ON FAMILIES OF ALGEBRAIC CURVES WITH AUTOMORPHISMS

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Abstract. We exploit the Fuchsian group representation of the monodromy data for a Riemann surface with genus $g > 0$ as a cyclic branched covering of the sphere to construct a family of surfaces modelled on this structure as a quotient of the corresponding Teichmüller family of genus g. A universal property of the family is proved and some examples are given which indicate the nature of the general problem and the rationale underlying our approach.

Introduction

It is often valuable in the complex geometric setting to represent a family of genus g Riemann surfaces in terms of a single group action on a simply connected (total) space, where the group has a product structure incorporating the fundamental groups of base and fibre; such a representation is a consequence of the exact homotopy sequence for surface bundles. Even in simple cases, however, this process is not easy to make explicit geometrically and the relationship between geometric information such as monodromy and topological invariants is often poorly understood.

It is well known that the moduli space functor of isomorphism classes of compact Riemann surfaces (or nonsingular algebraic curves) of given genus g is not representable, at least in the category of algebraic varieties or complex manifolds. Grothendieck showed that in order to make this a representable functor one has to rigidify the problem by fixing a marking on each Riemann surface S, an extra topological structure which serves to distinguish it from all neighbouring structures. In [Gro] it is proved that the functor of isomorphism classes of Teichmüller marked surfaces is represented by the Teichmüller curve $\mathcal{V} = \{V_g \to T_g\}$; thus any holomorphic family of marked surfaces of genus g over a complex base manifold, $\mathcal{E} = \{E \to B\}$, is expressible up to isomorphism by pullback of the universal genus g family $\mathcal{V}$ through a holomorphic map $\phi : B \to T_g$, such that $\phi * (\mathcal{V}) \cong \mathcal{E}$. As an appendix in that work, Serre showed that if, instead of marking the fundamental group $\pi_1(S)$, one fixes a level $\ell \geq 3$ and uses the homology group $H_1(S, \mathbb{Z}/\ell\mathbb{Z})$, then one still obtains a representable functor giving rise to the *universal family of curves with level ℓ structure*, $\mathcal{C}_g(\ell)$).

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The results of [Gro] were extended to surfaces with punctures (or marked points) by Engber ([Eng]), working along similar lines, and by Earle([E]), who employed the tools of Teichmüller theory and complex geometry. Later Earle and Fowler([E-F]) generalised their results to families of open Riemann surfaces with base an arbitrary complex Banach manifold, by strengthening the procedure for making a fibration rigid.

The purpose of this article is to obtain appropriate modular families of surfaces which carry a specific type of automorphism within the Teichmüller framework and to deduce that the functor of isomorphism classes of cyclic covers of $\mathbb{P}^1$ with given ramification structure at a marked finite set of branch points is also representable (Theorems 5.2, 5.3). We give examples showing that this property cannot hold in general for such families if the quotient surface is not $\mathbb{P}^1$.

For cyclic covers of degree n with $r \geq 3$ branch points, the universal curve representing this functor will be a complex fibre space with base the affine space Ω^{r-3}, which denotes $\mathbb{C}^{r-3}$ with the diagonal set $\Delta = \{\lambda_i = 0, 1, \lambda_j; i \neq j\}$ removed, and with fibre over a point $(\lambda_3, ..., \lambda_{r-1})$ the Riemann surface with affine algebraic equation of the form

$$y^n = x^{d_1}(x-1)^{d_2}(x-\lambda_3)^{d_3}...(x-\lambda_{r-1})^{d_{r-1}},$$

where ∞ is a point of branching with index d_r These families arise in many contexts, for instance in the foundational paper of A. Kuribayashi [Kuri] on symmetric Teichmüller -families and more recently in the work of R. Holzapfel, P. Deligne and G.D. Mostow and others [Holz], which study number-theoretic aspects of the monodromy problem for certain differential operators of hypergeometric type associated with families of algebraic curves. They discover certain compactifications of our families in the 2 dimensional case ($r = 5$) which arise from arithmetic quotients of the complex unit ball first studied by E. Picard and his students.

1. The universal family of complex tori

As a simple prologue, to indicate the nature of the problem and exhibit our methods, we examine briefly the well known case of genus 1. Consider the Riemann surface family given by complex tori $E_\tau = \mathbb{C}/\Lambda_\tau$, with τ in the upper half plane $\mathbb{H}$ and Λ_τ the lattice $\mathbb{Z} + \mathbb{Z}\tau$. There is a $\mathbb{Z}_2$-cyclic covering $E_\tau \to \mathbb{P}^1$ induced by the elliptic involution automorphism $J(z) = -z$ which fixes the four points of order two $0, \frac{\tau}{2}, \frac{1+\tau}{2}$, and $\frac{1}{2}$. In our terminology, the Riemann surfaces E_τ endowed with this extra data are a Teichmüller family of *4-pointed symmetric surfaces* over $\mathbb{H}$.

Thus we have a fibre space $\mathbf{E} = (\mathbb{H} \times \mathbb{C})/\mathbb{Z}^2$ over $\mathbb{H}$, where the lattice group $L = \mathbb{Z}^2$ acts on $\mathbb{H} \times \mathbb{C}$ by $(n, m)(\tau; z) = (\tau; z + n + m\tau)$, and the fibre over τ is precisely E_τ.

Corresponding to the four fixed points of J, we have four holomorphic sections of the family $\mathbf{E} \to \mathbb{H}$, given by

$$s_1(\tau) = 0, \qquad s_2(\tau) = \frac{\tau}{2}, \qquad s_3(\tau) = \frac{1+\tau}{2}, \qquad s_4(\tau) = \frac{1}{2}.$$

The modular group $\Gamma = SL(2, \mathbb{Z})$ acts (discretely and properly) on $\mathbb{H}$ as Möbius transformations $A(\tau) = \frac{a\tau+b}{c\tau+d}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Two points τ, $\tau' \in \mathbb{H}$ are in the same $SL_2(\mathbb{Z})$-orbit if and only if the complex tori E_τ and E'_τ are isomorphic. This action extends to an action of Γ on the fibre space $\mathbf{E}$ by the following rule for the semidirect product group $\Gamma \cdot L$ on the universal cover $\mathbb{H} \times \mathbb{C}$:-

$$[A,(m,n)](\tau;z) = (A(\tau);(c\tau+d)^{-1}(z+m\tau+n))$$

Here the group extension $\Gamma \cdot L$ is specified by the action of Γ on the normal subgroup L via the rule

