

BELYI'S THEOREM FOR COMPLEX SURFACES

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To W. J. Harvey, teacher and friend, on his 65th birthday

Abstract. Belyi's theorem states that a compact Riemann surface C can be defined over a number field if and only if there is on it a meromorphic function f with three critical values. Such functions (resp. Riemann surfaces) are called *Belyi functions* (resp. *Belyi surfaces*). Alternatively Belyi surfaces can be characterized as those which contain a proper Zariski open subset uniformized by a *torsion free subgroup of the classical modular group* $\mathbb{P}SL_2(\mathbb{Z})$. In this article we establish a result analogous to Belyi's theorem in complex dimension two. It turns out that the role of Belyi functions is now played by (*composed*) *Lefschetz pencils* with three critical values while the analogous to torsion free subgroups of the modular group will be certain *extensions of them acting on a Bergmann domain of* \mathbb{C}^2 . These groups were first introduced by Bers and Griffiths.

1. Introduction. Let C be a compact Riemann surface, that is a complex algebraic curve. The (by now) well-known theorem of Belyi states that C can be defined over a number field if and only if there is a meromorphic function $f: C \rightarrow \mathbb{P}^1$ with three critical values [Bel]. Such functions (resp. Riemann surfaces) are often called *Belyi functions* (resp. *Belyi surfaces*). Belyi's theorem has attracted much attention ever since Grothendieck noticed in his *Esquisse d'un Programme* [Groth] that it implies amazing interrelations between algebraic curves defined over number fields and a certain class of graphs embedded in a topological surface which he named *dessins d'enfants*. (See for instance the survey article [JS] or the conference proceedings [SL]). From the point of view of uniformization theory, Belyi surfaces can be characterized as those algebraic curves which contain a *proper Zariski open subset which can be uniformized by a torsion free subgroup of the classical modular group* $\mathbb{P}SL_2(\mathbb{Z})$. The goal of this article is to establish a result analogous to Belyi's theorem in dimension 2, that is for complex surfaces. It will turn out that in this case the role of Belyi functions is going to be played by *composed Lefschetz pencils* or, as we shall call them, *Lefschetz functions* with three critical values (Theorems 1, 2 and 3). As for the second point of view, the corresponding uniformizing groups will arise as *extensions of a genus zero torsion free subgroup of the classical modular group* $\mathbb{P}SL_2(\mathbb{Z})$ by a (punctured) surface group *acting on a Bergmann domain of* \mathbb{C}^2 in the manner introduced by Bers and Griffiths (Theorem 4).

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2. A criterion for a variety to be defined over $\overline{\mathbb{Q}}$. Let $X \subset \mathbb{P}^n(\mathbb{C})$ be an irreducible projective variety. We shall say that X is defined over a field $K \subset \mathbb{C}$ if there is a finite collection of homogeneous polynomials with coefficients in K

$$\left\{ P_\alpha(X_0, \dots, X_n) = \sum \alpha_\nu X_0^{\nu_0} \cdots X_n^{\nu_n} \right\}_\alpha$$

whose zero set $Z(P_\alpha)$ is X . We shall say that X can be defined over K if it is isomorphic to a variety defined over K .

Likewise, we shall say that a morphism $f: X \rightarrow Y$ between irreducible varieties $X \subset \mathbb{P}^n(\mathbb{C})$ and $Y \subset \mathbb{P}^r(\mathbb{C})$ is defined over K if X and Y are defined over K and there is an open cover $\{U_j\}$ of X such that $f|_{U_j} \equiv (F_{j,0}, \dots, F_{j,r})$ for some homogeneous polynomials $F_{j,k} = F_{j,k}(X_0, \dots, X_n)$ with coefficients in K . We will say that $f: X \rightarrow Y$ can be defined over K if it is equivalent to a morphism $f_0: X \rightarrow Y$ defined over K . Here f and f_0 being equivalent means that there are automorphisms $h_1: X \simeq X$ and $h_2: Y \simeq Y$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow h_1 & & \downarrow h_2 \\ X & \xrightarrow{f_0} & Y. \end{array}$$

We are interested in the question of whether a given variety X can be defined over a number field, or equivalently, over $\overline{\mathbb{Q}}$, the field of algebraic numbers. Let $Gal(\mathbb{C}) = Gal(\mathbb{C}/\mathbb{Q})$ denote the group of all field automorphisms of \mathbb{C} . For given $\sigma \in Gal(\mathbb{C})$ and $a \in \mathbb{C}$, we shall write a^σ instead of $\sigma(a)$. We shall employ same rule to denote the obvious action induced by σ on the projective space $\mathbb{P}^n(\mathbb{C})$, the ring of polynomials $\mathbb{C}[X_0, \dots, X_n]$, etc. Namely, for a point $x = (x_0, \dots, x_n) \in \mathbb{P}^n(\mathbb{C})$ we put $x^\sigma = (x_0^\sigma, \dots, x_n^\sigma)$, for a polynomial $P = \sum a_\nu X_0^{\nu_0} \cdots X_n^{\nu_n}$ we write $P^\sigma = \sum a_\nu^\sigma X_0^{\nu_0} \cdots X_n^{\nu_n}$, for a morphism $f: X \rightarrow Y$ given by a collection of local expressions $\{(F_{j,0}, \dots, F_{j,r})\}$ we let $f^\sigma: X^\sigma \rightarrow Y^\sigma$ be the morphism given by $\{(F_{j,0}^\sigma, \dots, F_{j,r}^\sigma)\}$, etc. Note that we have $X^\sigma = Z(P_\alpha^\sigma)$.

Of course, the chief criterion to detect if X can be defined over a given number field K is *Weil's criterion for rationality* [We]. However if, as in our case, we only intend to know whether or not X can be defined over $\overline{\mathbb{Q}}$ then the following weaker characterization will be much easier to handle.

Criterion 1. [Gon] The following conditions relative to an irreducible variety $X \subset \mathbb{P}^n(\mathbb{C})$ (resp. a morphism $f: X \rightarrow Y$ between irreducible projective varieties

defined over a number field) are equivalent:

- (i) X (resp. $f: X \rightarrow Y$) can be defined over a number field.
- (ii) The family $\{X^\sigma\}_{\sigma \in \text{Gal}(\mathbb{C})}$ (resp. $\{f^\sigma: X^\sigma \rightarrow Y^\sigma\}_{\sigma \in \text{Gal}(\mathbb{C})}$) contains only finitely many isomorphism classes of complex projective varieties (resp. of morphisms).
- (iii) The family $\{X^\sigma\}_{\sigma \in \text{Gal}(\mathbb{C})}$ (resp. $\{f^\sigma: X^\sigma \rightarrow Y^\sigma\}_{\sigma \in \text{Gal}(\mathbb{C})}$) contains only countably many isomorphism classes of complex projective varieties (resp. of morphisms).

3. Belyi's theorem. Let now S be a *complex surface*, that is a compact holomorphic manifold of complex dimension 2. Naturally, we shall say that S can be defined over a number field, if it is biholomorphic to a projective surface defined over $\overline{\mathbb{Q}}$. Note that a complex surface need not be algebraic, that is need not be isomorphic to a projective surface.

