [8]).

Definition 2.2. We shall denote by C_w the curve given by equation (2.1), and by f_w the polynomial $x(x-1)\cdot \cdots \cdot (x - w_{2g+2})$.

The *hyperelliptic involution* is given in C_w by $J(x,y) = (x, -y)$. Its action induces the *hyperelliptic function* $\pi : C_w \to \frac{C_w}{<J>} \simeq \hat{\mathbb{C}}$, given by $\pi(x,y) = x$.

From the point of view of uniformization, there is a torsion free Fuchsian group $K = K_w$ such that the universal covering map $p : \mathbb{D} \to C_w$ is represented by the obvious projection $\mathbb{D} \to \mathbb{D}/K \simeq C_w$. Likewise, the map $X := \pi \circ p : \mathbb{D} \to \hat{\mathbb{C}}$ corresponds to a projection of the form $\mathbb{D} \to \mathbb{D}/\Gamma \simeq \hat{\mathbb{C}}$, where Γ is a Fuchsian group generated by $2g+2$ order two elements, say $\gamma_1, \ldots, \gamma_{2g+2}$. The group K is then the subgroup generated by all products $\gamma_i\gamma_j$. A possible strategy to determine Γ is based on the use of the following differential operator:

Definition 2.3. Given a holomorphic function φ, its *Schwarzian derivative* is given by $S(\varphi)(z) = \dfrac{\varphi'''(z)}{\varphi'(z)} - \dfrac{3}{2}\left(\dfrac{\varphi''(z)}{\varphi'(z)}\right)^2$.

Two important features of S are its behaviour with respect to composition and the fact that it characterizes Möbius transformations. Namely

- $S(\varphi) \equiv 0 \Leftrightarrow \varphi$ is a Möbius transformation.
- $S(\varphi \circ \phi)(z) = \phi'(z)^2 S(\varphi)(\phi(z)) + S(\phi)(z)$.

Despite the fact that the inverse of the map X is a multivalued function from $\hat{\mathbb{C}}$ to $\mathbb{D}$, its Schwarzian derivative $S(X^{-1})$ is single valued. The reason is that any two branches f, g of X^{-1} are related by $f = \gamma \circ g$ for some $\gamma \in \Gamma$ and therefore the above two properties of S yield $S(f)(z) = S(g)(z)$.

Now a fundamental fact (see e.g. [6]) is that Γ, that is the covering group of the map $X : \mathbb{D} \longrightarrow \widehat{\mathbb{C}}$, is obtained as the monodromy group of the second order Fuchsian differential equation

$$y''(z) + \frac{1}{2}(S(X^{-1})(z))y(z) = 0. \tag{2.2}$$

The quotient $Y(z) = \frac{y_1(z)}{y_2(z)}$, where y_1, y_2 are two linearly independent solutions of equation (2.2), verifies $S(Y)(z) = S(X^{-1})(z)$ (see [6]). Therefore $Y = T \circ X^{-1}$ for a Möbius transformation T, again by a combination of the two properties of S mentioned above.

If we could determine $S(X^{-1})$ in terms of the curve C_w, we could then obtain Γ (and X up to Möbius transformation), and therefore K. However the determination of $S(X^{-1})$ in terms of w has turned out to be a very elusive problem for more than seventy years.

Let $w \in W^{2g-1}$, and let S_w denote the Schwarzian derivative $S(X^{-1})$, where X is as in the previous section. By examining the local behaviour at the branch points of X, namely $0, 1, \infty, w_4, \ldots, w_{2g+2}$, it can be shown (see [13], p. 37) that

$$S_w(z) = \frac{3}{8}\left(\frac{h(z)}{f_w(z)} + \frac{1}{z^2} + \frac{1}{(z-1)^2} + \sum_{i=4}^{2g+2} \frac{1}{(z-w_i)^2}\right), \tag{2.3}$$

where h is a certain polynomial of the form $h(z) = -2gz^{2g-1} + c^w_{2g-2}z^{2g-2} + \cdots + c^w_1 z + c^w_0$, and f_w is given in definition 2.2.