$$(m,n) \mapsto (m,n)A = (am+cn, bm+dn).$$

Now consider the level-2 congruence subgroup $\Gamma(2)$ of $SL(2,\mathbb{Z})$, comprising matrices $A \equiv Id \pmod 2$. Each orbit of this subgroup in $\mathbb{H}$ consists of points whose corresponding Riemann surfaces are isomorphic as pointed symmetric surfaces, ie. two points τ and τ' are in the same $\Gamma(2)$-orbit if and only if the corresponding surfaces E_τ and E'_τ are related by an isomorphism which commutes with the automorphism J and preserves all the points of order two. Accordingly, the universal family of symmetric surfaces of genus 1 we seek should have as base the 3-pointed sphere $\mathbb{H}/\Gamma(2)$. However, in order to achieve the corresponding family of complex tori, simply lifting the action of $\Gamma(2)$ to the fibre space $\mathbf{E}$ is inadequate, because on examining the extended group action of $\Gamma(2) \cdot L$, one sees that the (central) element $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ of $\Gamma(2)$, while acting trivially on $\mathbb{H}$, induces the automorphism J on each fibre torus, so that the family so obtained has fibres of genus zero, $E_\tau/\langle J\rangle \sim \mathbb{CP}^1$. This shows that we must find a way to remove the action of the element $-I$, while keeping the lattice L-part of the action. To be more precise, in the notation of later sections, we denote the group which represents the action of $\Gamma(2)$ on $\mathbf{E}$ by $P(J)$: what we need to find is a subgroup $P'(J)$ of $P(J)$ such that on the one hand $P'(J)$ is isomorphic to $P(J)/\langle J\rangle$, so that the factor space of its action on the base $\mathbb{H}$ is still $\mathbb{H}/\Gamma(2)$, but on the other hand $P'(J)$ does not contain the central element J induced by the matrix $-I$. In fact the existence of a universal pointed symmetric family of cyclic covers is equivalent to the existence of such a subgroup (Theorem 5.4).

In this example, a subgroup $P'(J)$ exists because $\Gamma(2)/\langle \pm Id\rangle$ is a free group. We could take, for instance, the group generated by the elements $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$. However, there is a more systematic way to reach this goal which illustrates our approach in the general case:

Each surface E_τ is an elliptic curve with equation $y^2 = x(x-1)(x-\lambda(\tau))$, where $\lambda : \mathbb{H} \to \mathbb{C}$ is the Legendre modular function. The elliptic involution J representing the action of the element $-I \in \Gamma(2)$ is given by $J(x,y) = (x,-y)$, which implies that one way to exclude J and obtain the desired group $P'(J)$ is to consider the subgroup of $\Gamma(2)$ consisting of those elements whose action on the (algebraic model) of the family $\mathbf{E}$ leaves invariant the y-coordinate. One sees that this subgroup is the kernel of an order 2 character of the group $\Gamma(2)$. A similar approach succeeds in the case of cyclic covers of $\mathbb{P}^1$, as we show in section 3.

2. Background: modular spaces and groups

2.1. Fuchsian groups and Teichmüller spaces Let G be a Fuchsian group acting on the upper half plane, here denoted U, with compact orbit space U/G. Then G has the following structure:-

$$\text{Generators} \qquad \gamma_1, \dots, \gamma_r; a_1, b_1, \dots, a_g, b_g.$$

$$\text{Relations} \qquad \begin{cases} \gamma_1^{m_1} = \gamma_2^{m_2} = \cdots = \gamma_r^{m_r} = 1 \\ \gamma_1\gamma_2 ... \gamma_r \prod_{i=1}^{g} [a_i, b_i] = 1, \ \ [a_i, b_i] = a_i^{-1} b_i^{-1} a_i b_i. \end{cases} \tag{2-1}$$

The integers $m_1, m_2, \dots, m_r$ (all different from 1) are called *the periods* of the group. If U_G denotes the upper half-plane with the fixed points of elliptic elements of G removed, then U_G/G is a Riemann surface of genus g, namely U/G, with r punctures at the ramification points, one for each conjugacy class of finite cyclic subgroup generated by γ_i. Such a surface is said to be *of type* (g, r). ([Harv 1],[J-S]).

The *Teichmüller space* $T(G)$ of the group G is the space of quasiconformal (qc) self-mappings f of U which fix 0, 1 and ∞ and such that $G' = fGf^{-1}$ is Fuchsian, modulo those whose (continuous) extension to $\partial U = \widehat{\mathbb{R}} = \mathbb{R} \cup \infty$ is the identity map: note that qc-mappings always have such an extension to ∂U. It is well known that this equivalence relation serves to pick out the homotopy class of f viewed as a homeomorphism between $S = U/G$ and $S' = U/G'$. The class of f will be denoted by $[f]$.

The *extended modular group* of G, *mod*(G), is defined to be the group of quasiconformal self-mappings h of U such that $hGh^{-1} = G$, where two such mappings h_1 and h_2 are identified if their extensions to ∂U are equal. Clearly, the Fuchsian group G is itself a normal subgroup of *mod*(G). The *modular group Mod*(G) of G is defined as the quotient group *mod*$(G)/G$. The class of h in *Mod*(G) will be denoted by $\{h\}$, and called the *mapping class* of h.

The *Bers fibre space* over $T(G)$ is an open contractible subset $F(G)$ of $T(G) \times \mathbb{C}$ enjoying the following properties:-

(i) Each fibre subset $U_t = \{z \in \mathbb{C}; (t, z) \in F(G)\}$ is a Jordan domain in $\mathbb{C}$, and the quotient space U_t/G is the Riemann surface represented by the point $t \in T(G)$.
(ii) G acts properly discontinuously on $F(G)$ as a group of fibre-preserving biholomorphic maps

$$\gamma(t, z) = (t, \rho_t(\gamma)(z)) \text{ for all } \gamma \in G \text{ and } (t, z) \in F(G),$$

and $z \mapsto \rho_t(\gamma)(z)$ is a Möbius transformation for every $t \in T(G)$.
(iii) The projection $p_1 : (t, z) \to t$ of $F(G)$ onto $T(G)$ induces a holomorphic map π from the quotient manifold $V(G) = F(G)/G$ onto $T(G)$ with fibre $S_t = U_t/G$. In the case where $G \cong \pi_1(S)$, a closed surface of genus g, this structure $\pi : V(G) = V_g \to T(G) = T_g$ determines a holomorphic family of compact genus g surfaces known as the Teichmüller family (or Teichmüller curve).