Recall that S is termed *minimal* when it does not contain genus zero Riemann surfaces with self-intersection -1 . These are called *exceptional* or (-1) -*curves*. A (-1) -curve E can always be *contracted* (or *blown down*) to a point in the sense that there is a complex surface S_1 and a holomorphic map $\pi: S \rightarrow S_1$ which maps E to a point $\pi(E) \in S_1$ and is biholomorphic off E . The map $\pi: S \rightarrow S_1$ is the *blow-up* of S_1 at $x = \pi(E)$. Given an arbitrary complex surface S , a minimal surface S_{\min} can be obtained by a sequence of contractions

$$S = S_0 \rightarrow S_1 \rightarrow \cdots \rightarrow S_n = S_{\min}.$$

A minimal surface obtained in this way is called *a minimal model of S* .

Suppose that S is defined over $\overline{\mathbb{Q}}$ and let $E \subset S$ be an exceptional curve, then E is also defined over $\overline{\mathbb{Q}}$ [Gon]. Now, since blowing down an exceptional curve is an operation in Algebraic Geometry that works over any algebraic closed field, there can be no doubt that the contracted surface S_1 and the point $x_1 = \pi(E) \in S_1$ are both defined over $\overline{\mathbb{Q}}$. As same argument can be applied at every next step, we see that the following result holds:

PROPOSITION 1. *If a complex surface S is defined over a number field then so is any minimal model of S .*

We thus see that a complex surface S can be defined over a number field if and only if it can be obtained out of a minimal surface S_{\min} defined over $\overline{\mathbb{Q}}$ by a finite sequence of blow-ups centered at points also defined over $\overline{\mathbb{Q}}$. Therefore in our search for a Belyi criterion for complex surfaces we can restrict ourselves to minimal ones.

Recall that by classical results of Bertini given a surface $S \subset \mathbb{P}^n$ there is a *pencil of hyperplanes* $\{H_\lambda = \lambda_0 H_0 + \lambda_1 H_1\}_\lambda$ with $\lambda = (\lambda_0, \lambda_1) \in \mathbb{P}^1$, so that the hyperplane sections $S_\lambda = S \cap H_\lambda$ are *generically* nonsingular connected algebraic

curves and $S = \cup_{\lambda \in \mathbb{P}^1} S_\lambda$. Moreover, the rule “ $f(x) = \lambda \in \mathbb{P}^1$ if and only if $x \in S_\lambda$ ” defines a meromorphic function $f \in \mathcal{M}(S)$ with nonempty base locus $B = \cap_{\lambda \in \mathbb{P}^1} H_\lambda = Z(H_0, H_1)$. By results of Lefschetz $f: S \dashrightarrow \mathbb{P}^1$ can be chosen so as to satisfy the following requirements (see [Lam]):

- (i) $f: S \setminus B \rightarrow \mathbb{P}^1$ is a holomorphic submersion outside a finite set of critical points $\{x_1, \dots, x_d\}$, no two of them in a same fibre, which therefore correspond bijectively to the critical values $q_1 = f(x_1), \dots, q_d = f(x_d)$,
- (ii) at each critical point x_i , f is locally of the form $(z_1, z_2) \rightarrow z_1^2 + z_2^2$, and
- (iii) at each base point b_k , f is locally of the form $(z_1, z_2) \rightarrow z_1/z_2$.

Definition 1. A Lefschetz pencil on a complex surface S is a meromorphic function $f \in \mathcal{M}(S)$ which has a nonempty base locus $B = \{b_1, \dots, b_r\}$ and satisfies conditions (i) to (iii) above.

The closures of the (open) fibres of $f: S \setminus B \rightarrow \mathbb{P}^1$ are given by including the points $\{b_1, \dots, b_r\}$. By condition (iii) each base point b_k is nonsingular in every completed fibre $f^{-1}(t) \cup B \subset S$. If $\pi: \tilde{S} \rightarrow S$ is the blow-up of S at b_1, \dots, b_r then f induces a well-defined morphism $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$ called the *associated Lefschetz fibration*. For each $t \in \mathbb{P}^1$, π induces an isomorphism between $\tilde{f}^{-1}(t)$ and $f^{-1}(t) \cup B$. It follows that the fibration $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$ comes equipped with r sections, one for each base point. It is also known that all nonsingular fibres $\tilde{f}^{-1}(t)$ (resp. $f^{-1}(t)$) with $t \neq q_i$ are connected compact (resp. r times punctured) Riemann surfaces of a same genus g called the *genus of the pencil* (see e.g. [GS], 8.1). We note that \tilde{S} will never be minimal as it has at least r exceptional curves. If none of the exceptional curves of \tilde{S} lies in a fibre, $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$ is called *relatively minimal*. When this is not the case, a relatively minimal fibration can be achieved by blowing down all (-1) -curves contained in some fibre. We shall denote by $\hat{\pi}: \tilde{S} \rightarrow \hat{S}$ this contraction map and by $\hat{f}: \hat{S} \rightarrow \mathbb{P}^1$ the relatively minimal fibration so obtained. One has the following commutative diagram

$$\begin{array}{ccc} \tilde{S} & \rightarrow & \hat{S} \\ & \searrow & \downarrow \\ & & \mathbb{P}^1, \end{array}$$

that is, $\tilde{f} = \hat{\pi} \circ \hat{f}$.

We also recall that S is called *rational* (resp. *ruled*) if it is bimeromorphically equivalent to \mathbb{P}^2 (resp. $C \times \mathbb{P}^1$, for some Riemann surface C). Rational and ruled surfaces are always algebraic, so above we can replace *bimeromorphic* by *birational* equivalence. A ruled surface is rational if and only if $C \equiv \mathbb{P}^1$. A surface S is called *geometrically ruled* if there is a smooth surjective morphism $p: S \rightarrow C$ onto a curve C whose fibres are all isomorphic to \mathbb{P}^1 . Geometrically ruled surfaces are always minimal and, with the single exception of $\mathbb{P}^1 \times \mathbb{P}^1$,

admit exactly one ruling. Conversely, apart from \mathbb{P}^2 , any minimal ruled surface is geometrically ruled.

3.1. Lefschetz pencils on minimal surfaces. Throughout this section we will employ the following notation. For each base point $b_i \in S$ of a Lefschetz pencil $f: S \dashrightarrow \mathbb{P}^1$ we shall write $\tilde{E}_i = \pi^{-1}(b_i)$ to refer to the exceptional divisors of the corresponding blow up morphism $\pi: \tilde{S} \rightarrow S$. For a given curve $F \subset S$ we shall denote by $\pi^*F \subset \tilde{S}$ (resp. $\pi^{-1}F \subset \tilde{S}$) the full (resp. strict) transform of F .

