Submitted exclusively to the London Mathematical Society
DOI: 10.1112/S0000000000000000

A NOTE ON THE ACTION OF THE ABSOLUTE GALOIS GROUP ON DESSINS

ERNESTO GIRONDO AND GABINO GONZÁLEZ-DIEZ

Abstract

We show that the action of the absolute Galois group on dessins d'enfants of given genus g is faithful, a result that had been previously established for $g = 0$ and $g = 1$.

1. *Belyi surfaces, dessins d'enfants, and the absolute Galois group*

One of the fundamental results in Riemann surface theory is the fact that every compact Riemann surface can be thought of as an algebraic curve and every meromorphic function as a rational function. A compact Riemann surface C is said to be *defined* (or, better, *definable*) over some field $\Bbbk$ if it is isomorphic (as complex manifold) to the Riemann surface C_F associate to an algebraic curve $F(x, y) = 0$ with coefficients in $\Bbbk$. A fundamental problem is then to characterize the definability over some given field. For example, a Riemann surface can be defined over the reals exactly when it has an anticonformal involution.

Belyi proved (see [**1**]) that when a surface C can be defined over a number field (thus over $\overline{\mathbb{Q}}$), then it has a meromorphic function f with only three branching values, what is nowadays called a *Belyi function* (accordingly (C, f) is then termed a *Belyi pair*). The reciprocal statement is also true and follows from previous work of Weil: if $f : C \to \widehat{\mathbb{C}}$ ramifies only over $\{0, 1, \infty\}$ then C and f are definable over $\overline{\mathbb{Q}}$. Definability over number fields is therefore characterized by the existence of Belyi functions.

Part of the interest in this theory comes from the fact that the *absolute Galois group* $\mathrm{Gal}(\overline{\mathbb{Q}})$ acts on curves defined over $\overline{\mathbb{Q}}$. If $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}})$ and $C = C_F$, the action is defined by $\sigma(C) = C_{F^\sigma}$, where F^σ is obtained from F applying σ to its coefficients. One could thus obtain information about $\mathrm{Gal}(\overline{\mathbb{Q}})$ by understanding this action.

Grothendieck early pointed out the striking fact that Belyi pairs can be described in a very simple combinatorial way, as they correspond exactly to what he called *dessins d'enfants* (see [**2**]). A dessin is an embedding of a bipartite graph in a topological surface such that the complement of the graph is a union of simply connected cells. For a Belyi pair (C, f) the corresponding dessin $\mathcal{D}_f$ is given by the set $f^{-1}([0, 1])$. One can therefore study such an important object as $\mathrm{Gal}(\overline{\mathbb{Q}})$ by making it act on graphs so simple that they look like children's drawings (*dessins*

2000 *Mathematics Subject Classification* 14H55 (primary), 11R32 (secondary).

Both authors were supported by MCyT, research project BFM2003-04964

d'enfants), the action being defined by $\sigma(\mathcal{D}_f) = \mathcal{D}_{f^\sigma}$. A lot of work has been done in the last years in this direction (see [**5**], [**4**], and the references given there).

It is well known that the action of $\mathrm{Gal}(\overline{\mathbb{Q}})$ is faithful ondessins of genus 1, see [**6**], [**3**]. The action is also faithful on dessins on the sphere, even when restricted to the subclass formed by the trees (see [**6**]). Nevertheless, as far as we know, there is in the literature no proof of the faithfulness of the action of $\mathrm{Gal}(\overline{\mathbb{Q}})$ on genus g dessins if $g > 1$. The aim of this note is to provide such a proof.

2. *Faithfulness of the action of* $\mathrm{Gal}(\overline{\mathbb{Q}})$ *on dessins of genus g*

We now show that $\mathrm{Gal}(\overline{\mathbb{Q}})$ acts faithfully on genus g Belyi surfaces ($g \geq 2$). In fact it is enough to restrict to hyperelliptic surfaces.

Theorem 1. *Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}})$ be an element of the absolute Galois group, $\sigma \neq \mathrm{Id}$, and $a \in \overline{\mathbb{Q}}$ an algebraic number such that $\sigma(a) \neq a$. Let C_n ($n \in \mathbb{N}$) be the hyperelliptic curve*

$$C_n := \{y = (x-1)(x-2)\dots(x-(2g+1))(x-(a+n))\}.$$

Then, there is an n such that C_n^σ is not isomorphic to C_n.