(iv) The group *mod*(G), acts properly discontinuously on $F(G)$ as a group of biholomorphic automorphisms

$$h(t,z) = (h(t), h_t(z)), \qquad \text{for all } h \in mod(G) \text{ and } (t,z) \in F(G),$$

where $z \mapsto h_t(z)$ is a conformal map of U_t onto $U_{h(t)}$. When $h = \gamma \in G$, this action and the one described in (ii) coincide.
(v) The action of *mod*(G) on $F(G)$ projects via p_1 to an action on $T(G)$, whose restriction to the group G is trivial. Hence, we have a well defined action of *Mod*(G) on $T(G)$. The corresponding quotient space, the *moduli space* $T(G)/Mod(G)$ will be denoted by $\mathcal{M}(G)$. The quotient space of the parallel action of *mod*(G) on $F(G)$ is a singular fibre space $\mathcal{C}(G)$ over $\mathcal{M}(G)$. When $G = \pi_1(S)$, a compact surface of genus g, this fibre space is called the *modular curve* $\mathcal{C}_g$ of genus g. The fibre over a point $x = [t] \in \mathcal{M}_g$ is $S_x/\mathrm{Aut}(S_x)$ where $\mathrm{Aut}(S_x)$ is the full group of automorphisms of the surface S_x, represented here as the stabiliser in Mod_g of the point t.

The spaces $\mathcal{C}_g$ and $\mathcal{M}_g$ are complex orbifolds, each having finite coverings which are complex manifolds. For more details, the reader may consult a standard source such as ([Be],[Nag]).

When K is a Fuchsian group without torsion uniformising a compact surface S_0 of genus $g \geq 2$, so that K acts freely on U and U/K is isomorphic to S_0, then the points of the Teichmüller space $T(K)$, also denoted by $T(S_0)$ or T_g, can be represented as homotopy classes $[f]$ of topological maps $f : S_0 \to S$. Similarly the modular group *Mod*(K), or *Mod*(S_0) or *Mod*$_g$, is then defined as the group of homotopy classes $\{h\}$ of self-homeomorphisms of S_0. In these terms the action of an element $\{h\} \in Mod_g$ on T_g as in (v) can be simply described as $\{h\} \circ [f] = [f \circ h^{-1}]$.

2.2 Surfaces with an automorphism. In this section we summarise the analysis of surfaces with automorphisms in terms of Fuchsian groups, following [Harv 2]. Let K be a Fuchsian group uniformising a compact surface S_0 of genus $g \geq 2$ and let $\tau_0 \colon S_0 \to S_0$ be an automorphism of order n. Then there is a Möbius transformation u such that $u^{-1}Ku = K$ and u acting on U induces the self-mapping τ_0 on S_0. Hence $u^n \in K$ and writing $\Gamma = \langle K, u\rangle$ we have that $U/\Gamma = S_1$ is isomorphic to $S_0/\langle\tau_0\rangle$ and there is an exact sequence

$$1 \to K \to \Gamma \xrightarrow{\delta} \langle\tau_0\rangle \to 1. \tag{2-2}$$

In terms of a presentation as in (2-1) for the group Γ, the periods $\{m_i\}$ are the branching orders of the covering $S_0 \to S_1$, and u can be chosen as γ_1 (or, indeed, any γ_i) when $n = p$ is prime and $g(\Gamma) = 0$. In the general cyclic case, the element u will be given as a word $w(\gamma_i, a_j, b_k)$ with δ-image the generator τ_0.

If $\tau_0 \colon S_0 \to S_0$ has order n and fixes all r points $\{P_i\}$, so that all periods $m_j = n$, then in local coordinates z near any P_i we have $\tau(z) = \varepsilon_i z$ where ε_i is a primitive nth root of unity. Thus τ_0 is locally a rotation at P_i through angle $2\pi v_i/n$ where $(v_i, n) = 1$; it can be shown that $\delta(\gamma_i) = \tau^{-v_i}$ [Harv2]. In the general case, one has that $n = \mathrm{lcm}\{m_j\}$ and a similar analysis can be made for the various periodic points of τ_0.

Proposition A. *In the natural inclusion $\iota_* : T(\Gamma) \subset T(K)$ induced by $\iota : K \to \Gamma$ from (2-2), $\iota_* T(\Gamma)$ consists of the points $[f] \in T(K)$ such that the Riemann surface*

$S_1 = U/fKf^{-1}$ admits a cyclic group of automorphisms $\langle\tau_1\rangle$, with the same order, number of fixed (or, more generally periodic) points and rotation angles as $\langle\tau_0\rangle$; and such that the induced map $f : S_0 \to S_1$ satisfies $\langle\tau_1\rangle = f\langle\tau_0\rangle f^{-1}$ up to homotopy.

Notation.

The surface automorphism τ_0 provides a geometric realisation of the mapping class $\{u\} \in Mod_g$ which it determines. We denote the *fixed subspace* of $\tau_0 \sim \{u\}$ in the Teichmüller space $T_g = T(K)$ by $T_g(\tau_0)$, the fibre space $V(K)$ by V_g and the restriction of V_g to the base $T_g(\tau_0)(\cong T(\Gamma)$ via $\iota_*)$ by $V_g(\tau_0)$. It is known ([E-K]) that the rule which associates to each point $t \in T(\Gamma)$ the K-orbit of the fixed point of an elliptic generator $\rho_t(\gamma_i)$, $1 \leq i \leq r$, acting on the fibre set U_t is well defined and determines r disjoint holomorphic sections $\xi_i(t)$, $i = 1, \ldots, r$ of $V_g(\tau_0)$ over $\iota_* T(\Gamma)$.

By making use of Riemann's theory of theta functions, one can prove more explicit results when n is prime, as follows.

Proposition B. *Suppose that the model Riemann surface S_0 has an automorphism τ_0 of prime order and that the quotient surface $S_0/\langle\tau_0\rangle$ has genus zero. Then there exist meromorphic functions $x, y\colon V_g(\tau_0) \to \mathbb{C}$ whose restriction to each fibre S_t provides two generating meromorphic functions x_t,y_t satisfying the identity*

$$y_t^p = x_t^{d_1}(x_t - 1)^{d_2}(x_t - \lambda_3)^{d_3}...(x_t - \lambda_{r-1})^{d_{r-1}},$$

where the integers d_i are inverses (mod p) of the rotation numbers v_i of τ_0.

We note that x_t is characterised as the unique meromorphic function on S_t of degree p which has a single zero (resp. pole) of order p at $\xi_1(t)$ (resp. $\xi_r(t)$) and takes the value $x_t(\xi_2(t)) = 1$. We can then set

$$x_t(\xi_3(t)) = \lambda_3, ..., x_t(\xi_{r-1}(t)) = \lambda_{r-1},$$

For details, see ([Kuri],[Gon1],[G-H]).

