Moduli of Riemann surfaces with symmetry

G. González-Díez and W. J. Harvey

To Murray Macbeath on the occasion of his retirement

The moduli space $\mathcal{M}_g$ of Riemann surfaces with genus $g \geq 2$ contains an important subset corresponding to surfaces admitting non-trivial automorphisms. In this paper, we study certain irreducible subvarieties $\mathcal{M}_g(G)$ of this singular set, which are characterised by the specification of a finite group G of mapping-classes whose action on a surface S is fixed geometrically. In the special case when the quotient surface S/G is the sphere, we describe a holomorphic parameter function λ which extends the classical λ-function of elliptic modular theory, and which induces a birational isomorphism between the normalisation of $\mathcal{M}_g(G)$ and a certain naturally defined quotient of a configuration space $\mathbb{C}^n - \Delta$ where Δ is the discriminant set $\{z_i = z_j\ , \text{ for some } i \neq j\}$. Thus $\mathcal{M}_g(G)$ is always a unirational variety. We also show that in general $\mathcal{M}_g(G)$ is distinct from its normalisation, and construct a (coarse) modular family of G-symmetric surfaces over the latter space.

1. Teichmüller spaces and modular groups

First we introduce some of the necessary formalism. Let H_0 be a subgoup of the group $Aut(S_0)$ of automorphisms of a closed surface S_0 of genus $g \geq 2$; by a famous theorem of Hurwitz (see e.g. [19]), $Aut(S_0)$ is finite of order at most $84(g-1)$. We shall later concentrate on the case where the quotient surface S_0/H_0 is $\mathbb{P}^1$, the Riemann sphere, but results in §1 and §2 apply without this restriction.

DEFINITION. A *Riemann Surface with H_0-symmetry* is a pair (S, H) comprising a Riemann surface S with $H < Aut(S)$ such that (S_0, H_0) and (S, H) are topologically conjugate by some homeomorphism θ: $S_0 \to S$.

Two surfaces (S, H), (S', H') with H_0-symmetry are *H_0-isomorphic* if there is a biholomorphic mapping ϕ: $S \to S'$ such that $H' = \phi H \phi^{-1}$.

NOTATION. An H_0-isomorphism class is denoted by $\{S, H\}$ and the set of all H_0-isomorphism classes of surfaces with H_0-symmetry is denoted by $\widetilde{\mathcal{M}_g}(H_0)$.

We shall also need to consider the weaker equivalence relation of (non-equivariant) isomorphism for surfaces (S, H), (S', H') with H_0-symmetry; here there must be a biholomorphic mapping $\phi : S \to S'$ as before, but it is no longer required to satisfy the condition $H' = \phi H \phi^{-1}$. We shall denote by $\mathcal{M}_g(H_0)$ the set of all isomorphism classes of surfaces with H_0-symmetry.

There is a natural surjection $\widetilde{\mathcal{M}_g}(H_0) \to \mathcal{M}_g(H_0)$ between these two sets. Our primary purpose is to provide complex analytic structures for them which make this mapping a morphism of analytic spaces. Our approach rests on well-known results of Teichmüller theory which we now discuss briefly. Good references for the facts we need are [9], [22]. More details of our methods are given in earlier papers [14, 20].

Let T_g be the *Teichmüller space* of S_0. A point $t \in T_g$ is an equivalence class $[S, \theta]$, where $\theta : S_0 \to S$ is a *marking* homeomorphism, and two marked pairs (S, θ), (S', θ') are equivalent iff there is a biholomorphic $f : S \to S'$ such that θ' is isotopic to $f \circ \theta$.

If $b = \{b_1, \ldots, b_n\}$ is a finite subset of S_0, and $S_0^* = S_0 - b$ denotes the surface punctured at b, then the (stronger) equivalence relation obtained by requiring the isotopy between θ' and $f \circ \theta$ to fix the points of b determines the Teichmüller space $T_{g,n}$ of S_0^*, $n \geq 1$.

The group of mapping classes $Mod(S_0)$, viewed as the path components of the group of homeomorphisms of S_0, is denoted Mod_g if S_0 has genus g (or $Mod_{g,n}$ for S_0^*). This group operates on T_g (or on $T_{g,n}$) by the rule

$$[S, \theta] \xmapsto{\text{f}} [S, \theta \circ f] \quad .$$

By fundamental results of Bers [3], there is a canonical representation of each $T_{g,n}$ as a bounded domain in some $\mathbb{C}^N$, with $N = 3g - 3 + n$. Furthermore, the action of $Mod_{g,n}$ is by holomorphic isomorphisms and properly discontinuous [18], [20].