The $2g-1$ coefficients $c^w_0, c^w_1, \ldots, c^w_{2g-2}$ are classically known as the *accessory parameters.*

In 1929 [16] E.T. Whittaker conjectured that $S_w(z) = W_w(z)$, where

$$W_w(z) := \frac{3}{8}\left(\left(\frac{f'_w(z)}{f_w(z)}\right)^2 - \frac{2g+2}{2g+1}\frac{f''_w(z)}{f_w(z)}\right). \tag{2.4}$$

From now on we shall employ the following notation and terminology: The hyperelliptic Riemann surface C_w is a *Whittaker surface* if $S_w = W_w$, i.e. if C_w satisfies

Whittaker's conjecture. Accordingly, $w \in W^{2g-1}$ is a *Whittaker point* if C_w is a Whittaker surface. We will refer to W_w as *Whittaker rational function.*

Also, we shall denote by $\mathcal{W}$ the subset of points in W^{2g-1} corresponding to curves which satisfy Whittaker's conjecture and by $\widetilde{\mathcal{W}}$ its image in the moduli space of hyperelliptic curves W^{2g-1}/Σ_{2g+2}.

Remark 2.4. From equation 2.4, we find an expression for W_w that will be useful later on, namely

$$\begin{aligned} W_w(z) &= \tfrac{3}{8}\Big\{\tfrac{1}{z^2} + \tfrac{1}{(z-1)^2} + \textstyle\sum_{i=4}^{2g+2} \tfrac{1}{(z-w_i)^2} - \tfrac{2}{2g+1}\Big(\tfrac{1}{z(z-1)} \\ &\quad + \textstyle\sum_{i=4}^{2g+2}(\tfrac{1}{z} + \tfrac{1}{z-1})(\tfrac{1}{z-w_i}) + \sum_{i\neq j} \tfrac{1}{(z-w_i)(z-w_j)}\Big)\Big\} \end{aligned} \tag{2.5}$$

which, in turn, can be rewritten as

$$W_w(z) = \frac{3}{8}\left(\frac{q(z)}{f_w(z)} + \frac{1}{z^2} + \frac{1}{(z-1)^2} + \sum_{i=4}^{2g+2} \frac{1}{(z-w_i)^2}\right), \tag{2.6}$$

where $q(z) = -2gz^{2g-1} + q^w_{2g-2}z^{2g-2} + \cdots + q^w_1 z + q^w_0$.

Now, comparing equations 2.3 and 2.6 we find

$$S_w(z) - W_w(z) = \frac{A_{2g-2}(w)z^{2g-2} + \cdots + A_1(w)z + A_0(w)}{f_w},$$

where $A_i(w) = c^w_i - q^w_i$. It is known (see [11]) that c^w_i depends real-analytically on $w_4, \ldots, w_{2g+2}$. Note also that q^w_i is simply a polynomial in $w_4, \ldots, w_{2g+2}$.

Thus $\mathcal{W}$ can be described as the common zero set of the real-analytic functions A_i, namely:

$$\mathcal{W} = \{A_{2g-2}(w) = \cdots = A_0(w) = 0\} \subset W^{2g-1}.$$

These functions A_i are, in principle, complex valued functions, but when w is real $A_i(w)$ is also real. This is because the Schwarzian derivative behaves nicely under conjugation, namely $S_{\bar{w}}(\bar{z}) = \overline{S_w(z)}$ (see [9]).

This allows us to give a description of the locus of real Whittaker points in terms of real valued analytic functions. More precisely we have

$$\mathcal{W} \cap \mathbb{R}^{2g-1} = \{\text{Re } A_{2g-2} = \cdots = \text{Re } A_0 = 0\} \cap \mathbb{R}^{2g-1}.$$

3. Whittaker locus is compact

This section is devoted to the proof of our main result, namely

Theorem 3.1. *$\mathcal{W}$ is a compact real analytic variety.*

Note that w approaches Δ exactly when some of the w_i approaches $0, 1$ or ∞, or when w_j and w_k tend to get close to each other. Therefore, we have to study the behaviour of both S_w and W_w under degeneration (coalescing of points) of the branching value set of X, namely $\{0, 1, \infty, w_4, \ldots, w_{2g+2}\}$.