PROPOSITION 2. *Let $f: S \dashrightarrow \mathbb{P}^1$ be a Lefschetz pencil of genus $g \geq 1$ on a minimal surface S which is not rational or ruled. Then the associated Lefschetz fibration $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$ is relatively minimal.*

Proof. Suppose we have a (-1) -curve $\tilde{L} \subseteq \tilde{f}^{-1}(q) \subset \tilde{S}$ and set $L = \pi(\tilde{L}) \subseteq f^{-1}(q) \cup B \subset S$, hence L is also a genus zero Riemann surface such that $\pi^{-1}L = \tilde{L}$. Now if L contains m base points so that \tilde{L} meets m exceptional divisors $\tilde{E}_1, \dots, \tilde{E}_m$ we have $\pi^*(L) = \tilde{L} + \sum \tilde{E}_i$ and intersection theory gives $L^2 = \pi^*(L)^2 = (\tilde{L} + \sum \tilde{E}_i) \cdot (\tilde{L} + \sum \tilde{E}_i) = \tilde{L}^2 + 2m - m = m - 1$. Since S is minimal we must have $m \neq 0$, hence $L^2 \geq 0$. It follows that S is rational or ruled (see [BPV] V,4.3). \square

As for Lefschetz pencils on ruled surfaces the situation is as follows

PROPOSITION 3. *Let $f: S \dashrightarrow \mathbb{P}^1$ be a Lefschetz pencil of genus $g \geq 1$ over a minimal nonrational ruled surface $p: S \rightarrow C$. Then the associated Lefschetz fibration $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$ is either relatively minimal or else all its singular fibres contain a (-1) -curve.*

Assume we are in the second case, then the ruling morphism p restricts to bijections of the sets $\{x_i\}$ of critical points and $\{b_i\}$ of base points onto its common image $\{p(x_i)\} = \{p(b_i)\}$. Moreover we have:

(i) *Any such triple (S, p, f) arises in the following way. Blowing up $C \times \mathbb{P}^1$ at the points $(c_i, q_i) = (p(x_i), f(x_i))$ and then contracting in the surface \tilde{S} so obtained the strict transforms of the lines $L_i = \{c_i\} \times \mathbb{P}^1$ yields a surface isomorphic to S . Under this isomorphism the ruling and Lefschetz pencil structures $p: S \rightarrow C$ and $f: S \dashrightarrow \mathbb{P}^1$ correspond to projection of $C \times \mathbb{P}^1$ onto the first and the second factor respectively.*

(ii) *Conversely for any given finite subset $\{(c_i, q_i)\} \subset C \times \mathbb{P}^1$ with $c_i \neq c_j$ and $q_i \neq q_j$ for $i \neq j$, the procedure above yields a minimal surface S endowed with a non relatively minimal Lefschetz pencil structure $f: S \dashrightarrow \mathbb{P}^1$ and a ruled structure $p: S \rightarrow C$ induced by projection of $C \times \mathbb{P}^1$ onto the second and the first factor respectively.*

Proof. Let $p: S \rightarrow C$ be a minimal ruled surface with C of positive genus and $f \in \mathcal{M}(S)$ a nonrelatively minimal Lefschetz pencil. The situation is summarized

in the following diagram

$$\begin{array}{ccccc}
 \tilde{S} & \xrightarrow{\pi} & S & \xrightarrow{p} & C \\
 \widehat{\pi} \downarrow & & & & \downarrow f \\
 \widehat{S} & \xrightarrow{\widehat{f}} & \mathbb{P}^1 & &
 \end{array}$$

where $\tilde{f} = \widehat{f} \circ \widehat{\pi}$ and $\widehat{f}: \widehat{S} \rightarrow \mathbb{P}^1$ is relatively minimal.

We proceed as in the proof of Proposition 2. Suppose we have a (-1) -curve $\tilde{L} \subseteq \tilde{f}^{-1}(q)$ and set $L = \pi(\tilde{L}) \subset S$, a nonsingular rational curve such that $\pi^{-1}L = \tilde{L}$. If L contains m base points $b_i \in S$ we have seen that $L^2 = \pi^*(L)^2 = m - 1$. As we are assuming that S is not rational we conclude that L must contain only one base point $b_1 \in B$, for $L^2 > 0$ can only occur in a rational surface (see [BPV] V,4.3). Moreover L has to be a line of the ruling for otherwise L would be surjectively mapped onto C .

Now if $F_\lambda \subset S$ is a generic fibre of the pencil and $\pi^{-1}F_\lambda \subset \tilde{S}$ is its strict transform, we have:

$$\begin{aligned}
 L \cdot F_\lambda &= \pi^*(L) \cdot \pi^*(F_\lambda) = (\tilde{L} + \tilde{E}_1) \cdot \left(\pi^{-1}F_\lambda + \sum \tilde{E}_i \right) \\
 &= \tilde{L} \cdot (\pi^{-1}F_\lambda + \sum \tilde{E}_i) + \tilde{E}_1 \cdot \left(\pi^{-1}F_\lambda + \sum \tilde{E}_i \right) \\
 &= \tilde{L} \cdot \tilde{E}_1 + \tilde{E}_1 \cdot (\pi^{-1}F_\lambda + \tilde{E}_1) = 1 + 0 = 1.
 \end{aligned}$$

Since all lines are homology equivalent, we have $L_c \cdot F_\lambda = 1$ for any line $L_c = p^{-1}(c)$. This implies that the morphism p induces an isomorphism between the generic fibre and C . In particular C has also genus g .

Now suppose $\tilde{f}^{-1}(q)$ is a singular fibre not containing (-1) -curves and let T be an irreducible component. If $g > 1$, its genus γ must satisfy $1 \leq \gamma \leq g - 1$ while if $g = 1$, our component T must equal the total fibre $\tilde{f}^{-1}(q)$ which can only be a rational curve with one node. In any case we see that on the one hand T is not a line of the ruling whereas on the other it can not be surjectively mapped onto C which has genus g . We conclude that such a fibre cannot exist.

Now we claim that the correspondence $\tilde{L} \rightsquigarrow L = \pi(\tilde{L})$ (resp. $\tilde{L} \rightsquigarrow L \cap B$) provides a bijection between the set of singular fibres of $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$ and the set of lines of $p: S \rightarrow C$ intersecting B (resp. the set B). So, suppose that L_c is a line of the ruling with $L_c \cap B \neq \emptyset$. We want to show that its strict transform $\pi^{-1}L_c \subset \tilde{S}$ is a (-1) -curve inside some fibre. Now, as F_λ passes through any base point, another implication of the identity $L_c \cdot F_\lambda = 1$ obtained above is that L_c intersects B in exactly one point b_k . Moreover, a computation similar to the

one carried out before shows that

$$\begin{aligned}
 1 &= \pi^*(L_c) \cdot \pi^*(F_\lambda) = (\pi^{-1}L_c + \tilde{E}_k) \cdot \left(\pi^{-1}F_\lambda + \sum \tilde{E}_i \right) \\
 &= \pi^{-1}L_c \cdot \pi^{-1}F_\lambda + \pi^{-1}L_c \cdot \tilde{E}_k + \tilde{E}_k \cdot \pi^{-1}F_\lambda + \tilde{E}_k \cdot \tilde{E}_k \\
 &= \pi^{-1}L_c \cdot \pi^{-1}F_\lambda + 1 + 1 - 1.
 \end{aligned}$$

This implies that $\pi^{-1}L_c \cdot \pi^{-1}F_\lambda = 0$ which means that $\pi^{-1}L_c$ is contained in some fibre. The remaining statement that $\pi^{-1}L_c$ is a (-1) -curve follows from the identity $0 = L_c^2 = (\pi^{-1}L_c + \tilde{E}_k)^2$. This settles the first statement in our proposition.

