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Non-special divisors supported on the branch set of a p-gonal Riemann surface

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Abstract

A compact Riemann surface S is called *cyclic p-gonal* if it possesses an automorphism τ of order p such that the quotient $S/<\tau>$ has genus zero. It is well known that if p is a prime number and $Q_1, \ldots, Q_r \in S$ are the fixed points of τ then S has genus $g = \frac{p-1}{2}(r-2)$.
In this article we find a criterion to decide when a divisor of the form $D = Q_1^{d_1} \cdots Q_r^{d_r}$, with $\sum d_i = g$, is non-special.
The criterion is very easy to apply in practice since it only depends on the arithmetic of the local rotation numbers of τ at the points Q_i and the multiplicities of these points on the divisor D, i.e. the integers d_i.
Knowledge of the set of non-special divisors supported on the ramification set seems to be essential in all attempts to extend the classical Thomae formulae, which apply to hyperelliptic (i.e. 2-gonal) Riemann surfaces, to the case of p-gonal ones.

Notation. *Throughout this paper we use the following notation. Given an integer $n \in \mathbb{Z}$ we shall denote by $\overline{n} \in \{0, 1, ..., p-1\}$ and $[n] \in \mathbb{Z}/p\mathbb{Z}$ its remainder and its residue class modulo p, respectively; thus, we have $[\overline{n}] = [n]$.*

1.1 Introduction and statement of the main result

Among the many ways in which a compact Riemann surface S of genus $g \geq 1$ can be described are the *algebraic equation*

$$F(x, y) = \sum a_{ij} x^i y^i = 0$$

satisfied by any pair of meromorphic functions x and y generating the function field of S (*Riemann's existence theorem*) and the *Jacobian of* S defined as

$$J(S) = \frac{\mathbb{C}^g}{\mathbb{Z}^g \oplus \mathbb{Z}^g \cdot \Omega}$$

where $\Omega = (\int_{B_j} \omega_i)$ is a $g \times g$ matrix, called the *period matrix*, whose entries are the B-periods of the basis of holomorphic 1-forms $\omega_i, i = 1, \ldots, g$ which is dual to a chosen symplectic basis $\{A_j, B_j, j = 1, \ldots, g\}$ of the first homology group $H_1(S, \mathbb{Z})$ (*Torelli's theorem*).
One may therefore hope to express the coefficients a_{ij} of a certain algebraic equation for S in terms of *Riemann's theta* function $\theta(z, \Omega)$ evaluated at a suitable finite collection of points $z = a + b \cdot \Omega; a, b \in \mathbb{R}^g$ (*theta constants*) and, conversely, to obtain these theta constants, usually denoted $\theta \left[\begin{array}{c} a \\ b \end{array} \right] (0, \Omega)$, as a function of the coefficients a_{ij}.

This correspondence between the algebraic and the transcendental moduli theory of Riemann surfaces works very well at a theoretical level. But, of course, it is not clear at all how to materialize these ideas when an arbitrary Riemann surface is given.

However, it can be satisfactorily achieved for hyperelliptic Riemann surfaces. In fact the formulae performing this relationship go back to the work of Frobenius [Frobenius 1885] and Thomae [Thomae 1866], [Thomae 1870]. We refer to [Farkas-Kra 1980], chapter VII.4. for a modern account of this correspondence in one direction (*expressing the branch points as functions of the periods*) and to [Mumford,1983] and [Eisenmann-Farkas], in the other one (*Thomae's formula*).

Several authors have generalized these formulae to certain families of cyclic p-gonal Riemann surfaces with $p > 2$. The reader may consult the articles [Farkas 1996], [Gonzalez-Diez 1991], [Gonzalez-Harvey 1991], [Kuribayashi 1976] and [Matsumoto 2001] for formulae expressing the branch points of p-gonal surfaces as functions of the periods and [Bershadsky-Radul 1988], [Eisenmann-Farkas], [Enolski-Grava 2006], [Gonzalez-Harvey 1991] for generalizations of Thomae formulae to several kinds of p-gonal Riemann surfaces. To obtain these identities one of the key points is to detect sufficiently many suitable degree g non-special divisors supported on the ramification locus of the structural p-gonal morphism $S \to \widehat{\mathbb{C}}$ of the p-gonal Riemann surface S.

The aim of this paper is to provide a criterion to check when a given divisor of this kind is special.

We recall that a compact Riemann surface S is called *cyclic p-gonal* if it possesses an automorphism τ of order p such that the quotient $S/<\tau>$ has genus zero and so the natural map $S \to S/<\tau> \simeq \widehat{\mathbb{C}}$ provides a degree p, or p-gonal, morphism which ramifies at the points fixed by τ. Accordingly, the set of fixed points will be referred to as *the ramification (or branch) locus (or set).*

Throughout this paper we shall assume that p is a prime positive integer.

It is well known (see e.g. [Gonzalez-Diez 1991], [Harvey 1971]) that such a Riemann surface admits an algebraic model of the form

$$y^p = (x-a_1)^{m_1}...(x-a_r)^{m_r} \tag{1.1.1}$$

where

- $\sum m_i = np$, for some positive integer n
- $1 \le m_i \le p-1$
- The Riemann surface S consists of the affine points of the curve (1.1.1) plus p points at infinity.
- The cyclic group $<\tau>$ is generated by the automorphism $\tau(x,y) = (x, e^{2\pi i/p}y)$.
- The full fixed point set of τ is $Fix(\tau) = \{Q_1 = (a_1,0), \dots, Q_r = (a_r,0)\}$. The points at infinity get permuted by τ.
- The integer m_k is called the *rotation number* of τ at the point Q_k. The rotation number of τ at a fixed point Q is independent of the choice of the model (1.1.1) (because, locally, $\tau^{-1}(z) = e^{2\pi i m_k/p} \cdot z$).
- The genus of S is $g = \frac{p-1}{2}(r-2)$.