3.1. Behaviour of S_w under degeneration: Kra's theorem

Let $\{w(n) = (w_4(n), \ldots, w_{2g+2}(n))\} \in W^{2g-1}$ be a sequence, and let $X_n : \mathbb{D} \to \widehat{\mathbb{C}}$ be the regular covering map constructed as in section 2 from $C_{w(n)}$. The branch value set of X_n is $A_n = \{\lambda_1(n), \ldots, \lambda_{2g+2}(n)\}$, where $\lambda_1(n) \equiv 0, \lambda_2(n) \equiv 1, \lambda_3(n) \equiv \infty$, and the branching indices are all equal to 2.

Assume that $\lim_{n\to\infty} \lambda_j(n) = \lambda_j(\infty)$ exists for $j = 1, \cdots, 2g+2$. Denote, relabeling if necessary, $A_\infty = \{\lambda_1(\infty), \ldots, \lambda_N(\infty)\}$ the maximal subset that consists of distinct elements of $\{\lambda_1(\infty), \ldots, \lambda_{2g+2}(\infty)\}$ (the sequence could have gone to the boundary Δ).

Set $\mu_j = 2$ if a single sequence $\lambda_j(n)$ converges to $\lambda_j(\infty)$, and $\mu_j = \infty$ otherwise. Denote by $X_\infty : \mathbb{D} \longrightarrow \widehat{\mathbb{C}}$ the regular cover corresponding to this limit branching data. The following result is a particular case of Kra's theorem on degeneration of uniformizing connections (see page 603 of [11]):

Theorem 3.2 (Kra). *If $1/\mu_1 + \cdots + 1/\mu_N < N-2$, i.e. if $(0; \mu_1, \ldots, \mu_N)$ is a hyperbolic signature, then*

$$S(X_\infty^{-1})(z) = \lim_{n\to\infty} S(X_n^{-1})(z)$$

uniformly on compact subsets of $\widehat{\mathbb{C}}$.

As an application, one has:

Corollary 3.3. *Let $\{w(n) = (w_4(n), \ldots, w_{2g+2}(n))\} \in W^{2g-1}$ be a sequence such that $\lim_{n\to\infty} w_j(n)$ exists for $j = 4, \ldots, 2g+2$. Let $\lambda_1(n) = 0, \lambda_2(n) = 1, \lambda_3(n) = \infty$, and $\lambda_k(n) = w_k(n)$ for $4 \le k \le 2g+2$. Set $\{\lim_{n\to\infty} \lambda_j(n), 1 \le j \le 2g+2\} = \{\lambda_1(\infty), \ldots, \lambda_N(\infty)\}$, with $\lambda_1(\infty) = 0, \lambda_2(\infty) = 1, \lambda_3(\infty) = \infty$. Suppose further that $3 \le N < 2g+2$, and that the limits of $w_k(n)$ are not all equal in the case $N = 3$. Then, at least one of the following statements holds:*

i) $\lim_{n\to\infty} S(X_n^{-1})(1/z) = \frac{1}{2}z^2 +$ *higher order terms in z.*

ii) There exists $\lambda \in \hat{\mathbb{C}}$ such that

$$\lim_{n\to\infty} S(X_n^{-1})(z) = \frac{1/2}{(z-\lambda)^2} + \text{ higher order terms in } (z-\lambda).$$

Proof.- Note first that the condition on the set $\{\lim_{n\to\infty} \lambda_j(n), 1 \le j \le 2g+2\}$ implies that the limit signature $(0; \mu_1, \ldots, \mu_N)$ is necessarily hyperbolic. This is because $\mu_j = \infty$ for some j, hence we have

$$\sum_{j=1}^{N} \frac{1}{\mu_j} \le \frac{N-1}{2},$$

and this is strictly less than $N-2$ for $3 < N$. But for $N = 3$ at least two of the indexes $\{\mu_1, \mu_2, \mu_3\}$ equal ∞, hence $1/\mu_1 + 1/\mu_2 + 1/\mu_3$ equals either 0 or 1/2 (both smaller than $(N-2) = 1$). Therefore, we can apply Theorem 3.2 to study $\lim_{n\to\infty} S(X_n^{-1})$.