(i) We observe first that \hat{S} is a geometrically ruled surface, hence minimal, by means of the morphism $\hat{p} = p \circ \pi \circ \hat{\pi}^{-1}: \hat{S} \rightarrow C$ whose fibres $\hat{p}^{-1}(c)$ are $\hat{\pi}(\tilde{E}_k)$ if $c = p(b_k)$ and $\hat{\pi}(\pi^{-1}L_c)$ if $B \cap L_c = \emptyset$. Now we claim that, in fact, \hat{S} is isomorphic to $C \times \mathbb{P}^1$, the isomorphism being given by $\Phi(\hat{x}) = (\hat{p}(\hat{x}), \hat{f}(\hat{x}))$. Since \hat{S} is minimal this is tantamount to showing that Φ has degree 1. Now, for generic $(c, \lambda) \in C \times \mathbb{P}^1$, $\Phi^{-1}(c, \lambda)$ is nothing but the intersection of $\hat{\pi}(\pi^{-1}L_c)$ and $\hat{\pi}(\pi^{-1}F_\lambda)$ whose intersection number is

$$\begin{aligned}
 \hat{\pi}(\pi^{-1}L_c) \cdot \hat{\pi}(\pi^{-1}F_\lambda) &= \pi^{-1}L_c \cdot \pi^{-1}F_\lambda \\
 &= \pi^{-1}L_c \cdot (\pi^{-1}F_\lambda + \sum \tilde{E}_i) = L_c \cdot F_\lambda = 1.
 \end{aligned}$$

(ii) For the proof of this part we only need to reverse the arguments used above. If $\hat{\pi}: \tilde{S} \rightarrow C \times \mathbb{P}^1$ is the surface obtained by blowing up at $\{(c_i, q_i)\}$ and we denote now by \tilde{F}_λ and \tilde{L}_i the strict transform of $C \times \{\lambda\}$ and the exceptional divisor $\hat{\pi}^{-1}(c_i, q_i)$ respectively, then the fibration $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$ obtained as $\tilde{f} = p_2 \circ \hat{\pi}$, with p_k equals projection onto the k -th factor, has fibres $\tilde{f}^{-1}(\lambda)$ equal to \tilde{F}_λ if $\lambda \neq q_i$ and to $\tilde{F}_{q_i} \cup \tilde{L}_i$ otherwise. It is also true that the strict transform \tilde{E}_i of $E_i = \{c_i\} \times \mathbb{P}^1$ is a (-1) -curve as can be deduced from the identity $0 = E_i^2 = (\tilde{E}_i + \tilde{L}_i)^2$, so these curves can be blown down to obtain a contracting map $\pi: \tilde{S} \rightarrow S$. On the surface S the rational map $f = \tilde{f} \circ \pi^{-1}$ determines a Lefschetz pencil whose associated Lefschetz fibration is, of course, $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$. Similarly the morphism $p = p_1 \circ \hat{\pi} \circ \pi^{-1}: S \rightarrow C$ makes of S a geometrically ruled surface with fibres $p^{-1}(c)$ equals $\pi \circ \hat{\pi}^{-1}(\{c\} \times \mathbb{P}^1)$ if $c \neq c_i$ and $\pi(\tilde{L}_i)$ if $c = c_i$. This concludes the proof. \square

We now recall the following definition (see [HM]):

Definition 2. A curve C (resp. a r -pointed curve $(C; c_1, \dots, c_r)$) is called *stable* if it has only nodes as singularities and every rational component of the normalization of C has at least 3 points lying over singular (resp. singular and/or marked) points of C .

COROLLARY 1. *Let $f \in \mathcal{M}(S)$ be a Lefschetz pencil of genus $g \geq 1$ on a minimal surface different from \mathbb{P}^2 or a ruled surface. If $g > 1$ (resp. $g = 1$) then $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$*

(resp. $f: S \setminus B \rightarrow \mathbb{P}^1$) is a stable family of curves of genus g (resp. of r pointed curves of genus 1).

Proof. By Proposition 2 the only chance for a singular fibre F_{q_i} to be non-stable is to be isomorphic to a rational curve with one node, hence with Euler characteristic $\chi(F_{q_i}) = 1$. But, on the other hand, topologically, F_{q_i} is obtained from the generic fiber by contracting a simple loop so $\chi(F_{q_i}) = 2 - 2g + 1$. These equalities only match if $g = 1$. Now, since $r > 0$ such fibre is also stable in this case. \square

3.2. Characterization of surfaces defined over $\overline{\mathbb{Q}}$.

Definition 3. By a *Lefschetz function* we shall refer to a meromorphic function $h \in \mathcal{M}(S)$ obtained as composition of a Lefschetz pencil $f: S \dashrightarrow \mathbb{P}^1$ with a rational function $\beta: \mathbb{P}^1 \rightarrow \mathbb{P}^1$.

In classical terms, if $h = \beta \circ f$ is a Lefschetz function, then the morphism $\beta \circ \tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$ is *composed with* a Lefschetz fibration. The following results state that Lefschetz functions with three critical points play in the theory of complex surfaces a role analogous to that of Belyi functions in the theory of complex curves.

THEOREM 1. *The following statements relative to a minimal complex surface S different from any nonrational ruled surface are equivalent.*

- (a) S can be defined over a number field.
- (b) S admits a Lefschetz pencil $f \in \mathcal{M}(S)$ with critical values q_1, \dots, q_d in $\mathbb{P}^1(\overline{\mathbb{Q}})$.
- (c) S admits a Lefschetz function $h \in \mathcal{M}(S)$ with three critical values, say $0, 1, \infty$.

Proof. Let us prove first the equivalence between (b) and (c). Given a finite set of points $q_1, \dots, q_d \in \mathbb{P}^1(\overline{\mathbb{Q}})$, Belyi([Bel]) has produced a very simple algorithm to construct a function $\beta: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ which sends them, as well as all its critical points, to $\{0, 1, \infty\}$.