Let D be an integral divisor of degree g, that is $D = P_1^{d_1} \cdots P_r^{d_r}$ with $d_i \ge 0$ and $\sum d_i = g$. Recall that D is said to be special if there is a non constant function f whose set of poles is bounded by D or, as it is usually written, $f \in L(D^{-1})$.

The significance of the special divisors can be explained as follows. Let us identify the set of integral divisor of degree g with the g-fold symmetric product $S^{(g)}$, then, after choosing a base point $Q \in S$, there is a holomorphic map, *the Abel-Jacobi map*, from S^g to $J(S)$ defined by

$$A(D) = \sum_{i=1}^{r} d_i \int_Q^{P_i} (\omega_1, \cdots, \omega_g) \in J(S)$$

It is a classical result that this map is a birational map which fails to be an isomorphism precisely at the special divisors.

It is a trivial fact that if one of the multiplicities d_i is bigger or equal to p then D is special (see Proposition 1, ii) below. Therefore we can assume from the start that $0 \leq d_i < p$, for all $i = 1, \ldots, r$.

We can now state our criterion

Theorem 1

Let S be a compact Riemann surface and τ an automorphism of S of prime order p such that the quotient $S/ < \tau >$ has genus zero. Let $Fix(\tau) = \{Q_1, \ldots, Q_r\}$ be the fixed point set of τ and let us denote by m_k the rotation number of the point Q_k.

Then, for a divisor D of the form $D = Q_1^{d_1} \cdots Q_r^{d_r}$ with $0 \leq d_i \leq p-1$ and $\sum d_i = g$, the following four conditions are equivalent

(i) D is non-special.

(ii) $\sum_{i=1}^{r} \overline{d_i + m_i k} > g$ for every $k = 1, ..., p-1$.

(iii) $\sum_{i=1}^{r} \overline{d_i + m_i k} = g + p$ for every $k = 1, ..., p-1$.

(iv) $\sum_{i=1}^{r} \overline{d_i + m_i k} = g + p$ for $p-2$ integers $k \in \{1, ..., p-1\}$.

Moreover, if for a certain $k \in \{1, \cdots, p-1\}$ one has the inequality $\sum_{i=1}^{r} \overline{d_i + m_i k} \leq g$, then the function

$$y^k \prod_{i=1}^{r} (x - a_i)^{s_{i,k}}, \quad \text{with} \quad s_{i,k} = \frac{\overline{d_i + m_i k} - (d_i + m_i k)}{p} \quad ,$$

belongs to $L(D^{-1})$.

The proof of this theorem is carried out in section 2 while in section 3 we explicitly describe the set of non-special divisors supported on the ramification set for some particularly interesting cases. These include most of the families of p-gonal Riemann surfaces that appear in the literature in connection with problems involving theta constants.

Acknowledgement 1 *I would like to thank Hershel M. Farkas for kindly pointing out an error in my article [Gonzalez-Diez 1991] concerning the choice of a collection of non-special divisors (see 1.2.2). It was his observation what made me to initiate the search for a complete*

characterization of non-special divisors supported on the ramification set of p-gonal Riemann surfaces.

1.2 Proof of the criterion

We begin by recalling the most elementary facts and definitions relative to the group of divisors on a Riemann surface. For a detailed account the reader is referred to [Farkas-Kra,1980].

A *divisor* $\mathcal{U}$ on S is a formal symbol

$$\mathcal{U} = P_1^{s_1} \cdots P_r^{s_r}$$

with $P_j \in S, s_j \in \mathbb{Z} \setminus \{0\}$. The subset $\{P_1 \cdots P_r\} \subset S$ is called the *support* of $\mathcal{U}$. At times we will also allow $s_j = 0$ which, of course, means that P_j does not belong to the support. The integer s_j is called *the multiplicity* of the point P_j in the divisor $\mathcal{U}$.

The set of divisors carries a structure of abelian group under the obvious multiplicative law; the inverse of $\mathcal{U}$ being $\mathcal{U}^{-1} = P_1^{-s_1} \cdots P_r^{-s_r}$

The divisor of a meromorphic function $f \in \mathcal{M}(S)$ is defined by

$$(f) = \frac{P_1^{d_1} \cdots P_r^{d_r}}{R_1^{l_1} \cdots R_r^{l_r}} = P_1^{d_1} \cdots P_r^{d_r} R_1^{-l_1} \cdots R_r^{-l_r}; \quad d_i, l_i \geq 0$$

where the numerator (resp. denominator) stands for the zero (resp. pole) set of f, multiplicities being taken into account.

Given two divisors $\mathcal{U} = P_1^{d_1} \cdots P_r^{d_r}$ and $\mathcal{B} = P_1^{l_1} \cdots P_r^{l_r}$ we say that $\mathcal{B} \geq \mathcal{U}$ if $d_i \geq l_i; i = 1, \cdots, r$.

We also need to introduce the $\mathbb{C}$-vector space

$$L(\mathcal{U}) = \{f \in \mathcal{M}(S) : (f) \geq \mathcal{U}\}$$

whose dimension we shall denote by $r(\mathcal{U})$.

Let D be an integral divisor of degree g, that is $D = P_1^{d_1} \cdots P_r^{d_r}$ with $d_i \geq 0$ and $\sum d_i = g$. We say that D is *special* if $r(D^{-1}) > 1$, that is if there is a non constant function f whose set of poles is bounded by D.

The following proposition collects a list of well known facts relative to the function field of a p-gonal Riemann surface.

Proposition 1 *Let S be the Riemann surface defined by equation (1.1.1) and let $\mathcal{M}(S)$ be its function field. The following properties hold*
(i) $\mathcal{M}(S)$ is generated by the coordinate functions x and y.

