ON EXTREMAL RIEMANN SURFACES AND THEIR UNIFORMIZING FUCHSIAN GROUPS

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Abstract. Compact hyperbolic surfaces of given genus g containing discs of the maximum radius have been studied from various points of view. In this paper we connect these different approaches and observe some properties of the Fuchsian groups uniformizing both compact and punctured extremal surfaces. We also show that extremal surfaces of genera $g = 2, 3$ may contain one or several extremal discs, while an extremal disc is necessarily unique for $g \geq 4$. Along the way we also construct explicit families of extremal surfaces, one of which turns out to be free of automorphisms.

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1. Introduction. Extremal surfaces, that is hyperbolic surfaces containing discs of maximum radius in given genus, appear in the literature from different points of view: as *cycloidal groups* (Petterson and Millington, see [**13**]), as *central curves* (Macbeath [**12**]), from the point of view of generic polygon side pairings (Fricke and Klein [**4**] or Jorgensen and Näätänen [**11**]) and finally as genuine extremal surfaces (Bavard [**2**]). In this paper we study some properties of extremal surfaces and the groups which uniformize them. Its content is as follows.

In section 2 we gather together the above mentioned points of view and show that they are all equivalent. The key characterization is that extremal surfaces of genus g arise, exactly, as semiregular covers of the sphere with three branch points of order $(2, 3, 12g - 6)$. We also construct two explicit families X_g, Y_g of such surfaces.

In section 3 we show that the groups uniformizing extremal surfaces (resp. extremal surfaces with the center deleted, called *cycloidal*) are never normal in the triangle group of type $(2, 3, 12g - 6)$ (resp. in $\mathbb{PSL}_2(\mathbb{Z})$).

In sections 4, 5 we show that while extremal surfaces of genera $g = 2, 3$ may contain several extremal discs, for $g \geq 4$ extremal surfaces contain exactly one. In the case of genus 2 we, in fact, consider the eight extremal surfaces of genus 2, whose description goes back to Fricke and Klein ([**4**], see also [**11**]), and detect the number of centers each of them has. The uniqueness of discs when $g \geq 4$ is obtained by linking extremality to arithmeticity (of the uniformizing groups). This result was presented in the note [**6**] but we include it here for the sake of completeness.

In section 6 we use the fact that the surfaces Y_g contain a unique extremal disc to prove that they have trivial automorphism group, a property that while being

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generically satisfied, is not usually held by the surfaces one can explicitly handle. Other examples can be found in [**18**] from the algebraic curve point of view and in [**3**] much in our vein.

2. Extremal discs and extremal surfaces. Let S be a compact hyperbolic surface of genus g, and let $D(R)$ be a disc of radius R isometrically embedded in S. It goes without saying that S is understood to be equipped with the metric induced by the Poincaré metric of the upper half plane $\mathbb{H}$ —or the unit disc $\mathbb{D}$— via uniformization. We have

Theorem 1 (Bavard [2]) *Let R be the radius of a metric disc isometrically embedded into a compact Riemann surface S of genus g.*

Then $R \leq R_g := \cosh^{-1} 1/(2\sin\frac{\pi}{12g-6})$, which is the radius of the inscribed circle to the regular $(12g-6)$-gon. When the latter is attained, S has a regular $(12g-6)$-gon as Dirichlet domain and $D(R_g)$ is its inscribed disc.

We shall refer to discs of such radius R_g as *extremal discs* and, accordingly, surfaces containing extremal discs will be called *extremal surfaces*.

We may look at extremal surfaces from different points of view; this way we will encounter different characterizations of the concept of extremality. Besides the standard definitions on Fuchsian group theory, for which we refer to [**10**], we shall need to recall the following concepts.

• A Riemann surface is called *circular* [**12**] if it is uniformized by a Fuchsian group possessing a Dirichlet fundamental domain whose inscribed circle touches all its edges. If in addition the Dirichlet domain can be chosen to be obtained by a generic surface matching, that is an edge matching such that every vertex cycle has length 3, the surface is called a *central curve* [**12**].