We shall regard a subgroup $H_0 \subseteq Aut(S_0)$ as tantamount to a subgroup of $Mod(S_0)$, since by a theorem of Hurwitz an automorphism of S_0 that is homotopic to the identity must be trivial. By a result which goes back to W. Fenchel and J. Nielsen, the fixed point set in T_g of any such finite group $G \subset Mod(S_0)$ is a (complex) submanifold denoted by $T_g(G)$.[1] In the present terminology it was reformulated in [14] as follows.

THEOREM A. *$T_g(H_0)$ is the set of Teichmüller points $[S, \theta]$ such that S possesses a group of automorphisms H conjugate to H_0 by means of the homeomorphism θ: $S_0 \to S$.*

Because the action of the modular group on T_g is properly discontinuous, the quotient *moduli space* $\mathcal{M}_g$ carries an induced structure of complex analytic V- manifold, for which the canonical projection map p: $T_g \to \mathcal{M}_g$ is holomorphic. In fact $\mathcal{M}_g$ is a projective variety; it is worth noting that the $\underline{\lambda}$-functions which we describe later fit in naturally with the projective embedding originally constructed by Baily [1] using Jacobi varieties and the Lefschetz embedding theorem.

COROLLARY. *$\mathcal{M}_g(H_0)$ is the image of $T_g(H_0)$ under the projection p.*

The submanifold $T_g(H_0)$ is itself a Teichmüller space. To see this, let the quotient surface $R_0 = S_0/H_0$ have genus γ, let $b = \{b_1, \ldots, b_r\}$ be the point set over which the projection $S_0 \to R_0$ is ramified and denote by $T_{\gamma,r}$ the Teichmüller space of the punctured surface $R_0^* = R_0 - b$.

For each $[S, \theta] \in T_g(H_0)$, write R for the quotient surface S/H and R^* for the corresponding unramified subsurface. Then $\theta : S_0 \to S$ induces a homeomorphism $\theta^* : R_0^* \to R^*$, which defines a rule

$$[S, \theta] \stackrel{\psi}{\longmapsto} [R^*, \theta^*].$$

At the level of Teichmüller spaces, this is a bijection.

[1] The fact that $T_g(G)$ is non-empty for all finite G was proved by S. Kerckhoff [17].

THEOREM B. *The spaces $T_g(H_0)$ and $T_{\gamma,r}$ are biholomorphically equivalent via the mapping ψ.*

For a proof, see [18], [14], [23].

Not every element in Mod_g stabilizes $T_g(H_0)$. The modular group permutes the various finite subgroups H_0 by conjugation and the relevant group for our purposes is the *relative modular group* with respect to H_0, which is defined as the subgroup of those mapping classes that do stabilise $T_g(H_0)$; this is the normaliser of H_0 in Mod_g (see[20]). We denote it by $Mod_g(H_0)$.

For each $[S,\theta] \in T_g(H_0)$ with a marked symmetry group $H = \theta H_0 \theta^{-1}$, the rule $[S,\theta] \to \{S,H\}$ defines a mapping from $T_g(H_0)$ into $\widetilde{\mathcal{M}}_g(H_0)$ which we shall denote by π_1. This map is clearly surjective.

Let f be an H_0-equivariant homeomorphism of S_0 representing an element $\mathbf{f}$ of $\mathcal{M}_g(H_0)$. Then $\mathbf{f}([\mathbf{S},\theta]) = [\mathbf{S},\theta \circ \mathbf{f}]$ has marked symmetry group H_f, obtained *via* $H_f = (\theta \circ f)H_0(\theta \circ f)^{-1} = \theta H_0 \theta^{-1}$. Notice that the underlying Riemann surface S and its automorphism group H are unchanged: the change of marking by $\mathbf{f}$ induces a complementary change of marking for H. This implies that the images under π_1 of $[S,\theta]$ and $\mathbf{f}([\mathbf{S},\theta])$ coincide in $\widetilde{\mathcal{M}}_g(H_0)$.