2.3 Relative modular groups and their actions. A description of the various relevant modular subgroups from the point of view of Fuchsian groups follows:-

- $mod_g = mod(K)$
- $Mod_g = mod(K)/K$
- $mod_g(\tau_0) = mod(\Gamma, K) = mod(K) \cap mod(\Gamma)$
- $Mod(\Gamma, K) = mod(\Gamma, K)/K$
- $\mathcal{P}(\Gamma) = \{h \in mod(\Gamma) | h^{-1}\langle\gamma_i\rangle h = t_i^{-1}\langle\gamma_i\rangle t_i$ for some $t_i \in \Gamma; i = 1, ..., r\}$.
- $\mathcal{P}_g(\tau_0) = \mathcal{P}(\Gamma, K) = mod(K) \cap \mathcal{P}(\Gamma)$.
- $P_g(\tau_0) = P(\Gamma, K) = \mathcal{P}(\Gamma, K)/K$.

As in section 2.2, we choose $u = w(\gamma_j)$ an element of Γ representing the automorphism τ_0. Abusing notation somewhat, we usually also denote by τ_0 the corresponding mapping class element in mod_g and $Modg$ induced by u, since when K and Γ are regarded as subgroups of $\mathcal{P}(\Gamma, K)$, and Γ/K as a subgroup of $P(\Gamma, K)$, then we have $\Gamma/K = \langle\tau_0\rangle$ and $\{u\}$ is indeed the surface automorphism τ_0 viewed as a mapping class.

The groups $Mod(\Gamma, K)$ and $P(\Gamma, K)$ defined above will usually be denoted by $Mod_g(\tau_0)$ and $P_g(\tau_0)$ respectively. They are referred to as the *relative modular group* and the *pure modular group* for τ_0, respectively.

An equivalent topological way to think of $Mod_g(\tau_0)$ is as the group of mapping classes $\{f\}$ of homeomorphisms $f : S_0 \to S_0$ that normalise τ_0, so that $f \circ \langle\tau_0\rangle \circ f^{-1} = \langle\tau_0\rangle$. Similarly $P_g(\tau_0)$ is the normalising subgroup corresponding to homeomorphisms which in addition preserve each fixed point of τ_0. In particular, this implies that $P_g(\tau_0)$ is normal in $Mod_g(\tau_0)$.

Notation.

Within the action of Mod_g on T_g, which produces as quotient the moduli space $\mathcal{M}_g$ of surfaces of genus g , these two subgroups acting on $T_g(\tau_0)$ produce as quotient complex analytic (orbifold) spaces $T_g(\tau_0)/Mod_g(\tau_0)$ and $T_g(\tau_0)/P_g(\tau_0)$; we denote these by $\mathcal{M}_g(\tau_0)$ and $\mathcal{M}_g^{pure}(\tau_0)$ respectively, the latter space being a finite holomorphic (ramified) cover of the former ([G-H],[H-M]).

2.4 A crucial remark on the action of $mod_g(\tau_0)$. Now consider any element $h \in mod_g(\tau_0)$ acting on the Bers fibre space $F(K)$ as in 2.1. Write $h(t) = t'$ for the map induced on the base space, $t, t' \in T_g(\tau_0)$; the biholomorphism $h_t : U_t \to U_{t'}$ obtained by restriction to the fibre over t induces an isomorphism from S_t onto $S_{t'}$ which we also denote by h_t. In other words, the action of $mod_g(\tau_0)$ on $F(\Gamma)$ induces an action of $Mod_g(\tau_0)$, and hence of $P_g(\tau_0)$, on $V_g(\tau_0)$.

Remark. *We note, and this is perhaps the key observation of the paper, that although the mapping class τ_0 fixes $T_g(\tau_0)$, its action on $V_g(\tau_0)$ is not at all trivial; indeed it produces a biholomorphic symmetry T of the total space of order n. Therefore the quotient of the family $V_g(\tau_0)$ by the action of $P_g(\tau_0)$ is a family of Riemann spheres.*

In the next section, we shall see how to remedy this difficulty.

3. De-singularisation of the group action on $V_g(\tau_0)$

3.1 Action of the relative modular group. From now on, we assume that the model quotient surface $S_0/\langle\tau_0\rangle$ has genus 0 and, for simplicity, that the order of τ_0 is prime. Our purpose is to examine in more detail the action of the pure relative modular group $\mathcal{P}_g(\tau_0)$ on the fixed subspace of the universal family V_g of marked surfaces, and to identify a character of the group which ensues.

We consider first the element $\tau_0 \in \mathcal{P}_g(\tau_0)$. The action of τ_0 on $V_g(\tau_0)$ preserves each fibre surface S_t and induces an automorphism τ_t which fixes the points $\xi_1(t), ..., \xi_r(t)$: this follows immediately from the analysis in 2.2 and specifically Proposition A.

A general element of the pure group $\mathcal{P}_g(\tau_0)$ maps fibres to fibres and also preserves each section $\xi_i : T_g(\tau_0) \to V_g(\tau_0)$ for $i = 1, \dots, r$. Thus for each $h \in \mathcal{P}_g(\tau_0)$, the restriction to the $t-$fibre, $h_t : S_t \to S_{t'}$, sends $\xi_i(t)$ to $\xi_i(t')$. We observe that this biholomorphic map induces by the comorphism h_t^* an isomorphism between the fields of meromorphic functions $\mathcal{K}(S_t)$, $\mathcal{K}(S_{t'})$ on these two (holomorphically equivalent) Riemann surfaces, given by $F' \mapsto F = h_t^*(F') = F' \circ h_t$.

Proposition 3.1. *(i)* $(h_t)^*(x_{t'}) = x_t$.

(ii) For each $h \in mod_g(\tau_0)$ *there is a p-th root of unity* $\epsilon = \varepsilon(h)$ *such that* $(h_t)^*(y_{t'}) = \epsilon y_t$ *holds for all* $t \in T_g(\tau_0)$.

(iii) The rule $h \mapsto \varepsilon(h)$ *defines a character* ε *of* $\mathcal{P}_g(\tau_0)$ *with* $\varepsilon(\tau_0) \neq 1$.