Conversely, let $h = \beta \circ f \in \mathcal{M}(S)$ be a function with three critical values, where f is a Lefschetz pencil and β a rational function. We have to show that f can be assumed to have critical values in $\mathbb{P}^1(\overline{\mathbb{Q}})$. If β is a Möbius transformation there is nothing to prove. If, on the contrary, β has degree $n \geq 2$ then the Riemann-Hurwitz formula implies that it has at least two critical values and that when it has exactly two, say 0 and ∞ , they correspond to precisely two critical points of maximum branching order n . In this case pre-composing β with a suitable Möbius transformation M we get the rational function $\beta_0(z) = z^n$. Now $M^{-1} \circ f$ is a Lefschetz pencil with critical values q_i in $\overline{\mathbb{Q}}$ since each q_i is a n -th root of either $0, 1$ or ∞ . Finally, if β has three critical values Criterion 1 easily implies that it

is equivalent to a rational function $\beta_0: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ defined over $\overline{\mathbb{Q}}$ (see [Gon]). We thus have a commutative diagram as follows

$$\begin{array}{ccccc} S \setminus B & \xrightarrow{f} & \mathbb{P}^1 & \xrightarrow{\beta} & \mathbb{P}^1 \\ & & \downarrow h_1 & & \downarrow h_2 \\ & & \mathbb{P}^1 & \xrightarrow{\beta_0} & \mathbb{P}^1, \end{array}$$

where h_1 and h_2 are isomorphisms. We claim that replacing f by $h_1 \circ f$ solves the problem. To see this, we observe that if q_i is a critical value of f then we have $\beta(q_i) = \beta(c) \in \{0, 1, \infty\}$ for some critical point c of β . Now by the commutativity of the diagram we have $\beta_0(h_1(q_i)) = \beta_0(h_1(c))$. On the other hand, since $h_1(c)$ is a critical point of β_0 , both $h_1(c)$ and $\beta_0(h_1(c))$ lie in $\overline{\mathbb{Q}}$. This, in turn, implies that $h_1(q_i) \in \overline{\mathbb{Q}}$ as wanted.

The implication (a) \Rightarrow (b) is trivial. As we have recalled above Bertini's theory provides S with a Lefschetz pencil structure. Moreover if S is defined over $\overline{\mathbb{Q}}$ and we take our pencil of hyperplanes $\{H_\lambda\}_\lambda$ defined also over $\overline{\mathbb{Q}}$, then the corresponding Lefschetz pencil will have critical points x_i and critical values q_i defined over $\overline{\mathbb{Q}}$ too.

The proof of the remaining implication (b) \Rightarrow (a) consists of two parts, first proving that S is actually an algebraic surface and then that it can be defined over $\overline{\mathbb{Q}}$.

The first part is easy. In dimension two the property of being algebraic is a birational invariant; so it is enough to show that \tilde{S} , the surface obtained by blowing up S at the base points of the Lefschetz pencil f , is an algebraic surface. Now, according to the results of Kodaira on complex surfaces (see e.g. [BPV] VI, 4.1), a surface which fibres over \mathbb{P}^1 such as \tilde{S} is algebraic unless the associated Lefschetz fibration $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$ is a fibration of genus $g = 1$ with no sections, but Lefschetz fibrations do have sections, namely one for each point of the (nonempty) base locus of the Lefschetz pencil.

As for the second part, we may assume from the start that $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$ is relatively minimal, for, according to Proposition 2, exceptions may only occur if S is either a nonrational ruled surface (excluded case) or a rational surface. Now, a rational surface S is either \mathbb{P}^2 or one of the countably many ruled surfaces of Hirzebruch F_n (see [BPV] or [Bea]) which are known to be defined over $\overline{\mathbb{Q}}$ (or else, apply again Criterion 1).

We now study separately the cases when the fibre genus is $g > 1$ and when $g = 1$.

$g > 1$). We claim that the restriction of the associated Lefschetz fibration $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$ to the set of regular values $\mathbb{P}^1 \setminus \{q_i\}$ gives a nonlocally trivial family. If it were locally trivial, then the classifying map $\phi: \mathbb{P}^1 \setminus \{q_i\} \rightarrow \mathcal{M}_g$ which sends

each $s \in \mathbb{P}^1 \setminus \{q_i\}$ to the point in moduli space representing the curve $\tilde{f}^{-1}(s)$ would be a constant map. By Corollary 1 the map ϕ extends to a map $\mathbb{P}^1 \rightarrow \overline{\mathcal{M}}_g$ from \mathbb{P}^1 to the stable (or Deligne-Mumford) compactification of moduli space (see e.g. [HM]). Of course, this extension can only be the constant map. In other words, we would deduce that the entire family $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$ is locally trivial. On the other hand, since the automorphism group of a Riemann surface of genus greater than 1 is finite and because here the base of the family is simply connected, our fibration must be isomorphic to a trivial fibration $\mathbb{P}^1 \times F$ (cf. [BPV] III 18.4.b). The contradiction now follows from the fact that $\mathbb{P}^1 \times F$ is a minimal surface whereas \tilde{S} can never be so as it contains one exceptional curve for each base point b_i of the pencil f .

Now if q_1, \dots, q_d are defined over a number field, letting $Gal(\mathbb{C})$ act on $\tilde{f}: \tilde{S} \setminus \tilde{f}^{-1}\{q_i\} \rightarrow \mathbb{P}^1 \setminus \{q_i\}$ gives a collection of nonlocally trivial families with only finitely many distinct base spaces. We are therefore now in position to apply Parshin-Arakelov's theorem ([Arak]) as stated e.g. in Caporaso's article [Cap], to infer that in the collection $\tilde{f}^\sigma: \{\tilde{S}^\sigma \rightarrow \mathbb{P}^1\}_{\sigma \in Gal(\mathbb{C})}$ there are only finitely many isomorphism classes of fibrations. Hence Criterion 1 implies that \tilde{S} , and thus S , can be defined over $\overline{\mathbb{Q}}$.

$g = 1$). Let us consider the family $f: S \setminus B \rightarrow \mathbb{P}^1$ provided by our Lefschetz pencil. By Corollary 1 this is a stable family of r -times punctured elliptic curves. Therefore same argument as in the case $g > 1$ above, with the only difference that \mathcal{M}_g should be replaced by the moduli space of r -punctured (or, equivalently, r -pointed) elliptic curves $\mathcal{M}_{1,r}$ (see e.g. [HM] or [Nag]), shows that for this family too local triviality implies global triviality. But here again it is not possible for $f: S \setminus B \rightarrow \mathbb{P}^1$ to be a product fibration because this would violate the requirement (iii) in Definition 1.