Now, if $\lim_{n\to\infty}\lambda_k(n)=\infty$ for some k with $4\le k\le 2g+2$, then $\mu_3(\infty)=\infty$, which means that ∞ is a parabolic value. If, on the contrary, $\lim_{n\to\infty}\lambda_k(n)\ne\infty$ for every $k>3$, then, as $N<2g+2$, there is an index $j\ne 3$ such that $\mu_j=\infty$, which means that in this case $\lambda=\lambda_j(\infty)$ is a parabolic value. A straightforward computation in local coordinates similar to the one needed to obtain equation 2.3 gives i) in the first case and ii) in the second one (see e.g. [11] 2.3.1) □

3.2. Behaviour of W_w under degeneration

We now study the effect of coalescing of branch points on Whittaker rational function W_w. Thus, let $w(n)=(w_4(n),\dots,w_{2g+2}(n))$ be a sequence in W^{2g-1} that converges to some point $w(\infty):=(w_4(\infty),\dots,w_{2g+2}(\infty))\in\hat{\mathbb{C}}^{2g-1}$, where $\lim_{n\to\infty}w_j(n)=w_j(\infty)$.

Let $W(n)=W_{w(n)}$, and $W_\infty=\lim_{n\to\infty}W(n)$.

Lemma 3.4. *If $k-1$ of the coordinates of $w(\infty)$ equal ∞, then*

$$W_\infty(1/z)=\left(\frac{6k+6gk-3k^2}{16g+8}\right)z^2+\textit{higher order terms in } z.$$

Also, if $k-1$ of the coordinates of $w(\infty)$ equal λ, where $\lambda=0$ or 1, or if k of the coordinates of $w(\infty)$ equal $\lambda\ne 0,1$, then

$$W_\infty(z)=\left(\frac{6k+6gk-3k^2}{16g+8}\right)\frac{1}{(z-\lambda)^2}+\ \textit{higher order terms in } (z-\lambda).$$

Proof.- The result follows easily taking the corresponding limits in expression 2.5. □

3.3. The proof of Theorem 3.1

We will use Corollary 3.3 and Lemma 3.4 to compare the behaviour of S_w and W_w under degeneration, which corresponds to approaching the diagonal Δ in $\mathbb{C}^{2g-1}$. We will need the following elementary

Lemma 3.5. *The equation*

$$\frac{6k+6gk-3k^2}{16g+8}=\frac{1}{2}$$

admits no solution in the variables $(g,k)\in\mathbb{N}^2$ with $g>1$.

Proof.- The original equation is equivalent to

$$g=\frac{3k^2-6k+4}{6k-8},$$

which in turn yields

$$g=-1+\frac{3k^2-4}{6k-8}.$$

It follows that an integer solution must have k even, say $k = 2j$. But then

$$\frac{3k^2-4}{6k-8} = \frac{3j^2-1}{3j-2} = j + \frac{2j-1}{3j-2}$$

is an integer only for $j = 1$, as $\frac{2j-1}{3j-2} < 1$ when $j > 1$. Now, $j = 1$ yields $k = 2, g = 1$, and we are done. □

Lemma 3.6. *Let $w \in W^{2g-1}$, and T a Möbius transformation such that the set $\{0, 1, \infty, w_4, \ldots, w_{2g+2}\}$ is sent via T to $\{0, 1, \infty, \tilde{w}_4, \ldots, \tilde{w}_{2g+2}\}$. Denote $\tilde{w} = (\tilde{w}_4, \ldots, \tilde{w}_{2g+2})$. Then w is a Whittaker point if and only if so is $\tilde{w}$.*

Proof.- It follows from the fact that Whittaker rational function behaves as a Schwarzian derivative with respect to composition with Möbius transformations (see [13] and Proposition 2.4 in [7]). Therefore

$$S_w(z) - W_w(z) = T'(w)^2 \left[S_{\tilde{w}}(T(z)) - W_{\tilde{w}}(T(w))\right],$$

and the proof is complete. □

Proof of Theorem 3.1.- We already know that $\mathcal{W}$ is an analytic space (see remark 2.4) so we only have to show that it is compact. If it were not, there would be a sequence of Whittaker points $w(n) \in \mathcal{W} \subset W^{2g-1}$ that accumulates in $w \in \Delta$. Take then a subsequence, labeled again $w(n)$, tending to w.

Suppose that the coordinates $w_4, \ldots, w_{2g+2}$ of w are not all identical or, if they are, that $w_4 = \cdots = w_{2g+2} \notin \{0, 1, \infty\}$. We have $S_{w(n)} \equiv W_{w(n)}$ for all n, as $w(n)$ are Whittaker points. On the other hand we can apply Corollary 3.3 and Lemma 3.4 to obtain information about $\lim_{n\to\infty} S_{w(n)}$ and $\lim_{n\to\infty} W_{w(n)}$ separately. In this way we see that Lemma 3.5 yields a contradiction.

