Smooth Double Coverings of Hyperelliptic Curves

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Abstract. It is well known that the number of *smooth* double coverings of an irreducible complex algebraic curve C is $2^{2g}-1$. Assume that C is hyperelliptic, say $C: y^2 = \prod_{d=1}^{2g+2}(x-\mu_d)$. It was proved by Bujalance (extending previous work of Farkas) that, in this case, the set of smooth double covers of C splits into the disjoint union of subsets $\sum_p$, $p = 0, ..., \left[\frac{g-1}{2}\right]$, each one consisting of curves $\widetilde{C}$ which are simultaneously double covers (now ramified) of some curve $\overline{C}$ of genus p.

Here we prove, firstly, that the curves $\overline{C}$ arising in this way are also hyperelliptic; in fact the hyperelliptic curves $\overline{C}: y^2 = \prod_{d\in A}(x-\mu_d)$, where A ranges among the subsets of even cardinality of $\{1, ..., 2g+2\}$, and, secondly, that $\widetilde{C}$ can be recovered as the fibre product of C and $\overline{C}$ over $\mathbb{P}^1$. This, in turn, allows us to provide explicit equations for all smooth double covers of a given hyperelliptic curve.

1. Introduction

Throughout this article we will use the same term curve *to refer to an affine algebraic curve, its complete non singular model and its associated compact Riemann surface. And so, the expressions* birational map *and* Riemann surface isomorphism *will be used as synonymous. By the symbol $|A|$ we shall mean the number of elements of a finite set A.*

It is well known that a given compact Riemann surface C admits exactly $2^{2g}-1$ *smooth,* or *unramified,* double coverings, corresponding to the $2^{2g}-1$ group epimorphisms of its fundamental group (or, equivalently, its first homology group) onto $\mathbb{Z}_2$, the cyclic group of order 2. Here, we shall consider the case in which C is a hyperelliptic curve of genus g, hence given by an equation of the form

$$C: y^2 = \prod_{d=1}^{2g+2}(x-\mu_d). \tag{1}$$

It was first proved by Farkas ([**2**]) that, in this case, among these $2^{2g}-1$ covers there are precisely $\binom{2g+2}{2}$ of them which are again hyperelliptic. Then,

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Bujalance ([**1**]) classified the whole set of smooth double covers of a hyperelliptic curve C by showing that it splits into the disjoint union of $1+\left[\frac{g-1}{2}\right]$ subsets Σ_p, $p=0,...,\left[\frac{g-1}{2}\right]$, each one consisting of $\binom{2g+2}{2p+2}$ ($\frac{1}{2}\binom{2g+2}{2p+2}$, if $p=\frac{g-1}{2}$) curves which are $p-$hyperelliptic; that is, each curve $\widetilde{C}$ in Σ_p is also a (now ramified) double cover of some curve $\overline{C}$ of genus p. (Such curves are called *$p-$hyperelliptic.* When more precision is needed we will say that $\widetilde{C}$ is a $p-$hyperelliptic covering of $\overline{C}$). Bujalance's result was later reproved by Farkas (see [**3**]).

Here we prove the following

Theorem 1. *Let C be the hyperelliptic curve given by equation (1). Then*
i) Every unramified double cover of C is isomorphic to a space curve $\widetilde{C}_A$ given by

$$\widetilde{C}_A:\begin{cases} y^2=\prod_{d=1}^{2g+2}(x-\mu_d) \\ \\ z^2=\prod_{d\in A}(x-\mu_d)\end{cases}$$

where A ranges among all nonempty proper subsets of even cardinality of $X=\{1,...,2g+2\}$. The covering map being given by projection onto the (x,y)-coordinates.
ii) Two such curves $\widetilde{C}_A$ and $\widetilde{C}_B$ are isomorphic coverings of C if and only if $B=A$ or $B=A^C$, the complement of A in X. In the second case, the isomorphism is given by

$$\begin{array}{ccc}\widetilde{C}_A & \to & \widetilde{C}_{A^C}\\ (x,y,z) & \mapsto & (x,y,\pm y/z).\end{array}$$