Therefore, if q_1, \dots, q_d are defined over a number field, we may invoke again Arakelov's theorem (now in the transcendental form given by Imayoshi and Shiga in [IS]) to conclude that as σ ranges in $Gal(\mathbb{C})$ the families $f^\sigma: S^\sigma \setminus B^\sigma \rightarrow \mathbb{P}^1 \setminus \{q_i^\sigma\}$ give rise to only finitely many biholomorphism classes. This means that if for each $\sigma \in Gal(\mathbb{C})$ we denote by J_σ the j -invariant of the elliptic fibration $\tilde{f}^\sigma: \tilde{S}^\sigma \rightarrow \mathbb{P}^1$ defined by sending $s \in \mathbb{P}^1 \setminus \{q_i^\sigma\}$ to the Jacobi modular invariant of the elliptic curve $(\tilde{f}^\sigma)^{-1}(s)$, then, up to equivalence, we only get a finite number of rational functions. Now according to Kodaira (see e.g. [BPV]) a relatively minimal elliptic fibration without multiple fibres such as ours is determined, up to finitely many choices, by its j -invariant. We, therefore, may apply again Criterion 1 to conclude the proof of the theorem. \square

For nonrational ruled surfaces Theorem 1 takes the following form:

THEOREM 2. *Let $p: S \rightarrow C$ be a minimal ruled surface which is not rational. The following statements are equivalent.*

(a) *S can be defined over a number field.*

(b) *The curve C can be defined over $\overline{\mathbb{Q}}$ and S admits a Lefschetz pencil $f \in \mathcal{M}(S)$ with critical points $x_1, \dots, x_d \in S$ so that the critical values $f(x_1), \dots, f(x_d) \in \mathbb{P}^1$ and the points $p(x_1), \dots, p(x_d) \in C$ are also defined over $\overline{\mathbb{Q}}$.*

(c) *The curve C can be defined over $\overline{\mathbb{Q}}$ and S admits a Lefschetz function with three critical values of the form $h = \beta \circ f$ for some Lefschetz pencil $f \in \mathcal{M}(S)$ as in (b).*

Proof. The proof of the equivalence between (b) and (c) is the same as in Theorem 1.

The implication (a) \Rightarrow (b) is easy. As in the proof of Theorem 1 Bertini's theory provides S with a Lefschetz pencil with critical points x_i and critical values $q_i = f(x_i)$ defined over $\overline{\mathbb{Q}}$. Moreover, the fact that the ruling morphism $p: S \rightarrow C$ is unique up to an automorphism of the base C (see e.g. [Bea]) allows us to apply again Criterion 1 to the collection of morphisms $\{p^\sigma: S^\sigma \rightarrow C^\sigma\}_{\sigma \in \text{Gal}(\mathbb{C})}$ to infer that the base curve C must also be defined over $\overline{\mathbb{Q}}$. It remains to be shown that so must be the points $p(x_i) \in C$. Suppose $p(x_1)$ were not defined over $\overline{\mathbb{Q}}$. Then there would be plenty of elements $\sigma \in \text{Gal}(\mathbb{C})$ leaving invariant S , the points x_i and the curve C but not the point $p(x_1)$. This way we would have infinitely many rulings $p^\sigma: S \rightarrow C$ of the surface S . This is impossible if C has genus greater than one, in which case it only has finitely many automorphisms. If the genus of C is one, then we can chose $p(x_1)$ to be defined over a number field, say the origin of the elliptic curve C . Now if $p(x_2)$ were not defined over $\overline{\mathbb{Q}}$ we could use a similar argument to prove the existence of infinitely many automorphisms of the pointed curve $(C, p(x_1))$. This is again impossible.

As for the proof of (b) \Rightarrow (a), we first note the the argument given in Theorem 1 to prove that S is an algebraic surface is still valid in this case. Thus, what remains to be seen is that S can be defined over $\overline{\mathbb{Q}}$. But here again the argument given in Theorem 1 to prove this same fact remains valid as long as our pencil $f \in \mathcal{M}(S)$ is a relatively minimal pencil of genus $g > 0$. If this were not the case then $f \in \mathcal{M}(S)$ would be either a genus zero pencil in which case S would be a rational surface, hence defined over \mathbb{Q} , or a pencil of the form described in part (i) of Proposition 3 in which case the arithmeticity of the surface S would follow from that of the curve C and the points $q_i = f(x_i) \in \mathbb{P}^1$ and $p(x_i) \in C$. □

In view of Proposition 1 and the comments that follow it the above results can be re-formulated as follows:

THEOREM 3. *A complex surface S can be defined over a number field if and only if it is either a minimal surface admitting a Lefschetz function $h = \beta \circ f$ with only three critical values or it is obtained from one such by a finite sequence of blow-ups centered at points with coordinates in $\overline{\mathbb{Q}}$. In the case of ruled surfaces $p: S \rightarrow C$ we must in addition require the curve C and the image on it of the critical points of the Lefschetz pencil f to be defined over a number field too.*

4. Belyi's theorem via Griffiths uniformization. If $\beta: C \rightarrow \mathbb{P}^1$ is a Belyi function on a complex curve C , then the restriction $\beta: C \setminus \Sigma \rightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}$, with $\Sigma = \beta^{-1}(\{0, 1, \infty\})$, is a smooth cover isomorphic to one of the form $\mathbb{H}/\Gamma \rightarrow \mathbb{H}/\Gamma(2)$, where \mathbb{H} is the upper half plane, $\Gamma(2)$ is the level 2 principal congruence subgroup of the classical modular group $\mathbb{P}SL_2(\mathbb{Z})$ and Γ is a finite index subgroup of $\Gamma(2)$. This simple fact readily leads to the following formulation of Belyi's theorem in terms of uniformization.

PROPOSITION 4. *A compact Riemann surface C is a Belyi surface if and only if there is a finite set $\Sigma \subset C$ such that $C \setminus \Sigma$ is isomorphic to a quotient of the form \mathbb{H}/Γ where Γ is a torsion free finite index subgroup of $\mathbb{P}SL_2(\mathbb{Z})$.*

This idea of looking at algebraic curves defined over $\overline{\mathbb{Q}}$ through their particular uniformizing Fuchsian groups goes back to Shabat-Voevodsky and to Grothendieck himself. Proposition 4 and other results of this kind (see e.g. [CIW], [Gon], [JS] and [SV]) may be regarded as a manifestation of how the arithmetic nature of an algebraic curve is reflected in that of its (nonnecessarily torsion free or co-compact) uniformizing groups. This phenomenon can be also paralleled in the 2-dimensional case. Now the role of Fuchsian uniformization of algebraic curves is going to be played by Griffiths uniformization of algebraic surfaces.

4.1. Uniformization of certain Zariski open sets of an algebraic surface.

A domain \mathbb{B} in \mathbb{C}^2 is called a *Bergman domain* if it is the set of pairs (t, z) such that $t \in \mathbb{H}$, and $z \in D_t$ where D_t is a bounded Jordan domain whose boundary curve admits a parametric representation

$$z = W(t, e^{i\theta}), \quad 0 \leq \theta \leq 2\pi,$$

W being, for each fixed θ , a holomorphic function of t (see [Bers]). It is clear that by choosing an isomorphism between the upper half plane and the unit disc \mathbb{D} we can obtain an equivalent domain \mathbb{B}_1 in which \mathbb{H} is replaced by \mathbb{D} . We shall say that \mathbb{B} is bounded if \mathbb{B}_1 is.