• A subgroup of the modular group $\mathbb{P}\mathrm{SL}_2(\mathbb{Z})$ is termed *cycloidal* [**13**] if it has only one conjugacy class of parabolic elements.

• A (possibly ramified) covering of Riemann surfaces $f : X \to Y$ is called *semiregular* if every two points belonging to the same fiber have the same multiplicity. Clearly regular coverings, also called normal coverings, are semiregular.

We can now state our first result (compare with [**2**]).

Theorem 2 *Let S be a compact Riemann surface of genus g, and let $N = N(g) = 12g-6$. The following statements are equivalent:*

1) S is extremal.

2) S is uniformized by a Fuchsian group K which is an index N subgroup of a triangle group $\Gamma(2,3,N)$.

3) S is a central curve.

4) S is the compactified space of the quotient surface $\mathbb{H}/K_*$, where K_* is a torsion free cycloidal subgroup of the modular group.

5) There exists a semiregular covering $f : S \longrightarrow \widehat{\mathbb{C}}$ with degree N and ramified over three points with indexes $2, 3, N$.

Proof: We start by proving that 1) $\Leftrightarrow$ 2):

Assume that S is an extremal surface, and let K be the Fuchsian group uniformizing S. Then K admits the regular N-gon P as fundamental domain, and by the Gauss-Bonnet formula the angle subtended by a vertex equals $\frac{2\pi}{3}$.

Consider the natural projection $\mathbb{D} \longrightarrow \mathbb{D}/K \simeq S$, and let $p \in S$ be the center of the extremal disc. Then the Dirichlet polygons centered at the points in the fiber above p form a tessellation of the hyperbolic disc $\mathbb{D}$ by regular polygons of angle $\frac{2\pi}{3}$ and $12g-6$ sides. The group K can be recovered as the group of isometries of $\mathbb{D}$ which preserve this tessellation and are compatible with the side identifications of the polygons. But it is clear that these isometries leave also invariant the subtessellation by quadrilaterals Q of angles $\pi/2, \pi/2, 2\pi/3$ and $2\pi/N$ obtained by drawing the lines which connect the center of the polygons to the midpoints of each side. The group which preserves the latter is the triangle group $\Gamma(2,3,N)$, hence $K < \Gamma(2,3,N)$.

Conversely let S be uniformized by a subgroup of index N in the group $\Gamma(2,3,N)$ with fundamental domain given by a quadrilateral Q as above. Then a fundamental domain for S, which is a regular N-gon, can be obtained by taking the reunion of N copies of Q around the vertex of angle $2\pi/N$.

The equivalence between 2) and 3) is the content of p. 139-140 of [**12**].

To prove 2) $\Rightarrow$ 4), suppose that S is uniformized by a subgroup K of the triangle group $\Gamma(2,3,N)$. Consider the natural quotient maps $\mathbb{D} \xrightarrow{\pi} \frac{\mathbb{D}}{K} \xrightarrow{f} \frac{\mathbb{D}}{\Gamma(2,3,N)}$, and denote by $\overset{\circ}{\mathbb{D}}$ the open subset of $\mathbb{D}$ obtained by removing the branch points of $f \circ \pi$.