(ii) The divisors of the functions $x - a_i$ and y are

$$(x - a_i) = \frac{Q_i^p}{\infty}, \text{ and } (y) = \frac{Q_1^{m_1}, \ldots, Q_r^{m_r}}{\infty^n}$$

where ∞ stands for the integral divisor of degree p supported on the p points that S possesses at infinity.
(iii) The divisor of a meromorphic function of the form

$$f = f(x) = \prod_{i=1}^{r}(x - a_i)^{s_i} \prod_j (x - b_j)^{t_j}; \; s_i, t_j \in \mathbb{Z}, \; a_i \neq b_j$$

is

$$(f) = \prod_{i=1}^{r} Q_i^{ps_i} \cdot \prod_j B_j^{t_j} \cdot \infty^{-(\sum s_i + \sum t_j)}$$

where B_j is the divisor of degree p given by:

$$B_j = \prod_{k=1}^{p} (b_j, e^{2\pi k/p} \sqrt[p]{\prod_i (b_j - a_i)^{m_i}}).$$

Definition 1 *Let D be a divisor of the form $D = Q_1^{d_1} \cdots Q_r^{d_r}$ with $0 \leq d_i \leq p-1$ and $\sum d_i = g$. Associated to D we define the following objects*
(i) A collection of integers $s_{i,k}; i = 1, ..., r, k = 0, 1, ..., p-1$ defined by

$$s_{i,k} = \frac{\overline{d_i + m_i k} - (d_i + m_i k)}{p} \tag{1.2.1}$$

(ii) A collection of meromorphic functions $f_k = f_k(x), \; k = 1, ..., p-1$ defined by

$$f_k = \prod_{i=1}^{r} (x - a_i)^{s_{i,k}} \tag{1.2.2}$$

In proving Theorem 1 the following simple lemma will be useful

Lemma 1 *Let $U = Q_1^{s_1} \cdots Q_r^{s_r}$ be a divisor supported in the ramification locus. Then we have the following direct sum decomposition*

$$L(U) = (L(U) \cap \mathbb{C}(x)) \oplus (L(U) \cap \mathbb{C}(x)y) \oplus ..... \oplus (L(U) \cap \mathbb{C}(x)y^{p-1})$$

Proof Let $\tau^* : \mathcal{M}(S) \to \mathcal{M}(S)$ denote the map induced by τ on the function field. This is a linear automorphism with (distinct) eigenvalues $1, e^{2\pi i/p}, ..., e^{2\pi i(p-1)/p}$. Therefore $\mathcal{M}(S)$ decomposes as the direct sum of its corresponding eigenspaces, namely

$$\mathcal{M}(S) = \mathbb{C}(x) \oplus \mathbb{C}(x)y \oplus ..... \oplus \mathbb{C}(x)y^{p-1}$$

Now, as by hypothesis $\tau(U) = U$, $L(U)$ is an invariant $\mathbb{C}$-linear subspace, hence we have a corresponding diagonal decomposition for $L(U)$ as required. □

For the divisor $U = D^{-1}$, D as in Theorem 1, Lemma 1 implies the following crucial observation.

Corollary 1 *Let D be as in Theorem 1. Then D is a special divisor if and only if there is a function $h \in L(D^{-1})$ of the form $h = f(x)y^k$, for some $f(x) \in \mathbb{C}(x)$ and some $k \geq 1$.*

Proof By Lemma 1 if there is a non constant function $h \in L(D^{-1})$ then there is a non constant function $h_k \in L(D^{-1})$ of the form $h_k = f(x)y^k$. Thus we only have to rule out the case in which $h = h_0 = f(x)$. So, let us write $h = f$ as in Proposition 1,*iii*). Then, if $h \in L(D^{-1})$, we must have

$$\prod_{i=1}^{r} Q_i^{ps_i} \cdot \prod_{i=1}^{r} B_j^{t_j} \cdot \infty^{-(\sum s_i + \sum t_j)} \geq Q_1^{-d_1} \cdots Q_r^{-d_r}$$

From here we infer the following inequalities
1) $ps_i \geq -d_i$, hence $s_i \geq 0$
2) $t_j \geq 0$ and
3) $-(\sum s_i + \sum t_j) \geq 0$, hence $s_i = t_j = 0$.
Thus, h would be a constant function. □

Lemma 2 *Let $f = \prod_{i=1}^{r}(x - a_i)^{s_i} \prod_j (x - b_j)^{t_j}$ as in Proposition 1,iii) and D as in Theorem 1. Then $fy^k \in L(D^{-1})$ if and only if the following numerical conditions hold*
(i) $t_j \geq 0$
(ii) $ps_i \geq -d_i - m_i k$
(iii) $-(\sum s_i + \sum t_j) \geq nk$

Proof $(fy^k) = \prod_{i=1}^{r} Q_i^{ps_i} \cdot \prod_{i=1}^{r} B_j^{t_j} \cdot \infty^{-(\sum s_i + \sum t_j)} (\frac{Q_1^{m_1 k} .....,Q_r^{m_r k}}{\infty^{nk}}) =$

$\prod_{i=1}^{r} B_j^{t_j} \prod_{i=1}^{r} Q_i^{ps_i+m_ik} \infty^{-(nk+\sum s_i+\sum t_j)}$. This divisor is bigger than $D^{-1} = Q_1^{-d_1} \cdots Q_r^{-d_r}$ if and only if conditions (i), (ii) and (iii) hold. □

Corollary 2 *There exists $f \in \mathbb{C}(x)$ such that $fy^k \in L(D^{-1})$ if and only if there are integers $s_i \in \mathbb{Z}$ satisfying the following conditions*
(i) $ps_i \geq -d_i - m_ik$
(ii) $\sum s_i \leq -nk$
Moreover, in this situation f can be chosen to be $f = \prod_{i=1}^{r}(x-a_i)^{s_i}$.