By a *Bers transformation* of a Bergman domain $\mathbb{B} \subset \mathbb{H} \times \mathbb{C}$ we shall mean a holomorphic isomorphism $g(t, z) = (\widehat{g}(t), g_t(z))$ where $\widehat{g} \in \mathbb{P}SL_2(\mathbb{R})$ is a real Möbius transformation and $g_t: D_t \rightarrow D_{\widehat{g}(t)}$ is a biholomorphic map. If G is a group of Bers transformations acting freely on \mathbb{B} , there is an obvious short exact sequence

$$1 \hookrightarrow K \rightarrow G \xrightarrow{\rho} \Gamma \rightarrow 1,$$

where the epimorphism ρ is defined by $\rho(g) = \widehat{g}$. We see that while Γ is a Fuchsian group, the group K acts freely on each simply connected region D_t as a group K_t of biholomorphic transformations whose quotient space D_t / K_t is a Riemann surface. We will say that G is a *Griffiths extension* of Γ if for each

$t \in \mathbb{H}$ the surface D_t / K_t is of finite hyperbolic type, that is, a Riemann surface of genus p with r punctures subject to the restriction $2p - 2 + r > 0$.

It is clear that G gives rise to a holomorphic fibration $f: \mathbb{B} / G \rightarrow \mathbb{H} / \Gamma$ whose fibre over a coset $[t] \in \mathbb{H} / \Gamma$, $t \in \mathbb{H}$, is the Riemann surface D_t / K_t . We shall be interested in the case in which Γ is a Fuchsian group of finite type or, as they are also called, of *finite volume*.

Definition 4. We shall say that a 2-dimensional complex manifold U admits a Griffiths uniformization if its holomorphic universal cover $\tilde{U} \rightarrow U$ is isomorphic to a bounded Bergman domain of \mathbb{C}^2 and its covering transformations group G is a Griffiths extension of some finite volume Fuchsian group.

Griffiths uniformization theorem for algebraic surfaces states that every algebraic surface contains a Zariski open set which admits a Griffiths uniformization ([Gri], see also [Bers]).

4.2. Characterization of surfaces defined over $\overline{\mathbb{Q}}$ via Griffiths uniformization.

THEOREM 4. *A minimal nonruled surface $S \subset \mathbb{P}^n(\mathbb{C})$ can be defined over a number field if and only if it contains a Zariski open set U admitting a Griffiths uniformization such that the uniformizing group G is a Griffiths extension of a finite index torsion free subgroup of $\mathbb{P}SL_2(\mathbb{Z})$ of genus zero.*

Proof. Let S be a nonsingular surface defined over $\overline{\mathbb{Q}}$ and $f: S \dashrightarrow \mathbb{P}^1$ a Lefschetz pencil with base locus $B = \{b_1, \dots, b_r\}$, critical points $\{x_1, \dots, x_d\}$ and critical values $\{q_1, \dots, q_d\}$ in $\overline{\mathbb{Q}}$. Then, by restriction to the regular values, one obtains a family of r -punctured nonsingular curves $f: U \rightarrow \mathbb{P}^1 \setminus \{q_i\}$, where $U \subset S$ is the Zariski open set $U = (S \setminus B) \setminus f^{-1}\{q_i\}$. Moreover the genus p of the fibres must be strictly positive, for otherwise the surface would be ruled.

Let $\Gamma \subset \mathbb{P}SL_2(\mathbb{R})$ be the Fuchsian group uniformizing $\mathbb{P}^1 \setminus \{q_i\}$. Then the results in [Gri] imply that U admits a Griffiths uniformization such that the covering transformations group G is a Griffiths extension of Γ by a group K of type (p, r) . We would like to have $\Gamma \subset \mathbb{P}SL_2(\mathbb{Z})$. To achieve this we recall that thanks to Belyi ([Bel]), we can construct a rational function $\beta: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ which sends $\{q_i\}$, as well as all its critical points, to $\{0, 1, \infty\}$, thereby inducing the following smooth cover of pointed Riemann surfaces $\mathbb{P}^1 \setminus \beta^{-1}\{0, 1, \infty\} \rightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}$. It follows that the group Γ_1 uniformizing $\mathbb{P}^1 \setminus \beta^{-1}\{0, 1, \infty\}$ is a subgroup of the group uniformizing $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ which is the principal congruence subgroup $\Gamma(2) \subset \mathbb{P}SL_2(\mathbb{Z})$. Now to settle the “only if” part of the theorem we merely have to replace U by the Zariski open set $U_1 = (S \setminus B) \setminus f^{-1}(\beta^{-1}\{0, 1, \infty\}) \subset U$ and Γ by the group Γ_1 .

Conversely, let us assume that our surface S contains a Zariski open set $U \subset S$ admitting a Griffiths uniformization $U \simeq \tilde{U} / G$, where G is a Griffiths

extension of a finite index torsion free subgroup $\Gamma \subset \mathbb{P}SL_2(\mathbb{Z})$ of genus zero by a group of type (p, r) with $2p - 2 + r > 0$. Then, as noted above, the action of G induces a holomorphic fibration of Riemann surfaces of finite hyperbolic type $f: U \rightarrow \mathbb{P}^1 \setminus \{q_i\}$ with $\mathbb{P}^1 \setminus \{q_i\}$ isomorphic to \mathbb{H} / Γ . We claim that, after applying a suitable Möbius transformation, the values $\{q_i\}$ lie in $\overline{\mathbb{Q}}$. To see this, we note that the natural projection $\mathbb{H} / \Gamma \rightarrow \mathbb{H} / \mathbb{P}SL_2(\mathbb{Z})$ extends to a Belyi function $\beta: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ which sends $\{q_i\}$ to $\{0, 1, \infty\}$. Now, as an easy application of Criterion 1 (see [Gon]), one can see that β is equivalent to a rational function β_0 defined over $\overline{\mathbb{Q}}$, therefore the values $\{q_i\} \subset \beta_0^{-1}(\{0, 1, \infty\})$ must also lie in $\overline{\mathbb{Q}}$ as claimed.

Now, by the results of Imayoshi on holomorphic families of Riemann surfaces ([Ima2], see also [Ima1] and [Ima3]), $f: U \rightarrow \mathbb{P}^1 \setminus \{q_i\}$ extends to a holomorphic map $\hat{f}: \hat{U} \rightarrow \mathbb{P}^1$, where \hat{U} is a compact normal surface bimeromorphic to S such that $U = \hat{U} \setminus Z$, for certain analytic subspace Z .

Let now $\pi: Y \rightarrow \hat{U}$ be a resolution of singularities of \hat{U} (see e.g. [BPV] III.6.1). Then Y is a surface bimeromorphic to \hat{U} , hence to the algebraic surface S . Therefore Y is a projective surface birationally equivalent to S . By Chow's theorem $\pi^{-1}(Z)$ is a Zariski closed subset of Y , and hence $V := \pi^{-1}(U)$ is a Zariski open subset of Y . Moreover, since U does not contain singular points, π induces a holomorphic isomorphism between U and V .

Next we apply the GAGA principle to infer that the map $f_1 = \hat{f} \circ \pi: Y \rightarrow \mathbb{P}^1$ is a regular map. It then follows that for any $\sigma \in Gal(\mathbb{C})$ we have a well defined family of pointed algebraic curves $f_1^\sigma: V^\sigma \rightarrow \mathbb{P}^1 \setminus \{q_i^\sigma\}$.