iii) Every unramified double cover $\widetilde{C}_A$ is both p and q-hyperelliptic with $p=\frac{|A|}{2}-1$ and $q=(g-1-p)$. More precisely, $\widetilde{C}_A$ is a p (resp. q)-hyperelliptic covering of the hyperelliptic curve C_A (resp. C_{A^C}) given by

$$C_A: z^2=\prod_{d\in A}(x-\mu_d) \qquad \textit{and} \qquad C_{A^C}: z^2=\prod_{d\in A^C}(x-\mu_d).$$

iv) (Bujalance, [**1**]) *Let us denote by Σ the set of all equivalence classes of smooth double covers of C. Then Σ is the disjoint union of subsets Σ_p, $p=0,...,\left[\frac{g-1}{2}\right]$, each one consisting of equivalence classes of curves which are p-hyperelliptic. For every $p\leq\left[\frac{g-2}{2}\right]$, we have $|\Sigma_p|=\binom{2g+2}{2p+2}$ while, for $p=\frac{g-1}{2}$, $|\Sigma_p|=\frac{1}{2}\binom{2g+2}{2p+2}$.*

Corollary 2. *Any hyperelliptic curve given by $C:y^2=f(x)$ as in* (1)*, such that $f(x)$ lies in $\mathbb{Q}[x]$ and splits over $\mathbb{Q}$ into the product of two polynomials of even degree, admits a smooth double cover which is also defined over the rational numbers.*

2. Proof of the Theorem

The key point in proving this theorem is the observation that, by definition (see e.g. [**4**]), the curve $\widetilde{C}_A$ is nothing but the fibre product $C_A \underset{\mathbb{P}^1}{\times} C$ determined by the diagram

$$(2) \qquad \begin{array}{ccc}\widetilde{C}_A=C_A\underset{\mathbb{P}^1}{\times}C & \overset{\pi_A^2}{\to} & C\\ \downarrow \pi_A^1 & & \downarrow f_2\\ C_A & \overset{f_1}{\to} & \mathbb{P}^1\end{array}$$

where the maps f_i are the corresponding hyperelliptic morphisms $f_1(x,z) = x$, $f_2(x,y) = x$ and the maps π_A^i are defined by $\pi_A^1(x,y,z) = (x,z)$ and $\pi_A^2(x,y,z) = (x,y)$.

Since every branching value of f_1 is also a branching value of f_2 with same branching order 2 at all ramification points, we may conclude that π_A^2 is an unramified double covering of C (see e.g. [**6**], p. 116). Furthermore, $\widetilde{C}_A$ has to be connected, for otherwise the restriction of the degree two maps π_A^i to each connected component $T \subset \widetilde{C}_A$ could only have degree one, hence T would be simultaneously isomorphic to C_A and C, a contradiction.

We observe that the number of curves $\widetilde{C}_A$ we obtain this way by letting A vary among all proper non empty subsets of even cardinality of $X = \{1, ..., 2g+2\}$ equals $\sum_{r=1}^{r=g} \binom{2g+2}{2r} = \sum_{r=1}^{r=g} \left(\binom{2g+1}{2r-1} + \binom{2g+1}{2r} \right) = = \sum_{k=1}^{k=2g} \binom{2g+1}{k} = (1+1)^{2g+1} - 2 = 2\left(2^{2g} - 1\right)$, among which $\binom{2g+2}{2p+2}$ of them are $p-$hyperelliptic coverings of its corresponding curve C_A.

Now we address the question of whether two such coverings $\widetilde{C}_A$ and $\widetilde{C}_B$ are isomorphic (as coverings of C). Thus, let us assume that we have an isomorphism $\phi : \widetilde{C}_A \to \widetilde{C}_B$ such that $\pi_B^2 \circ \phi = \pi_A^2$. Then we must have $\phi(x,y,z) = (x,y,R(x,y,z))$ for certain rational function R satisfying $R^2 = \prod_{d \in B}(x - \mu_d)$. On the other hand, diagram (2) tells us that $\widetilde{C}_A$ is a $\mathbb{Z}_2 \oplus \mathbb{Z}_2-$Galois covering of $\mathbb{P}^1$ whose elements of order two are given by

$$\alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -z \end{pmatrix}, \ \beta \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ z \end{pmatrix} \text{ and } \alpha \circ \beta \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ -z \end{pmatrix}.$$