Following [**12**] we explicitely construct the universal cover of $\overset{\circ}{\mathbb{D}}$ by first considering the Riemann mapping h from the triangle $T(2,3,\infty) = \{z \in \mathbb{H} \text{ s.t. } |z| > 1, -\frac{1}{2} < \operatorname{Re}(z) < 0\}$ onto the triangle $T(2,3,N)$ and then extending it to the whole upper plane $\mathbb{H}$ by reflection across the sides (Schwarz's principle). This way the fundamental domain Q of $\mathbb{P}\mathrm{SL}_2(\mathbb{Z})$ (= two copies of $T(2,3,\infty)$) is bijectively mapped onto that of $\Gamma(2,3,N)$ and each of its translates $\alpha(Q)$, $\alpha \in \mathbb{P}\mathrm{SL}_2(\mathbb{Z})$ onto $k_\alpha \circ h(Q)$ for a uniquely defined $k_\alpha \in \Gamma(2,3,N)$ determined by the identity $h \circ \alpha = k_\alpha \circ h$. Therefore the union of the N translates of Q under the transformation $T(z) = z+1$, namely $\bigcup_{i=1}^{N} T^i(Q)$, is mapped onto the regular N- gon P (with the center removed). We thus see that the group K_* of covering transformations of $(\mathbb{H}, \pi \circ h)$, which consists of those $\alpha \in \mathbb{P}\mathrm{SL}_2(\mathbb{Z})$ such that $k_\alpha \in K$, admits $\bigcup_{i=1}^{N} T^i(Q)$ as fundamental domain. We also observe that K_* is torsion free for $\alpha(z) = z$ would imply $k_\alpha(h(z)) = h(z)$, contradicting the fact that K is torsion free. It follows that K_* is the required cycloidal subgroup.

The implication 4) $\Rightarrow$ 5) is obvious: the required covering $f : S \longrightarrow \widehat{\mathbb{C}}$ is obtained by compactifying the projection $\mathbb{H}/K_* \longrightarrow \mathbb{H}/\mathbb{P}\mathrm{SL}_2(\mathbb{Z})$.

The statement 5) $\Rightarrow$ 2) is a known result (see e.g. [**15**], [**5**]). $\square$

Remark 1 *(Monodromy)*
With the help of monodromy (see [**14**] for definitions and first properties) we can give the following useful combinatorial description of extremal surfaces:

Let $f : S \longrightarrow \widehat{\mathbb{C}}$ be the semiregular covering associated to the extremal surface S in the way described above. Let us denote by z_2, z_3, z_N the three branch values of f and by $\gamma_2, \gamma_3, \gamma_N$ the canonical loops encircling them that generate the fundamental group of $\widehat{\mathbb{C}} \setminus \{z_2, z_3, z_N\}$. The semiregularity of f forces the monodromy to satisfy $M(\gamma_k) = \sigma_k$, where σ_k is a product of $\frac{N}{k}$ disjoint k-cycles. As the only relation in $\Pi_1(\widehat{\mathbb{C}} \setminus \{z_2, z_3, z_N\})$ is $\gamma_3 = \gamma_N \gamma_2$ we see, by normalizing $\sigma_N = (1, 2, \ldots, N)$, that extremal discs are in one to one correspondence with the set of permutations σ_2 equal to the product of $\frac{N}{2}$ disjoint transpositions such that the composition $\sigma_N \sigma_2$ equals a product of $\frac{N}{3}$ disjoint 3-cycles. We identify σ_2 and σ_2' when $\sigma_2' = \sigma_N^j \sigma_2 \sigma_N^{-j}$, which geometrically corresponds to relabeling of sides of the N-gon. The corresponding extremal surface is obtained from σ_2 by pairing sides labeled k and $\sigma_2(k)$.

This remark shows that there are only finitely many generic surface matchings (or, equivalently, extremal discs up to isometry) and suggests a combinatorial approach to counting its number alternative to that employed in [**1**] (indeed see [**12**]). It turns out that this number grows exponentially with g. Here are some explicit examples of extremal surfaces:

Example 1: We shall denote by X_g $(g \geq 2)$ the extremal surface given by

$$\sigma_2 = \prod_{k=1}^{g}(-1+2k, 6g-4+2k)(2g+2-4k, 2g-1+2k)(4g-3+4k, 4g-2k)$$

$$\prod_{k=0}^{g-2}(12g-7-2k, 6g-4-2k)(8g-5-4k, 8g-2+2k)(10g-5-2k, 10g-2+4k)$$

Example 2: Y_g $(g \geq 3)$ will denote the extremal surface defined by

$$\sigma_2 = (1, 6g-2)(3g-4, 9g-7)(3g+3, 9g)(3g-3, 3g+1)(9g-1, 9g-5)(3g, 12g-6)$$

$$(9g-4, 2)(12g-7, 3g-2)(3, 9g-2)(3g+2, 9g-8)(3g+4, 9g-6)(3g-1, 6g-1)$$

$$(6g-3, 9g-3)\prod_{k=0}^{g-3}(9g+1+3k, 9g-9-3k)(3g-5-3k, 3g+5+3k)$$

$$\prod_{k=1}^{g-3}(12g-6-3k, 3+3k)(6g-2-3k, 6g-1+3k)$$

$$\prod_{k=0}^{g-4}(5+3k, 6g+1+3k)(6g-6-3k, 12g-10-3k)$$

(in both cases notation should be understood modulo $12g-6$).

This way we have produced a constructive proof of the following theorem.

Theorem 3 (Bavard [2]) *On each fixed genus g, there is a finite positive number of extremal discs and extremal surfaces (up to equivalence).*

Remark 2 It is a straightforward task to write down explicit generators for the groups K uniformizing these surfaces. Only observe that the hyperbolic transformation $\gamma_{k,l}$ which sends the k-th side to the l-th one can be written as $\gamma_{k,l} = R^{l-1} \circ L \circ R^{1-k}$, where L is the order 2 elliptic transformation with the midpoint of side 1 fixed, and R is the rotation around the origin through angle $\frac{2\pi}{N}$.

From the point of view of cycloidal groups, we can proceed as follows. With the notation of the proof of the part 2) $\Rightarrow$ 4) in theorem 2, we can define the homomorphism of groups $\psi : K_* \longrightarrow K$ given by $\alpha \longmapsto k_\alpha$.

It is not difficult to show that ψ is surjective: if $T(z) = z+1$ and $S(z) = -1/z$ are the usual generators of $\mathbb{P}\mathrm{SL}_2(\mathbb{Z})$, preimages for the generators $\gamma_{k,l}$ of K given above are afforded by $\overset{\circ}{\gamma}_{k,l} = T^{l-1} \circ S \circ T^{1-k}$. Moreover, the kernel of ψ agrees with the cyclic group generated by T^N and hence we have $K \simeq \frac{K_*}{<T^N>}$.

3. Some peculiarities of Fuchsian groups uniformizing extremal surfaces.
a) We maintain the same notation of theorem 2 so that K (resp K_*) is a Fuchsian group uniformizing an extremal surface (resp. a punctured extremal surface).

Proposition 1 *The inclusion $K < \Gamma(2,3,N)$ (resp. $K_* < \mathbb{P}\mathrm{SL}_2(\mathbb{Z})$) is never normal; in other words, the semiregular covering $f : S \longrightarrow \widehat{\mathbb{C}}$ associated to an extremal surface is never regular.*

Proof: Suppose that K (resp. K_*) is a normal subgroup of $\Gamma(2,3,N)$ (resp. $\mathbb{P}\mathrm{SL}_2(\mathbb{Z})$). Then the morphism $f : S \longrightarrow \widehat{\mathbb{C}}$ can be viewed as a quotient map $S \xrightarrow{f} \frac{S}{H} \simeq \widehat{\mathbb{C}}$, where $H \simeq \Gamma(2,3,N)/K$ (resp. $\simeq \mathbb{P}\mathrm{SL}_2(\mathbb{Z})/K_*$) is a group of automorphisms of S.