Now, since S is nonruled, so must be Y , and hence all the surfaces Y^σ , $\sigma \in Gal(\mathbb{C})$. Therefore none of the families $f_1^\sigma: V^\sigma \rightarrow \mathbb{P}^1 \setminus \{q_i^\sigma\}$ can be locally trivial. We can thus apply Arakelov's theorem as given in [IS] to conclude that as σ ranges in $Gal(\mathbb{C})$ the families $f_1^\sigma: V^\sigma \rightarrow \mathbb{P}^1 \setminus \{q_i^\sigma\}$ give rise to only finitely many biholomorphism classes.

Now, again by the results of Imayoshi, if $f_1^\sigma: V^\sigma \rightarrow \mathbb{P}^1 \setminus \{q_i^\sigma\}$ is biholomorphically equivalent to another fibration $f_1^\tau: V^\tau \rightarrow \mathbb{P}^1 \setminus \{q_i^\tau\}$, for some $\tau \in Gal(\mathbb{C})$, then any pair of compactifications of V^σ and V^τ , and in particular Y^σ and Y^τ , must be bimeromorphically equivalent ([Ima2], Theorem 5). Moreover, since Y^σ and Y^τ are algebraic surfaces, they must also be birationally equivalent. We thus see that in the collection $\{Y^\sigma\}_{\sigma \in Gal(\mathbb{C})}$ there are only finitely many birational classes of surfaces. Furthermore, Y^σ being nonruled, S^σ is the unique minimal model of Y^σ . It then follows that the collection of minimal surfaces $\{S^\sigma\}_{\sigma \in Gal(\mathbb{C})}$ gives rise to only finitely many biregular classes. By Criterion 1 we then conclude that S can be defined over $\overline{\mathbb{Q}}$ as was to be proved. \square

Remark 1. It should be noted that the condition imposed in Theorem 4 alone is not a sufficient condition for a minimal ruled surface $p: S \rightarrow C$ to be defined over a number field, even if the base curve C is. This is because, on the one hand, any such surface contains a Zariski open set $V \simeq (C \setminus \Sigma) \times \mathbb{P}^1$, with Σ a finite subset of C , therefore, it surely contains a smaller one of the form

$U \simeq (C \setminus \Sigma) \times (\mathbb{P}^1 \setminus \{0, 1, \infty\})$. The latter admits a Griffiths uniformization with uniformising group $G \simeq K \times \Gamma(2)$, where K is the Fuchsian group uniformising $(C \setminus \Sigma)$ and the action of G is the obvious product action on $\mathbb{H} \times \mathbb{D}$. But, on the other hand, the moduli space of minimal ruled surfaces over a given curve C of genus g is known to depend on $3g - 3$ complex parameters, thus, for mere cardinality reasons, not all of them can be defined over $\overline{\mathbb{Q}}$.

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REFERENCES

- [Arak] S. Arakelov, Families of algebraic curves with fixed degeneracies, *Izv. Akad. Nauk SSSR Ser. Mat.* **35** (1971), 1269–1293.
- [BPV] W. Barth, C. Peters, and A. Van de Ven, *Compact Complex Surfaces*, *Ergeb. Math. Grenzgeb.*, vol. 3, Springer-Verlag, Berlin, 1984.
- [Bea] A. Beauville, Surfaces algébriques complexes, *Astérisque* **54** (1978).
- [Bel] G. Belyi, On Galois extensions of a maximal cyclotomic field, *Math. USSR Izv.* **14** (1980), 247–256.
- [Bers] L. Bers, Uniformization, moduli, and Kleinian groups, *Bull. London Math. Soc.* **4** (1972), 257–300.
- [Cap] L. Caporaso, On certain uniformity properties of curves over function fields, *Compositio Math.* **130** (2002), 1–19.
- [CIW] P. Cohen, C. Itzykson, and J. Wolfart, Fuchsian triangle groups and Grothendieck dessins. Variations on a theme of Belyi, *Comm. Math. Phys.* **163** (1994), 605–627.
- [GS] R. Gompf and A. Stipsicz, *4-Manifolds and Kirby Calculus*, *Grad. Stud. Math.*, vol. 20, Amer. Math. Soc., Providence, RI, 1999.
- [Gon] G. González-Diez, Variations on Belyi's theorem, *Quart. J. Math.* **57** (2006), 355–366
- [Gri] P. Griffiths, Complex-analytic properties of certain Zariski open sets on algebraic varieties, *Ann. of Math. (2)* **94** (1971), 21–51.
- [Groth] A. Grothendieck, “*Esquisse d'un Programme*”, *London Math. Soc. Lecture Note Ser.*, vol. 242 (L. Schneps and P. Lochak, eds.), Cambridge Univ. Press, Cambridge, 1997.
- [HM] J. Harris and I. Morrison, *Moduli of Curves*, *Grad. Texts in Math.*, vol. 187, Springer-Verlag, Berlin, 1991.
- [Ima2] Y. Iwayoshi, Holomorphic families of Riemann surfaces and Teichmüller spaces. II, *Tôhoku Math. J. (2)* **31** (1979), 469–489.
- [Ima1] ———, Holomorphic families of Riemann surfaces and Teichmüller spaces, *Riemann Surfaces and Related Topics, Proceedings of the 1978 Stony Brook Conference*, *Ann. of Math. Stud.*, vol. 97, Princeton Univ. Press, 1981, pp. 277–300.
- [Ima3] ———, Holomorphic families of Riemann surfaces and Teichmüller spaces. III. Bimeromorphic embedding of algebraic surfaces into projective spaces by automorphic forms, *Tôhoku Math. J. (2)* **33** (1981), 227–247.
- [IS] Y. Iwayoshi and H. Shiga, A finiteness theorem for holomorphic families of Riemann surfaces, *Holomorphic Functions and Moduli II, MSRI*, vol. 28, 1998, pp. 207–219.
- [JS] G. Jones and D. Singerman, Belyi functions, hypermaps and Galois groups, *Bull. London Math. Soc.* **28** (1996), 561–590.
- [Lam] K. Lamotke, The topology of complex projective varieties after S. Lefschetz, *Topology* **20** (1981), 15–51.

- [Nag] S. Nag, *The Complex Analytic Theory of Teichmüller Spaces*, Wiley-Interscience, NY, 1998.
- [SL] L. Schneps and P. Lochak, *Geometric Galois Actions, 1; Around Grothendieck's "Esquisse d'un programme"*, 2; *The Inverse Galois Problem, Moduli Spaces and Mapping Class Groups*, London *Math. Soc. Lecture Note Ser.*, vols. 242–243, Cambridge Univ. Press, 1997.
- [SV] G. Shabat and V. Voevodsky, Equilateral triangulations of Riemann surfaces, and curves over algebraic number fields, *Soviet Math. Dokl.* **39** (1989), 38–41.
- [We] A. Weil, The field of definition of a variety, *Amer. J. Math.* **78** (1956), 509–524.