Proof. (i) $(h_t)^*(x_{t'}) = x_{t'} \circ h_t$ is a meromorphic function of degree p with a single zero (resp. pole) at $(h_t)^{-1}(\xi_1(t')) = \xi_1(t)$ (resp.$(h_t)^{-1}(\xi_r(t')) = \xi_r(t)$) and taking the value 1 at $(h_t)^{-1}(s_2(t')) = s_2(t)$. Thus, it is the function x_t.
(ii) Applying $(h_t)^*$ to the equation relating $x_{t'}$ and $y_{t'}$ (Proposition B) and taking into account part (i), we obtain

$$((h_t)^* y_{t'})^p = x_t^{d_1}(x_t - 1)^{d_2}(x_t - \lambda_3)^{d_3} \ldots (x_t - \lambda_{r-1})^{d_{r-1}},$$

which implies that $(h_t)^* y_{t'}$ differs from y_t by a p^{th} root of unity $\epsilon(h, t)$. But ε is independent of t by the continuity of $\varepsilon(h, t)$ with respect to the variable t.
(iii) It is routine to check that ε is a character by the formula for the composition action of $h_1 \circ h_2$. If it were trivial for the mapping class of τ_0, the automorphism τ_0 of the surface S_0 would fix the whole function field $\mathcal{K}(S_0)$ and hence would be the identity, which contradicts the definition of τ_0. ∎

3.2 The splitting homomorphism. We can now define the subgroup which will produce the desired nonsingular quotient fibre space.

Definition. *(a)* The *splitting subgroup for the automorphism* τ_0 of S_0 is

$$\mathcal{P}'(\tau_0) = Ker\ (\varepsilon) = \{h \in \mathcal{P}_g : (h_t)^* y_{t'} = y_t, \text{ for all } t \in T_g(\tau_0)\}.$$

(b) $P'(\tau_0) = \mathcal{P}'(\tau_0)/K \subset P_g(\tau_0)$.

It follows from Proposition 3.1 that the mapping class τ_0 does not belong to $P'(\tau_0)$.

Proposition 3.2. *The element* τ_0 *is central in* $P(\tau_0)$.

Proof. For any $h \in P(\tau_0)$ we have $\tau_0 \circ h = h \circ \tau_0^d$, for some integer d prime to p. Hence $\gamma_1 h = h\gamma_1^d$ modulo K , because from the description given in section 2 we have $\gamma_1 h = ht_1\gamma_1^d t_1^{-1}$. But it is also clear that $t_1\gamma_1^d t_1^{-1}\gamma_1^{-d} \in K$, because K is given as $Ker\{\delta : \Gamma \to \mathbb{Z}_p\}$, where $\mathbb{Z}_p$ denotes the cyclic group of p^{th}-roots of unity.

Now, applying the character ε to this identity, we deduce that $\tau_0^d = \tau_0$. ∎

These facts imply the crucial property we need for the group $P'(\tau_0)$:

Proposition 3.3. *The rule* $h \mapsto \tau_0^{-\varepsilon(h)} \circ h$ *defines an epimorphism* $\Theta \colon P(\tau_0) \to P'(\tau_0)$ *which induces the following split short exact sequence:*

$$1 \to \langle\ \tau_0\ \rangle \to P(\tau_0) \xrightarrow{\Theta} P'(\tau_0) \to 1. \tag{3-1}$$

Proof. The mapping is clearly surjective. It remains to prove that $\Theta \colon P(\tau_0) \to P'(\tau_0)$ is a group homomorphism, but this follows from Proposition 3.2. The rest is immediate. ∎

The splitting image of $P'(\tau_0)$ in $P(\tau_0)$ provides the group that will be used in the next section to construct the total space of our universal symmetric family of genus g surfaces.

Corollary 3.4. *There is a group isomorphism*

$$P(\tau_0) \to P'(\tau_0) \times \mathbb{Z}_n$$

given by the rule

$$h \to (\Theta(h), \varepsilon(h)).$$

Proof. This is immediate: the inverse map is $(h, k) \mapsto h \circ \tau_0^k$. ∎

A further simple consequence of this discussion is the fact that in any genus, the modular group Mod_g contains subgroups isomorphic to pure braid groups of the sphere.

Corollary 3.5. *For all values of the genus $g > 0$, the Teichmüller modular group contains subgroups isomorphic to the pure mapping class group of the sphere, $P_{0,r}$, for all integers r expressible as $2(g-1+p)/(p-1), p$ any prime. The values $r = 2g+2$, $p = 2$ and $r = g+2$, $p = 3$ occur in every genus.*

Proof.. For genus 1, we have seen already that $P_{0,4} = P'(J)$; see section 1. The subgroups which we produce in Mod_g satisfy the Riemann-Hurwitz branching formula for some (prime cyclic) Galois covering; they are the groups $P'(\tau_0)$ formed as above from the relative modular group quotients $P_g(\tau_0)/\langle\tau_0\rangle)$ of any cyclic genus g covering of the sphere. For instance, $Mod_{0,2g+2} \subset Mod_g$ for all $g > 0$, by letting τ_0 be the hyperelliptic involution J. ∎

Thus, for every value of $g > 0$, Mod_g contains a subgroup $P'(J)$ which is isomorphic to $P_{0,2g+2}$ ([G-H],[Gon]). This subgroup is already well known (as part of the pure hyperelliptic modular subgroup). The other type which occurs in every genus corresponds to order 3 symmetry and gives rise to a number of nonconjugate subgroups of Mod_g isomorphic to $P_{0,g+2}$: one of them, in genus 3, is associated with the family $y^3 = x(x-1)(x-\lambda_1)(x-\lambda_2)$ based on the Picard modular surface.

3.3 Two Examples. The following examples show that one of our hypotheses, that $S_0/\langle\tau_0\rangle$ has genus zero, is crucial for the above results. They also show that the integer d occurring in the proof of Proposition 3.2 is not generally equal to 1.

Example (i) Consider the Fermat curves X_{3d} with affine equation

$$x^{3d} + y^{3d} = 1$$

and the automorphisms

$$\tau(x, y) = (\omega x, \omega^2 y), \quad \text{with } \omega = e^{2\pi i/3}, \qquad \sigma(x, y) = (y, x).$$

One verifies directly that $\sigma^{-1} \circ \tau \circ \sigma = \tau^2$. Now σ defines a mapping class in $P_g(\tau)$ – in this case $P_g(\tau)$ coincides with $Mod_g(\tau)$, as τ has no fixed points – but it does not commute with τ, thereby contradicting Proposition 3.2.

Example (ii) Consider the Fermat curve X_4 of genus 3 with equation $x^4 + y^4 = 1$ and the automorphisms $\upsilon(x, y) = (x, -y)$ and $\sigma(x, y) = (x, iy)$. Of course $\sigma^2 = \upsilon$. Moreover σ and υ have the same fixed points, with coordinates $(\pm 1, 0)$ and $(\pm i, 0)$, so that $\sigma \in P_g(\upsilon)$. But the statement of Proposition 3.3 cannot hold. The reason is that, in this case, a splitting sequence (3-1) would induce, by restriction to the subgroup $< \sigma >$ of $P(\upsilon)$, a split sequence as follows:

$$1 \to \langle \upsilon \rangle \to \langle \sigma \rangle \stackrel{\Theta}{\to} H \to 1,$$

where H is the group $P'(\upsilon) \cap < \sigma >$ and we have identified τ_0 with υ as in section 2.2. Thus the image of H would be a subgroup of order 2 in $< \langle \sigma \rangle$, while on the other hand it would not contain the element υ.