The problem when $w_4 = \cdots = w_{2g+2} = x$ with $x = 0, 1$ or ∞ is that we cannot apply directly Corollary 3.3. Now, the cases $x = 1$ and $x = \infty$ are reduced to the case $x = 0$ by taking $T(z) = (z-1)/z$ or $T(z) = 1/z$ and applying Lemma 3.6.

Therefore, it is enough to deal with the case $w_k = 0$ for $k = 4, \ldots, 2g+2$. Passing to a subsequence we can assume that $\lim \frac{w_5(n)}{w_4(n)} = a \neq \infty$ (otherwise interchange the roles of w_4 and w_5). Now, if $T_n(z) = z/w_4(n)$, we have

$$T_n(\{0, 1, \infty, w_4(n), \ldots, w_{2g+2}(n)\}) = \left\{0, 1, \infty, \frac{1}{w_4(n)}, \frac{w_5(n)}{w_4(n)}, \cdots, \frac{w_{2g+2}(n)}{w_4(n)}\right\},$$

and thus Lemma 3.6 shows that $\{\tilde{w}(n)\}$, with

$$\tilde{w}(n) = \left(\frac{1}{w_4(n)}, \frac{w_5(n)}{w_4(n)}, \ldots, \frac{w_{2g+2}(n)}{w_4(n)}\right)$$

is also a sequence of Whittaker points.

Take a subsequence $\{w(n_m)\}$ for which the limits $\lim_{m\to\infty}(w_k(n_m)/w_4(n_m))$ exist for $k = 4, \ldots, 2g+2$. Thus, $\tilde{w}(n)$ tends to a point $\tilde{w} = (\tilde{w}_4, \ldots, \tilde{w}_{2g+2})$ for which $\tilde{w}_4 = \infty$ and $\tilde{w}_5 = a \neq \infty$. But this is already a contradiction, since we know by the first part of the proof that such a sequence of Whittaker points can't exist. □

4. Families of Whittaker surfaces

We can use Theorem 3.1 to obtain information about how a family of Whittaker surfaces may look like (or rather about how it can't look like). Our findings point in the same direction as the statement made by Rankin in [13] more than forty years ago to the effect that the conjecture *has not been proved for any algebraic equation containing irremovable arbitrary constants.*

Theorem 4.1. *Whittaker's conjecture cannot be satisfied for all members of a family of hyperelliptic curves* $\{C_\lambda\}_{\lambda\in\Lambda}$ *whose parameter space* $\Lambda \subset W^{2g-1}$ *verifies any of the following conditions:*

i) Λ *is unbounded.*

ii) Λ *is not relatively compact.*

iii) Λ *contains infinitely many points of a polynomially parametrized arc inside* W^{2g-1}.

iv) Λ *is a complex analytic subspace of* W^{2g-1} *of positive dimension.*

v) Λ *is an algebraic subset of* W^{2g-1} *of positive dimension.*

Proof.- Case v) is a particular case of iv). For iv), simply note that near a non singular point λ_0, the analytic space can be described as the image of a small ball $B \subset \mathbb{C}^r$, with $r \leq 2g-1$, by a holomorphic map

$$\begin{array}{rcl} B: & \longrightarrow & W^{2g-1} \\ z & \longmapsto & \lambda(z) \end{array}.$$

Now, use (see [11]) that the accessory parameters are not holomorphic but just real analytic functions of the branching points.

i) is a particular case of ii). With respect to ii), it is enough to note that if Λ is contained in the compact set $\mathcal{W}$, its closure must be compact.

As for iii), suppose $\Lambda \supset \{P(t_n)\}$, where $\{t_n\}$ is a nontrivial sequence in $\mathbb{R}$ and $P(t) = (P_4(t), \ldots, P_{2g+2}(t))$, where each P_k is an one variable polynomial with complex or real coefficients.