Proof. Suppose the theorem false, so that $C_n^\sigma \simeq C_n$ for all $n \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$ there is a Möbius transformation $M_n \in \mathbb{P}\mathrm{SL}_2(\mathbb{C})$ such that

$$M_n\left(\{1, 2, \dots, 2g+1, (a+n)\}\right) = \{1, 2, \dots, 2g+1, \sigma(a+n)\}.$$

We have:

(1) $M_n \in \mathbb{P}\mathrm{SL}_2(\mathbb{Q})$, since it maps three rational points to three rational points.
(2) $M_n(\{1, 2, \dots, 2g+1\}) = \{1, 2, \dots, 2g+1\}$ by (1), since $a + n \notin \mathbb{Q}$.
(3) $M_n(a+n) = \sigma(a+n) = \sigma(a) + n$, by (2).
(4) There are three distinct natural numbers p, q, r such that $M_p = M_q = M_r$. In fact, among all the transformations M_n there must be infinitely many coincidences, since by (2) $\{M_n;\ n \in \mathbb{N}\}$ must be a finite set.

We see therefore that

$$\left.\begin{array}{r} M_p(a+p) = \sigma(a) + p \\ M_p(a+q) = M_q(a+q) = \sigma(a) + q \\ M_p(a+r) = M_r(a+r) = \sigma(a) + r \end{array}\right\} \Rightarrow \left.\begin{array}{r} M_p(a+p) - \sigma(a) = p \\ M_p(a+q) - \sigma(a) = q \\ M_p(a+r) - \sigma(a) = r \end{array}\right\}$$

Let us then consider the Möbius transformation $M(z) := M_p(a+z) - \sigma(a)$. As $M(p) = p, M(q) = q$ and $M(r) = r$, it follows that $M = \mathrm{Id}$, and then $M_p(a+z) = z + \sigma(a)$, thus $M_p(z) = z + \sigma(a) - a$.

Now $1 + (\sigma(a) - a) = M_p(1) = l_1 \in \{1, 2, \dots, 2g+1\}$, the same applying to $2 + (\sigma(a) - a) = M_p(2) = l_2, \dots, (2g+1) + (\sigma(a) - a) = M_p(2g+1) = l_{2g+1}$. But then

$$(\sigma(a) - a) = (l_1 - 1) = (l_2 - 2) = \dots = (l_{2g+1} - (2g+1))$$

with $(l_1 - 1) \geq 0$ and $(l_{2g+1} - (2g+1)) \leq 0$. It follows that $l_i - i = 0$ for all i, thus $l_i = M_p(i) = i$ and $M_p = \mathrm{Id}$. Then $\sigma(a) + p = M_p(a+p) = a + p$ and $\sigma(a) = a$, a contradiction. □

References

1. G. Belyi, 'On Galois extensions of a maximal cyclotomic field', *Izv. Akad. Nauk SSSR, Ser. Mat.* 43:2 (1979) 269–276. English translation in *Math. USSR Izv.* 14 (1979), 247–256.
2. A. Grothendieck, 'Esquisse d'un programme', *Geometric Galois actions 1* 5–48. London Math. Soc. Lecture Note Ser. 242, Cambridge Univ. Press, Cambridge (1997).
3. G. Jones and D. Singerman, 'Belyĭ functions, hypermaps and Galois groups', *Bull. London Math. Soc.* 28 (1996) no. 6 561–590.
4. S. K. Lando and A. K. Zvonkin, 'Graphs on surfaces and their applications', *Encyclopaedia of Mathematical Sciences* 141. Low-Dimensional Topology, II. Springer-Verlag, Berlin (2004).
5. L. Schneps (ed.), 'The Grothendieck theory of dessins d'enfants', *London Mathematical Society Lecture Note Series* 200. Cambridge University Press, Cambridge (1994).
6. L. Schneps, 'Dessins d'enfants on the Riemann sphere', pages 47–77 in [**5**].

Ernesto Girondo
Gabino González-Diez
Departamento de Matemáticas
Universidad Autónoma de Madrid
Campus de Cantoblanco
28049 Madrid, Spain

ernesto.girondo@uam.es
gabino.gonzalez@uam.es