4. Families of Riemann surfaces

We now consider families of Riemann surfaces in a more formal way.

Definition 4.1.

(i) A *family of Riemann surfaces* of genus g is a holomorphic fibre space $\pi : V \to B$ (with π a proper holomorphic submersion) in which at each point $b \in B$ the fibre $V_b = \pi^{-1}(b)$ is a compact Riemann surface of genus g.

(ii) A *symmetric family of Riemann surfaces* of genus g is a family as in (i) together with an automorphism $T : V \to V$ of holomorphic fibre spaces over B which acts trivially on the base.

An *r-pointed symmetric family* of Riemann surfaces is a symmetric family $(\pi : V \to B, T)$, equipped with disjoint sections $s_i : B \to V$ for $i = 1, ..., r$ such that for each $b \in B$ the set of fixed points of the automorphism $T_{|V_b}$ is precisely the set $\{s_1(b), ..., s_r(b)\}$. Here disjoint means that for each $b \in B, s_i(b) \neq s_j(b), if i \neq j$.

A *morphism* between two pointed symmetric families $(\pi_1 : V_1 \to B_1; T_1; \{s_i^1\})$ and $(\pi_2 : V_2 \to B_2; T_2; \{s_i^2\})$ is given by a pair of holomorphic maps (H, h) such that the following diagram commutes:-

(4-1)
$$\begin{array}{ccc} V_1 & \xrightarrow{H} & V_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow{h} & B_2 \end{array}$$

and satisfying the following properties:

(1) H restricted to each fibre is a biholomorphism;
(2) $H \circ T_1 = T_2^k \circ H$ for some k prime to n;
(3) $H(s_i(b)) = s_i(h(b))$ for $i = 1, ..., r$ and all $b \in B_1$.

By a theorem of Ehresmann, one knows that for each point $b \in B$ there is a C^∞-trivializing open neighbourhood for the family, i.e. a neighbourhood U of b and a diffeomorphism $F : \pi^{-1}(U) \to U \times X$ making the following diagram commutative:-

(4-2)
$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{F} & U \times X \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ U & \xrightarrow{Id} & U \end{array}$$

We just mention here that results of M. Kuranishi, J.H. Hubbard, and others (see [E-F] for more details) strengthen this theorem for families $\pi : V \to B$ of compact

Riemann surfaces over a finite dimensional base: the trivialisation mapping $F : \pi^{-1}(U) \to U \times X$ enjoys the following extra properties:
(i) the fibre model X is a compact Riemann surface of genus g;
(ii) for each $x \in X$, the map $b \mapsto F^{-1}(b,x)$ is holomorphic from U into V;
(iii) for each $b \in U$, the map $x \mapsto F^{-1}(b,x)$ is quasiconformal.

For any local trivialization of our symmetric family $(\pi : V \to B, T)$, we put

$$(F \circ T \circ F^{-1})_{|\{b\}\times X} = \mathrm{Id} \times \tau_b$$

so that τ_b is the self-map of the fibre model X induced by the holomorphic symmetry T.
Note that once we have chosen a reference point $t \in B$ and employ the fibre $V_t = S_t$ as model surface X, then all topological information about surfaces in the family is naturally expressible in terms of this model fibre by means of the mappings in (iii) above.

In this way, we know that the family of diffeomorphisms $\{\tau_b = T_{|b},\ b \in U\}$ in a local trivialisation are isotopic to the same mapping class in $Mod_g = \mathcal{M}od(X)$, and this is true too for any connected base B. Furthermore, different trivialisations of the same family give rise to conjugate mapping classes. Note that, if we use as model fibre X our reference Riemann surface S_0 from before, the symmetry T is representable as the conformal automorphism τ_0 of the model S_0. We shall refer to the conjugacy class determined by the mapping class $\{\tau_0\}$ as the (topological) *model type* of the symmetry T.

We summarise this discussion in the following statement.

Proposition 4.1. *For each symmetric family (V, B, T), there is an associated topological model type determined by the automorphism $\tau_0 : S_0 \to S_0$ induced on the model fibre $S_0 = V_b$ for a chosen reference point $b \in B$.*

∎

Remark 4.2. The topological types which occur in this article, with τ_0 having prime order p and quotient surface the sphere, are characterised by the rotation angles at the r fixed points (see 2.2):-

Let $D = D_p$ be the subset of the projective space $\mathbb{P}^{r-1}(\mathbf{F}_p)$ over the field with p elements, defined as

$$D_p = \{(v_1, ..., v_r) / \sum v_i = 0, \prod l_i \neq 0\}.$$

Then the possible topological types are in bijection with the classes in D relative to the equivalence relation induced by the obvious action of the symmetric group S_n ([Gon2]).

Example 4.3. The Teichmüller family $\mathcal{V} = \{V_g(\tau_0) \to T_g(\tau_0)\}$ equipped with the automorphism $T : \mathcal{V} \to \mathcal{V}$ defined by the family of symmetries $\tau_t = \tau_0|_{S_t}$ on each fibre surface and the sections $\xi_1(t), ..., \xi_r(t)$ introduced in section 2 is an r-pointed symmetric family of Riemann surfaces, which we call the *Teichmüller pointed symmetric curve with type τ_0*.

Notes. 1) The essence of the construction of these Teichmüller families of symmetric surfaces is that the trivialization F of $V_g(\tau_0)$ can be chosen so that if we denote by

V_t the fibre over a point $t \in T_g(\tau_0)$, then t is the point of $T_g(\tau_0)$ defined as the homotopy class of the homeomorphism $S_0 \to V_t$ that F induces by restriction.
2) We usually make no distinction between the Teichmüller families which arise from two conjugate elements, τ_0 and $\tau_1 = \alpha\tau_0\alpha^{-1}$ with $\alpha \in Mod_g$, since they can be identified using the induced holomorphic map $\alpha_* : V_g \to V_g$.

5. The universal curve

5.1 Ordering the sections of a pointed family. Grothendieck's theorem on the universality of the Teichmüller family implies, when applied to the case of a symmetric r-pointed family over a contractible base, that the underlying holomorphic family of marked surfaces is induced by a mapping of the base into the corresponding fixed subspace in T_g. Furthermore, there is an automatic way to order the set of r sections on the pulled-back family which is compatible with our choice of sections ξ_j for the model surface S_0. Thus the only part of the following statements which remains to be proved concerns identification of the symmetry type and the associated sections of the family.