Proof If such integers s_i exist then one can take $f = \prod_{i=1}^{r}(x-a_i)^{s_i}$ (Lemma above with $t_j = 0$).

Conversely, if $f = \prod_{i=1}^{r}(x-a_i)^{s_i} \prod_j (x-b_j)^{t_j}$ is a function such that $fy^k \in L(D^{-1})$ then conditions (i) and (iii) of the previous lemma imply condition (ii) of this corollary. □

1.2.1 Proof of Theorem 1

1.2.1.1 (i)⇔ (ii)

By Corollary 1 it is enough to prove the following

Proposition 2 *Let $1 \leq k \leq p-1$. Then there exists $h_k \in \mathbb{C}(x)$ such that $h_k(x)y^k \in L(D^{-1})$ if and only if $\sum \overline{d_i + m_ik} \leq g$.*

Proof This is an easy application of Corollary 2. Suppose that $\sum_{i=1}^{r} \overline{d_i + m_ik} \leq g$ for some $k \in \{1, ..., p-1\}$. Then we would have $p\sum s_{i,k} = \sum \overline{d_i + m_ik} - \sum d_i - npk = \sum \overline{d_i + m_ik} - g - npk \leq -pnk$. On the other hand, by definition, $ps_{i,k} \geq -d_i - m_ik$. Thus, the two conditions in Corollary 2 for h_k to exist hold.
Conversely, if such function h_k exists then, by Corollary 2, there must be integers $\widetilde{s}_{i,k} \in \mathbb{Z}$ such that $p\widetilde{s}_{i,k} \geq -d_i - m_ik$ and $p\sum \widetilde{s}_{i,k} \leq -pnk$. Therefore we have $p(\widetilde{s}_{i,k} - s_{i,k}) \geq -d_i - m_ik - ps_{i,k} = -d_i - m_ik - (\overline{d_i + m_ik} - d_i - m_ik) = -\overline{d_i + m_ik} > -p$; in other words $\widetilde{s}_{i,k} \geq s_{i,k}$. From here we infer that $\sum \overline{d_i + m_ik} - g - pnk = \sum ps_{i,k} \leq \sum p\widetilde{s}_{i,k} \leq -pnk$ hence $\sum \overline{d_i + m_ik} - g \leq 0$ as wanted. □

The final statement in Theorem 1 follows at once from the final statement in Corollary 2.

1.2.1.2 (ii) $\Leftrightarrow$ (iii)

Clearly $[\overline{\sum(d_i + m_i k)}] = [\sum d_i + \sum m_i k] = [g + np] = [g]$, hence (ii) implies that

$$\sum_{i=1}^{r} \overline{d_i + m_i k} = g + pN_k \text{ for some positive integer } N_k \tag{1.2.3}$$

It remains to be seen that, for each k, $N_k = 1$. This obviously follows from the following lemma

Lemma 3 *For any divisor D such that*

$$D = Q_1^{d_1} \cdots Q_r^{d_r} \textit{ with } 0 \leq d_i \leq p-1 \textit{ and } \sum d_i = g$$

we have

$$\sum_{k=1}^{p-1} \sum_{i=1}^{r} \overline{d_i + m_i k} = (p-1)(g+p)$$

Proof For each $d = 0, 1, \cdots, p-1$ and $m = 1, \cdots, p-1$ we denote by $x_{d,m}$ the number of ramification points Q_i with rotation number m and multiplicity d on D. Before we proceed we observe that

$$\sum_{m=1}^{p-1} \sum_{d=0}^{p-1} x_{d,m} = r, \text{the total number of ramification points}$$

and

$$\sum_{m=1}^{p-1} \sum_{d=0}^{p-1} x_{d,m}.d = g, \text{ the degree of the divisor } D.$$

We then have

$$\begin{aligned}
\sum_{k=1}^{p-1}\sum_{i=1}^{r}\overline{d_i+m_ik} &= \sum_{k=1}^{p-1}\sum_{m=1}^{p-1}\sum_{d=0}^{p-1}x_{d,m}\cdot\overline{d+mk}\\
&= \sum_{m=1}^{p-1}\sum_{d=0}^{p-1}x_{d,m}\Big(\sum_{k=1}^{p-1}\overline{d+mk}\Big)\\
&= \sum_{m=1}^{p-1}\sum_{d=0}^{p-1}x_{d,m}\Big(\sum_{l=1}^{d-1}l+\sum_{l=d+1}^{p-1}l\Big)\\
&= \sum_{m=1}^{p-1}\sum_{d=0}^{p-1}x_{d,m}\Big(\frac{p(p-1)}{2}-d\Big)\\
&= r\frac{p(p-1)}{2}-g=(r-2)\frac{p(p-1)}{2}+p(p-1)-g\\
&= pg+p(p-1)-g=(p-1)(g+p)
\end{aligned}$$

□

1.2.1.3 (iii) $\Leftrightarrow$ *(iv)*

This clearly follows from the lemma above.

1.2.2 Corrigenda

Lemma 1 above is the correct version of Lemma 2.3 in my paper [Gonzalez-Diez 1991]. That lemma, as it stands, is incorrect. As a result the choice of non-special divisors D_i, V_i made in Proposition 2.1 of [Gonzalez-Diez 1991] with the property that D_i/V_i is the p-root of the divisor of the function $(x-a_i)/(x-a_r)$, i.e. such that $D_i/V_i = Q_i/Q_r$, should be modified so as to meet the criterion established in Theorem 1 above. This can be readily done, for instance, for the families of p-gonal Riemann surfaces

$$y^p = (x-a_1)...(x-a_{np})$$

defined by the rotation data $(1,\cdots,1)$, and

$$y^p = (x-a_1)...(x-a_n)\left((x-a_{r+1})...(x-a_{2n})\right)^{p-1}$$

corresponding to the rotation data $(1,\cdots,1,p-1,\cdots,p-1)$, whose non-special divisors have been listed in section 1.3 (see Proposition 3 and Corollary 3).