Now recall that f has degree N and ramifies with order N over a point $z_N \in \widehat{\mathbb{C}}$. Hence, the fiber above z_N is a single point $s \in S$ and the H-stabilizer of s agrees with the whole H. Since the stabilizer of a point is always cyclic, we conclude that H is a cyclic group whose order equals $\deg(f) = 12g - 6$. But this is not possible (see [**8**]). $\square$

b) In [**13**] the existence of cycloidal subgroups of the modular group was studied, and an explicit family of examples was given. It turns out that his family of cycloidal subgroups uniformize precisely the punctured surfaces obtained by removing the center of the extremal disc from our surfaces X_g (see part 3 of theorem 2).

c) We can ask whether the groups K_* are *congruence subgroups* of the modular group $\Gamma = \mathbb{P}\mathrm{SL}_2(\mathbb{Z})$. Let us assume so and let d be the least positive integer such that the *principal congruence subgroup* $\Gamma(d)$ is contained in K_*. Then by [**19**] we would have $d = [\mathrm{stab}_\Gamma(\infty) : \mathrm{stab}_{K_*}(\infty)] = [\mathbb{P}\mathrm{SL}_2(\mathbb{Z}) : K_*] = N$. We thus conclude:

Proposition 2 *The groups K_* are congruence subgroups of the modular group if and only if they contain the group $\Gamma(N)$.*

Remark 3 i) The Riemann-Hurwitz formula applied to the covering $\hat{\mathbb{H}}/\Gamma(N) \longrightarrow \hat{\mathbb{H}}/K_*$ provided by a hypothetical inclusion $\Gamma(N) < K_*$ would give the expression $g_N = 1 + \mu_N \frac{N-6}{12N}$ for the genus of $\mathbb{H}/\Gamma(N)$, where $\mu_N = [\mathbb{P}\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)]$. This is indeed the known expression for g_N (see e.g. [**10**]). Nevertheless, at present we do not have an answer to the question of whether K_* are congruence subgroups.

ii) The existence of non-congruence subgroups was already known to Klein and Fricke (see [**9**]). If no restriction is imposed on the number t of conjugacy classes of parabolic elements G. Jones [**9**] has shown the existence of infinitely many non-congruence subgroups in given genus.

4. Uniqueness of extremal discs when $\mathbf{g \geq 4}$. In this section we summarize the following result, that was proved in [**6**].

Theorem 4 *A hyperbolic surface of genus $g > 3$ contains at most one extremal disc.*

Proof: Suppose that S contains two extremal discs D_1 and D_2. From the characterization of extremal discs given in theorem 2 one sees that each of these discs corresponds to an inclusion of the uniformizing group K in a triangle group Γ_i $(i = 1, 2)$ of type $(2,3,N)$ such that $[\Gamma_i : K] = N = 12g - 6$.

Let now α be any isometry of $\mathbb{D}$ conjugating the two triangle groups (of same type) Γ_1, Γ_2 so that we have $\Gamma_1 = \alpha^{-1} \circ \Gamma_2 \circ \alpha$. We see that $K \subset (\Gamma_2 \cap \Gamma_1) = \Gamma_2 \cap \left(\alpha^{-1} \circ \Gamma_2 \circ \alpha\right)$, which shows that $\alpha \in \mathrm{Com}(\Gamma_2)$, the commensurator of Γ_2.

Now, when $g > 3$ then $N > 30$. It follows by a result of Takeuchi [**17**] that Γ_2 is not arithmetic, which in turn implies by a well known theorem of Margulis that $\mathrm{Com}(\Gamma_2)$ is Fuchsian. Finally, we use the fact proved by Singerman in [**16**] that groups of type $(2, 3, k)$ are maximal to deduce that $\mathrm{Com}(\Gamma_2) = \Gamma_2$; in other words, $\alpha \in \Gamma_2$, which means that we have $\Gamma_1 = \Gamma_2$. We conclude that $D_1 = D_2$. □

5. The cases $\mathbf{g = 2, 3}$ (non-uniqueness of discs). Case $\mathbf{g = 2}$:

Bavard gave in [**2**] an example of a genus 2 extremal surface containing a unique extremal disc. In fact it corresponds to our surface X_2 of Example 1. He also showed an example of a surface containing more than one disc.