It is clear that the index two subfields of $\mathbb{C}(\widetilde{C}_A) = \mathbb{C}(x,y,z)$ fixed by these elements are $\mathbb{C}(C) = \mathbb{C}(x,y)$, $\mathbb{C}(C_A) = \mathbb{C}(x,z)$ and $\mathbb{C}(C_{A^C}) = \mathbb{C}(x, y/z)$ respectively, while the index four subfield fixed by the whole group is $\mathbb{C}(x)$. Now, as $R^2 \in \mathbb{C}(x)$, $R = R(x,y,z)$ must lie in one of the index two subfields $\mathbb{C}(x,w)$ where w is one of the three functions y, z or y/z. In fact, R will generate such subfield; this is because if $R \in \mathbb{C}(x)$ then ϕ could not induce an isomorphism between the function fields of $\widetilde{C}_A$ and $\widetilde{C}_B$.

Thus we have $R = a(x) + b(x)\,w$, hence $R^2 = a(x)^2 + b(x)^2 w^2 + 2a(x)b(x)w$ which implies $w \in \mathbb{C}(x)$ unless $a(x)\,b(x) = 0$, which, since $R \notin \mathbb{C}(x)$, can only occur if $a(x) = 0$, which, in turn, implies the identity $\prod_{d \in B}(x - \mu_d) = R^2 = b^2(x)\,w^2 = b^2(x) \prod_{d \in H}(x - \mu_d)$, or equivalently, $b^2(x) = \dfrac{\prod_{d \in B}(x - \mu_d)}{\prod_{d \in H}(x - \mu_d)}$, where $H = X, A$ or A^C depending on whether $w = y, z$ or y/z. Since the rational function on the right hand side has neither zeros nor poles of order greater than one, we conclude that $b(x) = \pm 1$ and $B = H$. Now, the first possibility $B = X$ is in contradiction with our hypothesis $B \subsetneq X$, while the remaining two ones $B = A$ and $B = A^C$ are the cases contemplated in the statement of the theorem.

This shows that in our construction each covering appears exactly twice, thus our $2\left(2^{2g} - 1\right)$ curves $\widetilde{C}_A$ give rise to the total number $\left(2^{2g} - 1\right)$ of pairwise inequivalent coverings. This proves parts i), ii) and iii). Moreover, let us denote by $\widetilde{\Sigma}_p$ the set of curves $\widetilde{C}_A$ with $p = \frac{|A|}{2} - 1$. Then we have seen that each curve $\widetilde{C}_A \in \widetilde{\Sigma}_p$ gives rise to only one element in the set of isomorphic classes Σ_p since its replica $\widetilde{C}_{A^C}$ does not lie in $\widetilde{\Sigma}_p$ but in $\widetilde{\Sigma}_q$, except, of course, when $p = q$, that is when $p = \frac{g-1}{2}$. This proves the remaining part iv).

3. The case of hyperelliptic coverings

Let us look more closely at the case $p = 0$. These are precisely the curves with $A = \{i, j\}$, namely

$$\widetilde{C}_{i,j} : \begin{cases} y^2 = \prod_{d=1}^{2g+2}(x-\mu_d) \\ \\ z^2 = (x-\mu_i)(x-\mu_j) \end{cases}$$

Being hyperelliptic curves, one should like to find a hyperelliptic equation, that is an expression of type (1), for them.

If we reparametrize $\widetilde{C}_{i,j}$ by $(x, y, z) \to (x, y, t = z\diagup(x-\mu_i))$ we obtain a birationally equivalent space model of $\widetilde{C}_{i,j}$ given by the equations $y^2 = \prod_{d=1}^{2g+2}(x-\mu_d)$ and $t^2 = (x-\mu_j)\diagup(x-\mu_i)$.