As a general description of the set of non-special divisors which is valid for families corresponding to arbitrary rotation data $m_1, \cdots, m_r$ looks unmanageable (see Remark 1), the choices of D_i and V_i will have to be made for each family individually. Clearly, Theorem 1 above is the right tool to do that since it enables us to produce the list of non-special divisors of any given family (perhaps through a simple computer program, if p is large). Alternatively, one can set $D_i = Q_i U, V_i = Q_r U$ where U is a degree $(g-1)$ integral divisor such that D_i and V_i are non-special; a choice of such U can be made, since the degree g special divisors on S are a subvariety of $S^{(g)}$ of codimension at least 2 (see [Kuribayashi 1976], [Mumford,1983]). Note, however, that this general result does not guarantee that the divisor U can be always chosen with support in the ramification set (in fact, see Example 2); the consequence being that the points in $J(S)$ which -via the Abel-Jacobi map- correspond to the divisors D_i and V_i may not be points of order p of the abelian variety $J(S)$ as it would be the case otherwise.

1.3 Applications

1.3.1 A couple of interesting examples

Theorem 1 provides an easy recipe to check if an explicitly given divisor D is special. Let us see how it works in two concrete examples.

Example 1 *(Klein's Riemann surface of genus* 3*)*

$$y^7 = (x-a_1)(x-a_2)^2(x-a_3)^4$$

1) Let us check if the divisor $D = Q_1^2 Q_2 = Q_1^2 Q_2 Q_3^0$ is non-special.

We have to compute the integers $\overline{d_i + m_i k}$ for $i = 1, 2, 3$ and $k = 1, \cdots, 6$. In this case we have $p = 7, d_1 = 2, d_2 = 1, d_3 = 0$ and $m_1 = 1, m_2 = 2, m_3 = 4$. We get the following table of values

	$\overline{d_i + m_i}$	$\overline{d_i + m_i 2}$	$\overline{d_i + m_i 3}$	$\overline{d_i + m_i 4}$	$\overline{d_i + m_i 5}$	$\overline{d_i + m_i 6}$
$i=1$	3	4	5	6	0	1
$i=2$	3	5	0	2	4	6
$i=3$	4	1	5	2	6	3
$\sum \overline{d_i + m_i k}$	10	10	10	10	10	10

Thus, for all $k = 1, \cdots, 6$ we obtain $\sum \overline{d_i + m_i k} > 3 = g$ and therefore $D = Q_1^2 Q_2$ is not special.

2) Let us look now at the divisor $D = Q_1^3 = Q_1^3 Q_2^0 Q_3^0$

When we compute the integers $\overline{d_i + m_i 4}$ we find

$$\begin{aligned} \overline{d_1 + m_1 4} &= \overline{3 + 1 \cdot 4} = 0 \\ \overline{d_2 + m_2 4} &= \overline{0 + 2 \cdot 4} = 1 \\ \overline{d_3 + m_3 4} &= \overline{0 + 4 \cdot 4} = 2 \end{aligned}$$

hence $\sum \overline{d_i + m_i 4} = 3$ and so our criterion tells us that $D = Q_1^3$ is special. But not only that, the criterion also gives us a non-constant function whose pole divisor is bounded by D, namely the function

$$f_k y^k = (x-a_1)^{s_{1,4}}(x-a_2)^{s_{2,4}}(x-a_3)^{s_{3,4}} y^4 = (x-a_1)^{-1}(x-a_2)^{-1}(x-a_3)^{-2} y^4$$

whose divisor is $Q_1^{-3} Q_2 Q_3^2$.

3) It turns out that the integral divisors supported on the branch locus are precisely

$$D = Q_1^2 Q_2,\ Q_1 Q_3^2,\ Q_1 Q_2 Q_3,\ Q_2^2 Q_3$$

Example 2 *(the 7-gonal hyperelliptic Riemann surface of genus 3)*

$$y^7 = (x - a_1)(x - a_2)(x - a_3)^5$$

We want to find out which divisors of the form $D = Q_1^{d_1} Q_2^{d_2} Q_3^{d_3}$, with $d_1 + d_2 + d_3 = g$, are non-special, thus we have to compute the integers $\overline{d_i + m_i k}$ for $i = 1, 2, 3$ and $k = 1, \cdots, 6$. In this case we have $p = 7, m_1 = 1, m_2 = 1, m_3 = 5$ and $d_1 + d_2 + d_3 = 3$, hence $d_i \leq 3$.

For D to be non-special the following table of values must hold

	$\overline{d_i+m_i}$	$\overline{d_i+m_i2}$	$\overline{d_i+m_i3}$	$\overline{d_i+m_i4}$	$\overline{d_i+m_i5}$	$\overline{d_i+m_i6}$
$i=1$	$\overline{d_1+1}$	$\overline{d_1+2}$	$\overline{d_1+3}$	$\overline{d_1+4}$	$\overline{d_1+5}$	$\overline{d_2+6}$
$i=2$	$\overline{d_2+1}$	$\overline{d_2+2}$	$\overline{d_2+3}$	$\overline{d_2+4}$	$\overline{d_2+5}$	$\overline{d_2+6}$
$i=3$	$\overline{d_3+5}$	$\overline{d_3+3}$	$\overline{d_3+1}$	$\overline{d_3+6}$	$\overline{d_3+4}$	$\overline{d_3+2}$
$\sum\overline{d_i+m_ik}$	10	10	10	10	10	10