Suppose now that we have two pairs (S_1,H_1), (S_2,H_2) of surfaces with H_0-symmetry, related by a biholomorphic isomorphism $\phi : S_1 \to S_2$ such that $H_2 = \phi H_1 \phi^{-1}$. Choose two markings $\theta_j : S_0 \to S_j$, $j = 1,2$, so that each $[S_j,H_j]$ is a Teichmüller point in $T_g(H_0)$ lying over $\{S_j,H_j\}$. Then there is a homeomorphism $f : S_0 \to S_0$ making the diagram

$$\begin{array}{ccc} S_0 & \xrightarrow{\theta_1} & S_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \phi} \\ S_0 & \xrightarrow{\theta_2} & S_2 \end{array}$$

commute and compatible with $H_0 (= \theta_j^{-1} H_j \theta_j)$. Therefore f determines an element of $Mod_g(H_0)$ and we have proved the following statement.

PROPOSITION 1. *The mapping $\pi_1 : T_g(H_0) \to \widetilde{\mathcal{M}}_g(H_0)$ induces a natural bijection between the quotient space $T_g(H_0)/Mod_g(H_0)$ and $\widetilde{\mathcal{M}}_g(H_0)$.*

Since $T_g(H_0) \cong T_{\gamma,r}$ is a bounded domain in $\mathbb{C}^m$ where $m = 3\gamma - 3 + r$, it follows that $\widetilde{\mathcal{M}}_g(H_0)$ is a complex V-manifold of dimension m.

2. The relationship between $\widetilde{\mathcal{M}}_g(H_0)$ and $\mathcal{M}_g(H_0)$.

The first aim of this section is to prove that $\widetilde{\mathcal{M}}_g(H_0)$ is the normalisation of $\mathcal{M}_g(H_0)$. We shall use [11] as a basic reference for analytic spaces.

THEOREM 1. *$\mathcal{M}_g(H_0)$ is an irreducible subvariety of $\mathcal{M}_g$ and $\widetilde{\mathcal{M}}_g(H_0)$ is its normalisation.*

PROOF. Because $Mod_g(H_0)$ acts discontinuously with finite isotropy groups on $T_g(H_0) \cong T_{\gamma,r}$, a domain in $\mathbb{C}^m$, it follows from a theorem of Cartan [4] that $\widetilde{\mathcal{M}}_g(H_0)$ is a normal complex space. Also, by the discontinuity of Mod_g on T_g, the family of submanifolds $\{\mathbf{h}(T_g(H_0)),\ \mathbf{h} \in Mod_g\}$ is *locally finite*, that is each point of $T_g(H_0)$ has a neighbourhood in T_g intersecting only finitely many distinct subvarieties $\mathbf{h}(T_g(H_0))$.

We have already defined the natural mapping π: $\widetilde{\mathcal{M}}_g(H_0) \to \mathcal{M}_g$ whose image is precisely $\mathcal{M}_g(H_0)$. Thus if we check that:

(1) π is closed,

(2) π has finite fibres,

(3) π is injective outside a proper subvariety,

then by the Proper Mapping Theorem and the definition of normalisation the theorem will be proved.

Let us prove (2) and (3). The diagram below summarises the situation; the map $\pi_2 = \pi \circ \pi_1 : T_g(H_0) \to \mathcal{M}_g(H_0)$ is the restriction of p to $T_g(H_0)$.

$$
\begin{array}{ccccc}
T_g(H_0) & & \hookrightarrow & & T_g \\
\pi_1 \downarrow & & & & \downarrow p \\
\widetilde{\mathcal{M}}_g(H_0) & \xrightarrow{\ \pi\ } & \mathcal{M}_g(H_0) & \hookrightarrow & \mathcal{M}_g
\end{array}
$$

Two points in $T_g(H_0)$ with the same image in $\mathcal{M}_g(H_0)$ are of the form $[S,\, \theta]$ and $[S,\, \theta \circ h]$ with $\mathbf{h} \in Mod_g$. By Theorem A, $H = \theta H_0 \theta^{-1}$ and $H' = (\theta \circ h) H_0 (\theta \circ h)^{-1}$ are both subgroups of $Aut(S)$. Now, if $[S,\, \theta]$ and $[S,\, \theta \circ h]$ have different images in $\widetilde{\mathcal{M}}_g(H_0)$, then $\mathbf{h} \notin Mod_g(H_0)$ and so $\mathbf{h} H_0 \mathbf{h}^{-1} \neq H_0$. Hence necessarily $H \neq H'$. Since $Aut(S)$ is finite, there are only finitely many possibilities for $[S,\, \theta \circ h]$, which proves (2).