Writing the latter in the form $x = \left(\mu_j - t^2\mu_i\right)\diagup\left(1-t^2\right)$ allows us to provide a plane model for $\widetilde{C}_{i,j}$, namely

$$\begin{aligned} y^2 &= \prod_{d=1}^{2g+2}\left(\frac{\mu_j - t^2\mu_i}{1-t^2} - \mu_d\right) = \prod_{d=1}^{2g+2}\left(\frac{t^2(\mu_d-\mu_i)-(\mu_d-\mu_j)}{1-t^2}\right) \\ &= \frac{t^2(\mu_j-\mu_i)^2}{(1-t^2)^2}\prod_{d\neq i,j}\left(\frac{t^2(\mu_d-\mu_i)-(\mu_d-\mu_j)}{1-t^2}\right) \\ &= \left(\frac{\sqrt{\prod_{d\neq i,j}(\mu_d-\mu_i)}}{(1-t^2)^{g+1}}\right)^2(\mu_i-\mu_j)^2\, t^2 \prod_{d\neq i,j}\left(t^2 - \frac{\mu_d-\mu_j}{\mu_d-\mu_i}\right), \end{aligned}$$

which, by means of the reparametrization

$$(t, y) \to \left(t, \omega = \frac{\left(1-t^2\right)^{g+1}}{t(\mu_i-\mu_j)\sqrt{\prod_{d\neq i,j}(\mu_d-\mu_i)}}y\right),$$

is seen to be birationally equivalent to the equation $\omega^2 = \prod_{d\neq i,j}\left(t^2 - \frac{\mu_d-\mu_j}{\mu_d-\mu_i}\right)$, which is the model we were aiming for.

We can also trace what the covering map $\pi_A^2 : \widetilde{C}_{i,j} \to C$, $\pi_A^2(x, y, z) = (x, y)$ and the covering group $\langle\alpha\,(x, y, z) = (x, y, -z)\rangle$ look like in terms of the (t, ω) coordinates. As $x = \left(\mu_j - t^2\mu_i\right)\diagup\left(1-t^2\right)$ and $y = \frac{\omega t(\mu_i-\mu_j)\sqrt{\prod_{d\neq i,j}(\mu_d-\mu_i)}}{(1-t^2)^{g+1}}$, the covering group is given by $\langle\alpha(t, \omega) = (-t, -\omega)\rangle$ and the covering map by $(t, \omega) \to \left(\frac{\mu_j-t^2\mu_i}{1-t^2}, \frac{\omega t(\mu_i-\mu_j)\sqrt{\prod_{d\neq i,j}(\mu_d-\mu_i)}}{(1-t^2)^{g+1}}\right)$.

We also observe that by this map the set $\left\{\left(\pm\sqrt{\frac{\mu_d-\mu_j}{\mu_d-\mu_i}}, 0\right)\right\}_{d\neq i,j}$ of Weierstrass points of $\widetilde{C}_{i,j}$ maps onto $\{(\mu_d, 0)\}_{d\neq i,j}$, that is, the whole Weierstrass point set of C minus the points $(\mu_i, 0)$ and $(\mu_j, 0)$. On the other hand, it is known (see [**2**]) that there are only $\binom{2g+2}{2}$ unramified hyperelliptic coverings of C, thus precisely our curves $\widetilde{C}_{i,j}$. Summarizing, we have (*c.f.* [**5**]).

Corollary 3. *Let*

$$C : y^2 = \prod_{d=1}^{2g+2}(x-\mu_d)$$

be an arbitrary hyperelliptic curve of genus g. For each of the $\binom{2g+2}{2}$ pairs of Weierstrass points $\{P_i = (\mu_i, 0), P_j = (\mu_j, 0)\}$ there is a smooth hyperelliptic double *covering* $F_{ij} : \widetilde{C}_{ij} \to C$*, with*

$$\widetilde{C}_{ij} : y^2 = \prod_{k \neq i,j} \left(x^2 - \frac{\mu_k - \mu_j}{\mu_k - \mu_i} \right)$$

and

$$F_{ij}(x, y) = \left(\frac{\mu_j - \mu_i x^2}{1 - x^2}, xy(\mu_i - \mu_j) \frac{\sqrt{\prod_{d \neq i,j} (\mu_d - \mu_i)}}{(1 - x^2)^{g+1}} \right)$$

characterized, up to equivalence, by the property that P_i, P_j *are the only Weierstrass points of* C *which are not covered by Weierstrass points of the covering curve. The covering group being generated by* $\alpha(x, y) = (-x, -y)$*.*

All smooth hyperelliptic double coverings of C *arise in this way.*

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