Proposition 5.1. *Let $(\pi : V \to U; T; \{s_i\})$ be an r-pointed symmetric family of Riemann surfaces with U a contractible complex manifold and let $\tau_0 : S_0 \to S_0$ be an automorphism representing the topological type of T.*
(i) There is a map $h : U \to T_g(\tau_0)$ inducing by pullback a morphism (H, h) of symmetric families, as below:-

$$\begin{array}{ccc} V & \xrightarrow{H} & V_g(\tau_0) \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ U & \xrightarrow{h} & T_g(\tau_0). \end{array} \tag{5-1}$$

(ii) When the sections of $V \to U$ are ordered in accordance with those of $V_g(\tau_0)$ under the morphism (H, h), any other morphism (H_1, h_1) preserving this ordering is related to (H, h) as follows: $h_1 = \varphi \circ h$ and $H_1 = \Phi \circ H \circ \alpha$ where φ and Φ denote the action on $T_g(\tau_0)$ and $V_g(\tau_0)$ respectively of an element of $P_g(\tau_0)$, and α is an automorphism of the pointed symmetric family $(\pi : V \to U; T; \{s_i\})$ which fixes the base.

(iii) If $S_0/\langle\tau_0\rangle$ has genus zero, then any two pairs (H, h) and (H_1, h_1) are related by an element of $P'(\tau_0)$ up to post-composition with a power of τ_0. Namely, $h_1 = \varphi \circ h$ and $H_1 = \tau_0^k \circ \Phi \circ H$, where $\Phi \in \mathcal{P}'(\tau)$ acting on $V_g(\tau_0)$ induces φ on $T_g(\tau_0)$.

Proof. For a proof of Grothendieck's result, see for instance [Nag]. Parts (i) and (ii) then follow straightforwardly from earlier sections. To prove part (iii), we rewrite the φ given in (ii) as $\tau_0^d\phi$, with $\phi \in P'(\tau_0)$. This then implies that $H_1 = \tau_0^d \Phi H\alpha$, $\Phi \in \mathcal{P}'(\tau_0)$, and one sees that the symmetric family automorphism $\alpha = T^s$, for some integer s, by the rigidity condition imposed by the ordering of the $r \geq 3$ sections. Now the definition (4.1) of morphism between symmetric pointed families shows that $\Phi H \circ T = \tau_0^m \circ \Phi H$, which implies that H and H_1 are related by $H_1 = \tau_0^{sm+d}\Phi H$, $h_1 = \tau_0^{sm+d}\phi h$.

5.2 A smooth family over $\mathcal{M}_g^{pure}(\tau_0)$: the symmetry excluded. We continue with pointed symmetric families of type given by $\tau_0 : S_0 \to S_0$ such that $S_0/\langle\tau_0\rangle$

has genus zero. Recall from section 2.2 that the natural quotient of the universal Teichmüller curve family by the action of the modular group mod_g produces the modular curve $\mathcal{C}_g \to \mathcal{M}_g$, whose fibre at the point representing S is the surface $S/\mathrm{Aut}(S)$, hence *not* always of genus g. However from the results of section 4, it will follow that on taking suitable quotients of the Teichmüller symmetric family $\pi : V_g(\tau) \to T_g(\tau)$ we can achieve symmetric smooth modular families in this case.

Theorem 5.2. *(i) The action of the group $P_g(\tau)$ on $V_g(\tau_0)$ produces as quotient an analytic space $\mathcal{C}_{0,r}$ which determines a family of Riemann surfaces of genus 0 over $\mathcal{M}_g^{pure}(\tau_0)$. The sections $\xi_1, ..., \xi_r$ of the pointed symmetric family $V_g(\tau_0) \to T_g(\tau_0)$ induce sections $\mathcal{Z} = \{\zeta_1, ..., \zeta_r\}$ of $\mathcal{C}_{0,r}$, giving an r-pointed family of Riemann spheres over $\mathcal{M}_g^{pure}(\tau_0)$.*
(ii) The action of the group $P'(\tau)$ on $V_g(\tau_0)$ produces as quotient an analytic space $\mathcal{C}'_g(\tau_0)$ which is a family of Riemann surfaces of genus g over $\mathcal{M}_g^{pure}(\tau_0)$. The automorphism $T = \{\tau_t | t \in T_g(\tau_0)\}$ and ordered set of sections $\Xi = \{\xi_1, ..., \xi_r\}$ of the pointed symmetric curve $V_g(\tau_0) \to T_g(\tau_0)$ induce an automorphism τ and sections $\eta_1, ..., \eta_r$ of $\mathcal{C}'_g(\tau_0)$, giving an r-pointed symmetric family of Riemann surfaces of genus g over $\mathcal{M}_g^{pure}(\tau_0)$ with symmetry T modelled on the automorphism τ_0.
(iii) There is a natural morphism Φ_0 of pointed symmetric families

(5-2)
$$\begin{array}{ccc} \mathcal{C}'_g(\tau_0) & \xrightarrow{\Phi_0} & \mathcal{C}_{0,r} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ \mathcal{M}_g^{pure}(\tau_0) & \xrightarrow{Id} & \mathcal{M}_g^{pure}(\tau_0) \end{array}$$

The restriction of Φ_0 to each fibre S_t is the quotient map $\phi_t : S_t \to S/\langle \tau_t \rangle$.

Proof. The construction of these families follows that of the universal curve $\pi : \mathcal{C}_g \to \mathcal{M}_g$ from the Bers fibre spaces $F_g \to V_g = F_g/K \to T_g$, where $K = \pi_1(S_0)$. But the fibre at a point $\pi^{-1}(t)$ of $\mathcal{C}_g(\tau_0)$ is $S_t/\langle \tau_t \rangle \cong \mathbb{P}^1$, whereas that of $\mathcal{C}'_g$ is S_t, since the stabilising group in $\mathcal{P}'(\tau)$ for the Bers fibre disc U_t over t is $K_t = \rho_t(K) = \pi_1(S_t)$ as we saw in section 3.2.

5.3 A universal property for $\mathcal{C}'_g(\tau_0)$. We are now ready to prove our main result: the symmetric family $(\pi\colon \mathcal{C}'_g(\tau_0) \to M_g^{pure}(\tau_0), \tau)$, which becomes r-pointed when we choose an ordering on the set of sections Ξ, represents the functor of isomorphism classes of cyclic genus g covers of $\mathbb{P}^1$ with topological type τ_0.