This argument also shows that π fails to be injective only on the π_1-image of intersections $T_g(H_0) \cap \mathbf{h}(T_g(H_0))$ with $\mathbf{h} \in Mod_g - Mod_g(H_0)$. By the local finiteness this is a subvariety of $\widetilde{\mathcal{M}}_g(H_0)$, which proves (3).

For completeness we sketch the elementary property (1). Referring to the diagram above, it is sufficient to prove that if C is a closed subset of $T_g(H_0)$ then $\pi_2(C)$ is closed in $\mathcal{M}_g(H_0)$ or, equivalently, that the union of all $\mathbf{h}(C)$, $\mathbf{h} \in Mod_g$, is closed in T_g. Suppose that $y = \lim \mathbf{h}_n(x_n)$ with $x_n \in C$ and $\mathbf{h}_n \in Mod_g$. Taking N_y a small enough open set in T_g containing y such that N_y intersects only finitely many sets $\mathbf{h}(C)$, there is then a single set $\mathbf{h}_0(C)$ which contains an infinite subsequence of the $\{\mathbf{h}_n(x_n)\}$. Thus we have a sequence of points $x'_n \in C$ with $\mathbf{h}_0(x'_n) \to y$. But $T_g(H_0)$ is closed in T_g and $\mathbf{h}_0$ is an isometry in the Teichmüller metric, so it follows that $x'_n \to x \in C$. This completes the verification of the property (1). ∎

We next address the question whether $\widetilde{\mathcal{M}_g}(H_0)$ is biholomorphic to $\mathcal{M}_g(H_0)$. From the proof of the theorem we can see that these spaces are different if and only if there is a surface S whose automorphism group contains two subgroups H, H' that are conjugate topologically but not holomorphically. This situation occurs, for instance, when there is a surface S, which admits a larger group G of automorphisms containing a pair of (conjugate) subgroups H, H' such that $\langle H, H' \rangle = K$ is a proper subgroup of G with H, H' not conjugate in K. Usually a deformation of S may then be constructed which preserves the K-symmetry but destroys the G-symmetry. Examples are readily produced using the fact that for any finite group G there exist Riemann surfaces S with G as a group of automorphisms and such that the quotient surface S/G has arbitrarily given genus γ; see for instance [12]. An elementary example of this type is given later in this section (example 1).

Provided that the Teichmüller space $T_g(K)$ is not a point (the case $T_{0,3}$) and is *not* in the small list of types for which there is an isomorphism between Teichmüller spaces of surfaces with different signatures, we may conclude that the space $T_g(G)$ is *properly* contained in $T_g(K)$ for any proper overgroup $G > K$. This list is as follows (see [22] p.129 for details):

$$T_{0,6} \cong T_{2,0}\,, \qquad T_{0,5} \cong T_{1,2}\,, \qquad T_{0,4} \cong T_{1,1}\,.$$

The implication is that in general the locus of points $[S, \theta] \in T_g(K)$, with S admitting two automorphism groups $\theta H \theta^{-1}$ and $\theta H' \theta^{-1}$ which are conjugate only in some larger group than $\theta K \theta^{-1}$, forms an analytic subset Z of strictly lower dimension. Therefore the restriction of the mapping $\pi : \widetilde{\mathcal{M}_g}(H) \to \mathcal{M}_g(H)$ to the π_1-image of $T_g(K)$ is not injective since outside $\pi_1(Z)$ one has $\pi([S, H]) = \pi([S, H'])$. Thus π is not biholomorphic, and the variety $\mathcal{M}_g(H)$ is non-normal at all points in the image of $\pi_1(T_g(K) - Z)$.

Furthermore, using elementary facts on analytic spaces, we can conclude that since this subset of the non-normal points of $\mathcal{M}_g(H)$ is Zariski-open in $\mathcal{M}_g(K)$, and therefore dense, the whole subvariety $\mathcal{M}_g(K) = \pi_1(T_g(K)$ is non-normal because the non-normal set is necessarily closed ([11], p.128).

These arguments prove the following result.