Using the characterization 3 of theorem 2 and the results of [**4**] (see also [**11**]) we can prove that there are *exactly* eight extremal surfaces of genus 2 up to conformal or anti-conformal isometry of surfaces with marked extremal disc. Namely,

• *The extremal surfaces that occur in genus two are precisely the ones depicted in page* 267 *of* [**4**] *(see also* [**11**]*). No pair of these eight surfaces are isometrically equivalent as simple (i.e. unmarked) Riemann surfaces. The number of extremal discs each one possesses is* 2,1,2,2,2,4,2,2 *respectively.*

The proof of this statement is too long to be included here and can be found in [**7**]. We content ourselves with presenting an

Idea of the proof: There are two key ingredients.

a) To discover hidden discs in a given surface, we first look for isometries. Then their action on the explicit extremal disc centered at the center o of the polygon will uncover new extremal discs (except, of course, for the isometries fixing o).

For instance the rotation through angle π does not induce an isometry on surface VI of of [**4**], thus the hyperelliptic involution J does not fix o, hence $J(o)$ is a second center. On the other hand this surface has an isometry σ_3 induced by the rotation through angle $\frac{2\pi}{3}$, therefore, by known results relative to the automorphism group of surfaces of genus two, it must possess a further isometry τ of order 2 different from J, hence $\tau(o)$ is a third center, the fourth one being $J \circ \tau(o)$.

b) To ensure that in this way we have found the whole set of discs we use the fact (already employed by C. Bavard in the above mentioned work) that being the center of an extremal disc imposes on a point $z \in \mathbb{D}$ certain restrictions on the amount this point is displaced under the group K of isometries of $\mathbb{D}$ generated by the side pairing transformations. (For instance, $d(z, \gamma(z))$ must be $\geq 2R$, since z and $\gamma(z)$ serve as centers of two non overlapping discs of radius R inside $\mathbb{D}$).

c) The fact that these surfaces are pairwise non isomorphic follows from explicit knowledge of location of centers, group of isometries, and Weierstrass points.

Case $\mathbf{g = 3}$:

In [**1**] (see also [**12**]) it has been computed the exact number of (marked) extremal surfaces of genus 3, which turns out to be 927. We shall show that as in the case of genus 2, extremal surfaces of genus 3 may contain one or several extremal discs:

Unique disc.- It is not difficult to show that the surface X_3 of Example 1 contains only one extremal disc. This can be done using the techniques employed in [**2**].

Several discs.- Let us consider the surface S defined by the following side pairing in the 30-gon:

$$(1+k,10+k)(2+k,5+k)(3+k,7+k)(4+k,8+k)(6+k,9+k),\ k=0,10,20.$$

It is clear that S admits an order 3 rotation τ that fixes the points O and P, corresponding to the center of the polygon and the common vertex of sides 30 and 1.

The natural projection $\pi : S \longrightarrow \frac{S}{<\tau>}$ maps S into a complex torus E with branching over two points $\pi(O)$ and $\pi(P)$.

Let us write $E = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\sigma)$, $\mathrm{Im}(\sigma) > 0$. If one performs translation by $\frac{1}{2}(-\pi(O)-\pi(P))$ in E, then the branching values become w and $-w$, where $w = \frac{1}{2}(\pi(O)-\pi(P))$. We have:

Proposition 3 *The automorphism $T : E \longrightarrow E$ given by $T(z) = -z$ lifts to an isometry $\widetilde{T} : S \longrightarrow S$ which sends the point O to the point P. In particular P is the center of another extremal disc of S.*

Proof: Put $S^* = S \setminus \{O,P\}$, $E^* = E \setminus \{w,-w\}$. Then, the quotient map $\pi : S \longrightarrow E$ above, induces, by restriction, a non-ramified normal covering $\pi : S^* \longrightarrow E^*$ whose covering group, $<\tau>$, is isomorphic to the quotient $\Pi_1(E^*)/\pi_*(\Pi_1(S^*))$.