From the first column we infer that, in that case, $d_3 = 0, 1$. On the other hand, the 4-th column rules out the value $d_3 = 0$, hence we must have $d_3 = 1$. But then, the 5-th column shows that $d_i \neq 1, i = 1, 2$. The conclusion is that in this case we only have two non-special divisors, namely

$$Q_1^2 Q_3 \quad \text{and} \quad Q_2^2 Q_3$$

1.3.2 The case of equal rotation numbers

Let us consider the case in which all rotations numbers m_i agree. In fact, by replacing the automorphism τ by a suitable power τ^k we can assume that $m_i = 1$ for every i. That is, we can assume that S is a Riemann surface of genus $g = \frac{p-1}{2}(np-2)$ given by an equation of the form

$$y^p = (x - a_1)...(x - a_{np}) \tag{1.3.1}$$

Let D be a divisor of degree g supported on the branch locus. We can write it in the form

$$D = D_0^0 \cdot D_1 \cdot D_2^2 \cdot D_3^3 \cdot \dots \cdot D_{p-2}^{p-2} \cdot D_{p-1}^{p-1}$$

where

- $D_0 = Q_1 \cdots Q_{r_0}$ (and so $Q_1, \cdots, Q_{r_0}$ are the points which are omitted)
- $D_1 = Q_{r_0+1} \cdots Q_{r_0+r_1}$
- $D_2 = Q_{r_0+r_1+1} \cdots Q_{r_0+r_1+r_2}$

- $D_3 = Q_{r_0+r_1+r_2+1} \cdots Q_{r_0+r_1+r_2+r_3}$
- $\cdots$
- $D_{p-1} = Q_{r_0 \cdots + r_{p-2}+1} \cdots Q_{r_0 \cdots + r_{p-2}+r_{p-1}}$

We have the following relations among the non-negative integers r_i

$$r_0 + r_1 + r_2 + \cdots + r_{p-2} + r_{p-1} = pn$$

and

$$0 \cdot r_0 + 1 \cdot r_1 + 2 \cdot r_2 + \cdots + (p-2) \cdot r_{p-2} + (p-1) \cdot r_{p-1} = g$$

Let us now work out the integers $\overline{d_i + m_i k} = \overline{d_i + k}$; the last equality because we are now assuming that each m_i equals 1.
For $k = 1$ we get

$$\overline{d_i + m_i \cdot 1} = \begin{cases} \overline{0+1} & = & 1 & \text{(for the } r_0 \text{ points in } D_0) \\ \overline{1+1} & = & 2 & \text{(for the } r_1 \text{ points in } D_1) \\ \overline{2+1} & = & 3 & \text{(for the } r_1 \text{ points in } D_2) \\ \cdots & = & \cdots & \cdots \\ \overline{(p-3)+1} & = & p-2 & \text{(for the } r_{p-3} \text{ points in } D_{p-3}) \\ \overline{(p-2)+1} & = & p-1 & \text{(for the } r_{p-2} \text{ points in } D_{p-2}) \\ \overline{(p-1)+1} & = & 0 & \text{(for the } r_{p-1} \text{ points in } D_{p-1)} \end{cases}$$

According to our criterion for D to be non-special we need to have

$$1 \cdot r_0 + 2 \cdot r_1 + 3 \cdot r_2 + \cdots + (p-1) \cdot r_{p-2} + 0 \cdot r_{p-1} = g + p$$

For $k = 2$ we get

$$\overline{d_i + m_i 2} = \begin{cases} \overline{0+2} & = & 2 & \text{(for the } r_0 \text{ points in } D_0) \\ \overline{1+2} & = & 3 & \text{(for the } r_1 \text{ points in } D_1) \\ \overline{2+2} & = & 4 & \text{(for the } r_1 \text{ points in } D_2) \\ \cdots & = & \cdots & \cdots \\ \overline{(p-3)+2} & = & p-1 & \text{(for the } r_{p-3} \text{ points in } D_{p-3}) \\ \overline{(p-2)+2} & = & 0 & \text{(for the } r_{p-2} \text{ points in } D_{p-2}) \\ \overline{(p-1)+2} & = & 1 & \text{(for the } r_{p-1} \text{ points in} D_{p-1)} \end{cases}$$

As in the previous case for D to be non-special we need to have

$$2 \cdot r_0 + 3 \cdot r_1 + 4 \cdot r_2 + \cdots + 0 \cdot r_{p-2} + 1 \cdot r_{p-1} = g + p$$

We proceed in the same manner to obtain the corresponding equations for all $k = 1, \cdots, p-1$. For instance, the last one is

$$(p-1) \cdot r_0 + 0 \cdot r_1 + 1 \cdot r_2 + \cdots + (p-3) \cdot r_{p-2} + (p-2) \cdot r_{p-1} = g + p$$

This way we obtain a linear system consisting of $p+1$ linear equation in the unknowns r_0, $r_1, \cdots, r_{p-1}$. To solve it we employ the Gaussian elimination algorithm. The matrix corresponding to this linear system is

$$A = \left(\begin{array}{cccccc|c} 1 & 1 & 1 & \cdots & 1 & 1 & np \\ 0 & 1 & 2 & \cdots & p-2 & p-1 & g \\ 1 & 2 & 3 & \cdots & p-1 & 0 & g+p \\ 2 & 3 & 4 & \cdots & 0 & 1 & g+p \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p-2 & p-1 & 0 & \cdots & p-4 & p-3 & g+p \\ p-1 & 0 & 1 & \cdots & p-3 & p-2 & g+p \end{array}\right)$$