Theorem 5.3. *Let $\mathcal{V} = (\pi : V \to B; T; \{s_i\})$ be any r-pointed symmetric family of topological type τ_0 of order n, with $S_0/\langle \tau_0 \rangle$ of genus zero. Then, there is a finite-cyclic covering $\overline{B} \to B$ of degree d, with defining subgroup $< \tau_0^{n/d} >$ for some $d|n$, and a classifying map $\phi : \overline{B} \to \mathcal{M}_g(\tau_0)$ such that the family $\overline{\mathcal{V}}$ obtained from $\mathcal{V}$ by pullback admits a morphism to $\mathcal{C}'(\tau)$ of symmetric families over $\overline{B}$ as follows:*

(5-2)
$$\begin{array}{ccc} \overline{V} & \xrightarrow{\Phi} & \mathcal{C}'(\tau_0) \\ {\scriptstyle \pi^-}\downarrow & & \downarrow \\ \overline{B} & \xrightarrow{\varphi} & Mg^{pure}(\tau_0) \end{array}$$

Proof. We sketch the lines of the argument. Each point has a nbd $B_0 \subset B$ which is contractible, so that Theorem 5.1 applies to $V_{|B_0}$. To give a marking on one fibre S_0 in the family $V \to B$ is to define a map ϕ from the universal cover $\widetilde{B}$ into T_g, which must have image in some $T_g(\tau_0$. We then compose with projection p_g from T_g into the pure modular variety $\mathcal{M}_g^{pure}(\tau_0)$, calling the map $\varphi = p_g \circ \phi$. The smooth family $V_g(\tau_0)$ defined over $T_g(\tau_0)$ is marked by the exact sequence of (marked) Fuchsian groups (2-2) for the model surface. It then has an ordered set of canonical sections $\xi_i, ;, i = 1, \ldots, r$. This gives a pointed symmetric structure to the family we produce by pulling back each restricted $V-$subfamily over the images $\varphi(U)$ of a suitable finite covering by contractible open nbds $U \subset B$, $\Phi^*(V_g(\tau_0)|_{\varphi(U)}$. The $U-$based pieces are fitted together using analytic continuation in B. The choice of some power of the symmetry τ_0 needs to be made in patching the families together at each transition, since composing the modular family $(\mathcal{C}'(\tau_0), \mathcal{M}_g(\tau_0), \tau, \Xi)$ with any τ^k changes none of the data, but the map φ is otherwise uniquely defined, by Proposition 5.1(iii).

The monodromy of the family over B is then given by a homomorphism from $\pi_1(B)$ into the cyclic group $\mathbb{Z}_n$, which makes the structure induced by pulling back further to the corresponding smooth cyclic covering $\overline{B}$ of minimal degree. The kernel determines the degree of the covering $\overline{B}$. ∎

To complete the picture, we append some comments on uniqueness:-

1. The covering $\overline{B} \to B$ of lowest degree is unique.
2. The map φ is unique up to composition with τ_0^k.
3. The ordering of sections is unique up to composition with an automorphism α of the underlying symmetric family $\{V \to B, T\}$. A unique isomorphism of pointed families ensues if one considers two such orderings to be equivalent.

We note finally that Theorem 5.3 is equivalent to the existence of our pure symmetric mapping class subgroup $P'(\tau_0)$. For we have

Theorem 5.4. *The functor of isomorphic classes of pointed symmetric cyclic covers of given topological type τ_0 is representable if and only if $P(\tau_0)$ admits a subgroup Q which fits in a split exact sequence of the form*

$$1 \to \langle\tau_0\rangle \to P(\tau_0) \overset{\Theta}{\to} P(\tau_0)/\langle\tau_0\rangle \cong Q \to 1$$

Proof. Such a subgroup $Q \subset P(\tau_0)$ exists if and only if there is a corresponding subgroup $\mathcal{Q} \subset \mathcal{P}(\tau_0)$ with $\mathcal{Q} \cap \Gamma = K$. If so, then we can proceed as above. If not, then there is no possibility of constructing a smooth fibre space over the base $\mathcal{M}_g(\tau_0)$ by the action of a suitable subgroup of $\mathcal{P}(\tau_0)$, since it must contain K on the one hand but also excludes anything in $\mathcal{P}(\tau_0)$ whose $\delta-$image (in the terminolgy of (2-2)) is nontrivial. ∎

5.5 A remark on prime order coverings of $\mathbb{P}^1$. In the case $S_0/\langle\tau_0\rangle$ of genus 0, it is known (see [Gon1],[G-H]) that the base of the universal family, $M_g^{pure}(\tau_0)$, is $\Omega^{r-3} = \mathbb{C}^{r-3} - \Delta$ with Δ the diagonal set $\Delta = \{\lambda_i = 0, 1, \lambda_j; i \neq j\}$ removed, and the fibre over a point $(\lambda_1, ..., \lambda_{r-3})$ is the Riemann surface with algebraic equation

$$y^p = x^{d_1}(x-1)^{d_2}(x-\lambda_3)^{d_3}...(x-\lambda_{r-1})^{d_{r-1}}.$$

One could therefore introduce the universal family in an elementary way by suitably compactifying each fibre, avoiding all reference to Teichmüller theory. However Teichmüller methods would still be needed to prove that the family is indeed universal, that is, in order to construct morphisms of any other family to this one, we need the Grothendieck theorem even to treat the situation locally.

5.6 A final example. This is an illustration of theorem 5.3. Consider a holomorphic family $\mathcal{V} = \{\pi : V \to B\}$ with base the punctured disc $\mathbb{D}^*$ given by the equation:-

$$V_t = \{y^2 = x^{2g+1} - t^{2g+1}\} \cup \{\infty_t\}.$$

Adding in the points at infinity ∞_t in each fibre, we have a family of compact surfaces with automorphism of order 2 given by:

$$\alpha_t : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y \end{pmatrix}.$$

There are $2g+2$ sections, located at the $2g+1$ points $\xi_k(t) = (e^{2ki\pi/(2g+1)}t, 0)$ and $\xi_{2g+2}(t) = \infty_t$. This is a constant family locally, i.e. $V \cong \mathbb{D}^* \times V_1$, via the isomorphism sending $(x, y, t) \mapsto (x/t, y/t^{g+1/2}, t)$ but not globally. A double covering of the base is needed.

A further point to note in this example, relating to our comments on uniquenes in Theorem 5.3, is the existence of automorphisms of the family which project to nontrivial automorphisms of the base, and which correspond to permutations of the sections: there is an automorphism β of order $2g+1$ which permutes cyclically the first $2g+1$ sections ξ_k.

It is not hard to see that all automorphisms of pointed families with symmetry quotient $\mathbb{P}^1$ arise from permutations of the sections, as a result of the rigidity argument for automorphisms of $\mathbb{P}^1$ used earlier in 3.2.

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