THEOREM 2. *The modular subvariety $\mathcal{M}_g(H_0)$ is in general distinct from its normalisation $\widetilde{\mathcal{M}_g}(H_0)$.*

As illustration, we give two examples.

EXAMPLE 1. Let F_{2p} be the compact (Fermat) Riemann surface with affine algebraic equation

$$x^{2p} + y^{2p} = 1 \; ,$$

let H (respectively H') be the cyclic group generated by the involution $(x, y) \rightarrow (-x, y)$ (respectively by $(x, y) \rightarrow (x, -y)$) and let $K = \langle H, H' \rangle$. Then H and H' are not conjugate in K but in $G = Aut(F_{2p})$ they are conjugate by the automorphism $\alpha(x, y) = (y, x)$. Now F_{2p}/K has genus > 2; in fact F_{2p}/K is isomorphic to the surface F_p with equation $x^p + y^p = 1$, the isomorphism being given in affine coordinates by $\phi(x, y) = (x^2, y^2)$; F_p has genus $(p-1)(p-2)/2$ which is > 2 for $p \geq 4$.

Thus, by the discussion preceding theorem 2, the modular subvariety $\mathcal{M}_g(H)$, $g = (p-1)(2p-1)$, is not normal; in fact the point representing the Fermat surface F_{2p} is a non-normal point of this modular subvariety.

On the other hand, for certain types of surface with automorphism, the modular subvariety $\mathcal{M}_g(H)$ *is* itself normal.

EXAMPLE 2. Let S be a hyperelliptic surface of genus g, with $J : S \rightarrow S$ the hyperelliptic involution. Since J is the unique automorphism of S with order 2 having quotient $S/\langle J \rangle \equiv \mathbb{P}^1$, we obtain $\widetilde{\mathcal{M}_g}(\langle J \rangle) = \mathcal{M}_g(\langle J \rangle)$.

REMARK. Since $\mathcal{M}_g$ is a projective variety, the G.A.G.A. Principle implies that our complex-analytic results remain valid within the framework of complex algebraic geometry. Thus $\mathcal{M}_g(G)$ is also an irreducible algebraic subvariety of $\mathcal{M}_g$ by Chow's Theorem ([11] p.184). Furthermore, since the algebraic normalisation of a projective variety is again projective ([13] p.232) and therefore analytic, it follows from the uniqueness of the normalisation ([11] p.164) that $\widetilde{\mathcal{M}_g}(G)$ is also the algebraic normalisation of $\mathcal{M}_g(G)$.

3. The case of tori: Legendre's modular function.

We review the classical theory of moduli for elliptic curves from the point of view developed in the previous section. In genus 1, the Teichmüller space T_1 is the upper half plane U: any Riemann surface of genus 1 may be expressed as a complex torus $E = E_\tau = \mathbb{C}/\Lambda(\tau)$ with $\Lambda(\tau) = \mathbb{Z} + \mathbb{Z}\tau$ a lattice subgroup of the additive group $\mathbb{C}$ and $\tau \in U$. There is a standard involutory automorphism $J : E \rightarrow E$, given by the symmetry $z \rightarrow -z$ of $\mathbb{C}$ and so, writing $H = \langle J \rangle$, we have that $T_1(H) = T_1$. The quotient E/H is the projective line $\mathbb{P}^1$, with four ramification points $a_1, \ldots, a_4$ corresponding to the four fixed points of J (which are the points of order 2, the orbits of $0, \frac{\tau}{2}, \frac{1+\tau}{2}$, and $\frac{1}{2}$ under $\Lambda(\tau)$).

Let the orbit of the origin $\underline{0}$ be chosen as a base point of E and write $T_{1,1}$ for $T(E - \underline{0})$: this procedure renders the (flat) homogeneous space E into a hyperbolic surface, thereby placing the theory of moduli for E within the framework of Teichmüller spaces. Theorem B now captures the identification of T-spaces, $T_{1,1} \cong T_{0,4}$, in our earlier list.

This space may be identified with the upper half-plane U by associating to $\tau \in U$ the Teichmüller pair $[E_\tau, f_\tau]$, $f_\tau : E_i \rightarrow E_\tau$, where $E_i = \mathbb{C}/\Lambda(i)$ has been chosen as reference surface and f_τ is the projection of the real linear homeomorphism $L_\tau : \mathbb{C} \rightarrow \mathbb{C}$ which sends $1, i$ to $1, \tau$ respectively (see e.g. [22] 2.1.8).