Now, obvious elementary operations among the rows of the matrix A give the matrix

$$A_1 = \left(\begin{array}{cccccc|c} 1 & 1 & 1 & \cdots & 1 & 1 & np \\ 0 & 1 & 2 & \cdots & p-2 & p-1 & g \\ 1 & 1 & 1 & \cdots & 1 & 1-p & p \\ 1 & 1 & 1 & \cdots & 1-p & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1-p & \cdots & 1 & 1 & 0 \\ 1 & 1-p & 1 & \cdots & 1 & 1 & 0 \end{array}\right)$$

and then

$$A_2 = \left(\begin{array}{cccccc|c} 1 & 1 & 1 & \cdots & 1 & 1 & np \\ 0 & 1 & 2 & \cdots & p-2 & p-1 & g \\ 0 & 0 & 0 & \cdots & 0 & p & np-p \\ 0 & 0 & 0 & \cdots & p & 0 & np \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & p & \cdots & 0 & 0 & np \\ 0 & p & 0 & \cdots & 0 & 0 & np \end{array}\right)$$

and from here

$$A_3 = \left(\begin{array}{cccccc|c} 1 & 0 & 0 & \cdots & 0 & 0 & np-(p-1)n+1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & g - \frac{p(p-1)n}{2} + p - 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & n-1 \\ 0 & 0 & 0 & \cdots & 1 & 0 & n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & n \\ 0 & 1 & 0 & \cdots & 0 & 0 & n \end{array}\right),$$

Now the solutions of our linear system can be read off the matrix A_3. We find that

$$r_0 = n+1, r_1 = n, \cdots, r_{p-2} = n, r_{p-1} = n-1$$

In other words we have proved the following result

Proposition 3

Let S be a p-gonal Riemann surface whose branch locus consists of pn ramification points all of them having the same rotation number. Then, a degree g integral divisor D supported on the branch locus is non-special if and only if D is of the form

$$(R_1 \cdots R_n) \cdots (R_{(p-3)n+1} \cdots R_{(p-2)n})^{p-2} (R_{(p-2)n+1} \cdots R_{(p-1)n-1})^{p-1}$$

where $\{R_1 \cdots R_{(p-1)n-1}\}$ is any collection of $np-(n+1)$ ramification points.

Example 3 *(hyperelliptic case, $p = 2$)*

$$y^2 = (x-a_1)...(x-a_{2(g+1)})$$

In this case the non-special divisors are those of the form $D = Q_{i_1}.....Q_{i_g}$.

Example 4 *(the trigonal case, $p = 3$)*

$$y^3 = (x-a_1)...(x-a_{3n}) \tag{1.3.2}$$

The non-special divisors supported on the branch locus are those of the form

$$(Q_{i_1} \cdots Q_{i_n})(Q_{i_{n+1}} \cdots Q_{i_{2n-1}})^2$$

In [Eisenmann-Farkas] Thomae type formulae for Riemann surfaces of the form (1.3.2), with $n = 2$, have been investigated. In that case the relevant divisors take the form $(Q_{i_1} Q_{i_2}) Q_{i_3}^2$.

1.3.3 The case of two different rotation angles

Let us consider the Riemann surface S given by an equation of the form

$$y^p = (x-a_1)...(x-a_n)\left((x-a_1)...(x-a_{2n})\right)^{p-1} \tag{1.3.3}$$

Its genus is

$$g = (p-1)(n-1)$$

Let D be a divisor of degree g supported on the branch locus. Let r_d

(resp. t_d) be the number of points with rotation number 1 (resp. $p-1$) and multiplicity $0 \le d \le p-1$ on D. We have

$$\begin{cases} \sum_{d=0}^{p-1} r_d = n \\ \sum_{d=0}^{p-1} t_d = n \\ \sum_{d=0}^{p-1} d r_d + \sum_{d=0}^{p-1} d t_d = g \end{cases} \tag{1.3.4}$$

With this notation Theorem 1 tells us that a necessary and sufficient condition for our divisor D to be non-special is

$$\sum_{d=0}^{p-1} (\overline{d+k}) r_d + \sum_{d=0}^{p-1} (\overline{d+(p-1)k}) t_d = g+p \quad \text{for every } k = 1, ..., p-1. \tag{1.3.5}$$

All these $p+2$ conditions can be assembled together in a linear system which in Gaussian terminology can be represented as follows

$$\left(\begin{array}{cccccccc|c}
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & n \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & n \\
0 & 1 & \cdots & p-1 & 0 & 1 & \cdots & p-1 & g \\
1 & 2 & \cdots & 0 & p-1 & 0 & \cdots & p-2 & g+p \\
2 & 3 & \cdots & 1 & p-2 & p-1 & \cdots & p-3 & g+p \\
3 & 4 & \cdots & 2 & p-3 & p-2 & \cdots & p-4 & g+p \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
p-3 & p-2 & \cdots & p-4 & 3 & 4 & \cdots & 2 & g+p \\
p-2 & p-1 & \cdots & p-3 & 2 & 3 & \cdots & 1 & g+p \\
p-1 & 0 & \cdots & p-2 & 1 & 2 & \cdots & 0 & g+p
\end{array}\right)$$

Now, keeping fixed the first three rows and subtracting to each of the remaining rows the previous one we obtain

$$\left(\begin{array}{ccccccccc|c}
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & n \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 & n \\
0 & 1 & \cdots & 0 & 1 & 2 & \cdots & p-2 & p-1 & g \\
1 & 1 & \cdots & 1-p & p-1 & -1 & \cdots & -1 & -1 & p \\
1 & 1 & \cdots & 1 & -1 & p-1 & \cdots & -1 & -1 & 0 \\
1 & 1 & \cdots & 1 & -1 & -1 & \cdots & -1 & -1 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 1 & \cdots & -1 & -1 & -1 & \cdots & -1 & -1 & 0 \\
1 & 1 & \cdots & 1 & -1 & -1 & \cdots & -1 & -1 & 0 \\
1 & 1-p & \cdots & 1 & -1 & -1 & \cdots & p-1 & -1 & 0
\end{array}\right)$$