If $\{t_n\}$ is unbounded, then Λ is unbounded too and we apply i). If not, passing to a subsequence if necessary, we have $\lim_{n\to\infty} t_n = t_0$. If $P(t_0) \in \Delta$, apply ii). If $P(t_0) \in W^{2g-1}$ then consider the difference $S_{P(t)}(z) - W_{P(t)}(z)$, which (see section 2) has the form

$$S_{P(t)}(z) - W_{P(t)}(z) = \frac{A_{2g-2}(P(t))z^{2g-2} + \cdots + A_1(P(t))z + A_0(P(t))}{f_{P(t)}}.$$

Now, Λ is a set of Whittaker points, hence each function $\mathrm{Re}(A_k(P(t)))$ is a real analytic function near t_0 of which t_0 is limit of zeros. Therefore, if $P(\mathbb{R}) \subset W^{2g-1}$, then each function $\mathrm{Re}(A_k(P(t)))$ identically vanishes, and thus $\lim_{t\to\infty} \mathrm{Re}(A_k(P(t))) = 0$ for each k. Same argument applied to $\mathrm{Im}(A_k(P(t)))$ allows us to conclude that

$P(\mathbb{R})$, which is obviously unbounded in W^{2g-1}, would be a set of Whittaker points, thus contradicting i). If, on the contrary $P(\mathbb{R})$ intersects Δ, denote by t' the first $t > t_0$ such that $P(t) \in \Delta$ and apply the argument above to $P((t_0 - \epsilon, t'))$. The conclusion is now a consequence of ii). □

References

1 R. Brooks, H.M. Farkas and I. Kra, *Number theory, theta identities, and modular curves.* Extremal Riemann surfaces (San Francisco, CA, 1995), 125–154, Contemp. Math. **201**. Amer. Math. Soc., Providence, RI, 1997.

2 W. Burnside, *Note on the equation* $y^2 = x(x^4 - 1)$. Proc. London Math. Soc. **24** (1893), no. 1, 17–20.

3 D.V. Chudnovsky and G.V. Chudnovsky, *Computer algebra in the service of mathematical physics and number theory.* Computers in mathematics (Stanford, CA, 1986), 109–232, Lecture Notes in Pure and Appl. Math., 125, Dekker, New York, 1990.

4 D.P. Dalzell, *A note on automorphic functions.* J. London Math. Soc. **5** (1930), 280–282.

5 S.C. Dhar, *On the Uniformization of a special kind of algebraic curves of any genus.* J. London Math. Soc. **10** (1935), 259–263.

6 L.R. Ford, *Automorphic functions.* Chelsea Publishing Company, New York, 1951.

7 E. Girondo, G. González-Diez, *On a conjecture of Whittaker concerning uniformization of hyperelliptic curves.* Trans. Amer. Math. Soc. **356** (2004), no. 2, 691–702.

8 G. González-Diez, *Loci of curves which are prime Galois coverings of* $\mathbb{P}^1$. Proc. London Math. Soc., **62** (1991), no. 3, 469–489.

9 J.A. Hempel, *On the uniformization of the* n*-punctured sphere.* Bull. London Mat. Soc. **20** (1988), 97–115.

10 J. Hodgkinson, *Note on the uniformization of hyperelliptic curves.* J. London Math. Soc. **11** (1936), 185–192.

11 I. Kra, *Accesory parameters for punctured spheres.* Trans. Amer. Math. Soc. **313** (1989), no. 2, 589–617.

12 M. Mursi, *On the Uniformisation of Algebraic Curves of Genus* 3. Proc. Edinburgh Mat. Soc. **2** (1930), 101–107.

13 R.A. Rankin, *The differential equations associated with the uniformization of certain algebraic curves.* Proc. Roy. Soc. Edinburgh Sect. A **65** (1958), 35–62.

14 R.A. Rankin, *Sir Edmund Whittaker's work on automorphic functions.* Proc. Edinburgh Mat. Soc. **11** (1958), 25–30.

15 E.T. Whittaker, *On the connexion of algebraic functions with automorphic functions.* Phil. Trans. **192A** (1898), 1–32.

16 E.T. Whittaker, *On hyperlemniscate functions. A family of automorphic functions.* J. London Math. Soc. **4** (1929), 274–278.

17 J.M. Whittaker, *The uniformisation of algebraic curves.* J. London Math. Soc. **5** (1930), 150–154.

18 E.T. Whittaker, G.N. Watson, *A course of modern analysis. An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions.* 4th. ed., Cambridge U. Press, Cambridge, 1927.