Similarly the fact that $T_{1,1}(H)$ is the whole of $T_{1,1}$ implies, by the definition of relative modular group given in §1, that $Mod_1(H)$ is the modular group of genus 1, $SL(2,\mathbb{Z})$, so we have

$$\mathcal{M}_1(J) \equiv \mathcal{M}_1 \equiv U/SL(2,\mathbb{Z}) \, .$$

Here $SL(2,\mathbb{Z})$ acts by $A \cdot \tau = \dfrac{a\tau + b}{c\tau + d}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We identify τ with $[E_\tau, f_\tau]$ and A with the homeomorphism $f_A : E_i \rightarrow E_i$ characterised by the real-linear map $L_A(1) = ci + d, \quad L_A(i) = ai + b$. Then the Teichmüller modular group acts by the rule

$$f_A \cdot (E_\tau, f_\tau) = (E_\tau, f_\tau \circ f_A) = (E_{A\cdot\tau}, h_A \circ f_\tau \circ f_A)$$

where $h_A : E_\tau \rightarrow E_{A\cdot\tau}$ is the isomorphism induced by $h_A(z) = (c\tau + d)^{-1} z$. To see that this is a genuine group action, one checks directly that $h_A \circ f_\tau \circ f_A$ is just $f_{A\cdot\tau}$, so we have $f_A \cdot (E_\tau, f_\tau) = (E_{A\cdot\tau}, f_{A\cdot\tau})$ as it should be.

Next we focus attention on the level-2 congruence subgroup $\Gamma(2)$, comprising matrices $A \equiv Id \pmod 2$ in $SL_2(\mathbb{Z})$. The involution J corresponds to the central element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ of $\Gamma(2)$, which fixes every point of U. The quotient group $P\Gamma(2)$ acts faithfully on U and (see for instance [5], [16]) this action is free and discontinuous. In classical vein, we define a complex function λ on U by the rule

$$\lambda(\tau) = \{\wp(a_1), \wp(a_2); \wp(a_3), \wp(a_4)\},$$

where $\wp(z) = \wp_\tau(z)$ is the *Weierstrass* $\wp$*- function* of the lattice $\Lambda(\tau)$ and $\{-,-;-,-\}$ denotes *cross-ratio*; λ is the *Legendre modular function*, which is automorphic with respect to $P\Gamma(2)$ and induces an isomorphism

$$\lambda : \; U/P\Gamma(2) \to \mathbb{C} - \{0,1\}.$$

The famous modular invariant $j(\tau)$ may then be written as an invariant (degree 6) rational function of λ.

We shall need the following description of this classical theory in terms of the *universal family* $\mathbf{E}$ of tori over U; a brief account appears in [26]. This is a fibre space $\mathbf{E} = (U \times \mathbb{C})/\mathbb{Z}^2$ over U where $\mathbb{Z}^2$ acts on $U \times \mathbb{C}$ by $(n,m) \cdot (\tau; z) = (\tau; z + n + m\tau)$, so that the fibre over τ is precisely E_τ.

Corresponding to the four fixed points of J, we have the following four holomorphic sections of the family $\mathbf{E} \to U$,

$$s_1(\tau) = 0, \qquad s_2(\tau) = \frac{\tau}{2}, \qquad s_3(\tau) = \frac{1+\tau}{2}, \qquad s_4(\tau) = \frac{1}{2}.$$

By normalising the $\wp$-function we obtain a meromorphic function $x(\tau, z)$ on $\mathbf{E}$ which when restricted to each fibre gives rise to a function $x_\tau : E_\tau \to \mathbb{P}^1$, having these four points as branch points and with corresponding branch values

$$x_\tau(s_1(\tau)) = \infty\,, \quad x_\tau(s_2(\tau)) = 1\,, \quad x_\tau(s_3(\tau)) = 0\,, \quad x_\tau(s_4(\tau)) = \lambda(\tau).$$

Finally, the congruence group $\Gamma(2)$ can be characterised as the group of matrices A such that the corresponding mapping classes f_A introduced above preserve each of these four points; and $SL(2,\mathbb{Z})/\Gamma(2)$ is isomorphic to the subgroup stabilising s_1 of the symmetric group Σ_4 which permutes the $\{s_j\}$. This description will become relevant later on.

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