It is a well known fact of covering space theory that the (induced) mapping $T : E^* \longrightarrow E^*$ lifts to an automorphism $\widetilde{T}$ of S^*, and hence of S, if and only if $T_*(\pi_*(\Pi_1(S^*)))$ is a subgroup of $\pi_*(\Pi_1(S^*))$.

Instead of studying the action of T on the homotopy group directly, we address the more accesible question of understanding the action on the homology group $H_1(E^*)$. One finds (see lemma 1 below) that, at the homological level, $T_* = -\mathrm{id}$.

Now we recall that $H_1(E^*)$ is the abelianized group of $\Pi_1(E^*)$, that is $H_1(E^*) \simeq \Pi_1(E^*)/[\Pi_1(E^*),\Pi_1(E^*)]$, where, as usual, the brackets [,] stand for the commutator subgroup. Thus, the fact that the quotient $\Pi_1(E^*)/\pi_*(\Pi_1(S^*)) \simeq <\tau>$ is abelian, means that $[\Pi_1(E^*),\Pi_1(E^*)]$ is a subgroup of $\pi_*(\Pi_1(S^*))$.

With this in mind, for any $\gamma \in \Pi_1(S^*)$, we write $T_* = -\mathrm{id}$ as $T_*(\pi_*\gamma) = (\pi_*\gamma)^{-1}$, modulo $[\Pi_1(E^*),\Pi_1(E^*)]$. We deduce that $T_*(\pi_*(\Pi_1(S^*)))$ is contained in $\pi_*(\Pi_1(S^*))$, which proves the existence of a lift $\widetilde{T}$.

The fact that $\widetilde{T}(O) = P$ is a consecuence of the fact that $T(w) = -w$. $\square$

Lemma 1 *Let us denote by E the complex torus $E = \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\sigma)$, $Im(\sigma) > 0$. Let w be a point of E, and set $E^* = E \setminus \{w,-w\}$. Then, the action induced by T on $H_1(E^*)$ is multiplication by -1.*

Proof: Let us regard the surface E as a parallelogram with sides identified in the standard way. Let us denote by A and B the homology classes in E^* induced by its sides and by W (resp. $-W$) that induced by a small circle around the point w (resp. $-w$). Then direct inspection shows that $T_*(A) = -A$, $T_*(B) = -B$ and $T_*(W) = -W$. Since $H_1(E^*)$ is generated by A, B, and W, we are done. $\square$

6. An explicit family of extremal surfaces without automorphisms. In this section we shall prove that the extremal surfaces Y_g introduced in Example 2 lack isometries. The idea is that the uniqueness of extremal discs shown in theorem 4 almost implies that extremal surfaces of genus $g \geq 4$ have no automorphisms. More precisely, an automorphism of such a surface has to fix both the extremal disc and its center, and hence has to be a rotation; but it is a simple matter to check whether a rotation induces an automorphism of a polygon with given side pairing.

In this way, we get the following:

Theorem 5 *Y_g, $(g \geq 4)$ is a family of extremal surfaces each of which has trivial group of automorphisms.*

Proof: It can be directly checked looking at the permutation that defines Y_g that there are exactly 3 pairs of opposite sides which have been identified, namely $(1, 6g-2), (3g-4, 9g-7), (3g+3, 9g)$. Hence the only rotation that could induce an automorphism on Y_3 is the order 2 rotation. But side $3g-3$ is identified with side $3g+1$, whereas side $9g-6$ is identified with side $3g+4$ instead of $9g-2$. Therefore no rotation induces an automorphism on Y_g. $\square$

Note: The above result still holds for $g = 3$ but proving that Y_g has only one disc requires the use of techniques similar to those used in [**2**] or [**7**].

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