The next operations we perform are as follows: we leave the first two rows untouched, we erase the third row and subtract the first row to the remaining ones. This way we obtain

$$\left(\begin{array}{ccccccccc|c}
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & n \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 & n \\
0 & 0 & \cdots & -p & p-1 & -1 & \cdots & -1 & -1 & p-n \\
0 & 0 & \cdots & 0 & -1 & p-1 & \cdots & -1 & -1 & -n \\
0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 & -1 & -n \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 & -1 & -n \\
0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 & -1 & -n \\
0 & -p & \cdots & 0 & -1 & -1 & \cdots & p-1 & -1 & -n
\end{array}\right)$$

Now we first multiply the first row by p and the second one by (-1). Then, we add to the first row all the other p rows. We get

$$\left(\begin{array}{ccccccccc|c}
p & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -p & p \\
0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 & -1 & -n \\
0 & 0 & \cdots & -p & p-1 & -1 & \cdots & -1 & -1 & p-n \\
0 & 0 & \cdots & 0 & -1 & p-1 & \cdots & -1 & -1 & -n \\
0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 & -1 & -n \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 & -1 & -n \\
0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 & -1 & -n \\
0 & -p & \cdots & 0 & -1 & -1 & \cdots & p-1 & -1 & -n
\end{array}\right)$$

In our last step we subtract the second row to each of the rows below it. We finally obtain

$$\left(\begin{array}{cccccccccc|c}
p & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & -p & p \\
0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 & -1 & -1 & -n \\
0 & 0 & \cdots & -p & p & 0 & \cdots & 0 & 0 & 0 & p \\
0 & 0 & \cdots & 0 & 0 & p & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & p & 0 & 0 & 0 \\
0 & -p & \cdots & 0 & 0 & 0 & \cdots & 0 & p & 0 & 0
\end{array}\right)$$

Thus, we infer the following relations among the unknowns r_i, t_i

$$\left\{\begin{array}{lllll}
r_0 & + & -t_{p-1} & = & 1 \\
-r_{p-1} & + & t_0 & = & 1 \\
-r_{p-2} & + & t_1 & = & 0 \\
-r_{p-3} & + & t_2 & = & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-r_{p-k} & + & t_{k-1} & = & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-r_2 & + & t_{p-3} & = & 0 \\
-r_1 & + & t_{p-2} & = & 0
\end{array}\right. \Leftrightarrow \left\{\begin{array}{lll}
t_0 & = & 1 + r_{p-1} \\
t_1 & = & r_{p-2} \\
t_2 & = & r_{p-3} \\
t_3 & = & r_{p-4} \\
\cdots & \cdots & \cdots \\
t_k & = & r_{p-k-1} \\
\cdots & \cdots & \cdots \\
t_{p-2} & = & r_1 \\
t_{p-1} & = & r_0 - 1
\end{array}\right. \tag{1.3.6}$$

This allows full description of the set of non-special divisors

Proposition 4 *Let S be any p-gonal Riemann surface whose ramification set consists of $n \geq 2$ ramification points of rotation number 1 and n ramification points of rotation number $p-1$. Then, a degree g integral divisor D supported on the branch locus is non-special if and only if D is of the form*

$$\begin{aligned} D =&(T_1 \cdots T_{r_0-1})^{p-1} \\ &(R_{r_0} \cdots R_{r_0+r_1-1})(T_{r_0} \cdots T_{r_0+r_1-1})^{p-2} \\ &(R_{r_0+r_1} \cdots R_{r_0+r_1+r_2-1})^2 (T_{r_0+r_1} \cdots T_{r_0+r_1+r_2-1})^{p-3} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &(R_{r_0+r_1+\cdots+r_{p-2}} \cdots R_{r_0+r_1+\cdots+r_{p-1}-1})^{p-1} \end{aligned}$$

where $(r_0, r_1, \cdots, r_{p-1})$ *is any p-tuple of non negative integers such that*

$$r_0 \geq 1 \ , \ \sum_{d=0}^{p-1} r_d = n$$

and $\{T_1, \cdots, T_{r_1+\cdots+r_{p-2}-1}\}$ *(resp.* $\{R_{r_0} \cdots R_{r_0+r_1+\cdots+r_{p-1}-1}\}$*) is any choice of* $n - r_{p-1} - 1$ *(resp.* $n - r_0$*) ramification points with rotation number* $p-1$ *(resp.* 1*).*

In particular we have

Corollary 3 *Let* S *be as in Proposition 4.*

(i) The set of non-special divisors on S *supported on the ramification set such that all points in the support have multiplicity* $p-1$ *consists of the following divisors*

$$D = (R_1 \cdots R_a)^{p-1} (T_1 \cdots T_{n-1-a})^{p-1}$$

where $0 \leq a \leq n-1$ *and* $R_1 \cdots R_a$ *(resp.* $T_1 \cdots T_{n-1-a}$*) are arbitrary distinct points with rotation number* 1 *(resp.* $p-1$*).*

(ii) The set of non-special divisors on S *supported on the ramification set such that the multiplicity at all points but one of its support is* $p-2$ *consists of the following divisors*

$$D = R_1^{p-k-1} (R_2 \cdots R_{b+1})^{p-1} (T_1 \cdots T_{n-2-b})^{p-1} T_{n-b-1}^k$$

$$\Big(\textit{resp.} \quad V = T_1^{p-k-1} (T_2 \cdots T_{b+1})^{p-1} (R_1 \cdots R_{n-2-b})^{p-1} R_{n-b-1}^k \quad \Big)$$

where $1 \leq k \leq p-2, \quad 0 \leq b \leq n-2$ *and the* R_i*'s (resp. the* T_j*'s) are arbitrary distinct points with rotation number* 1 *(resp.* $p-1$*).*

In [Enolski-Grava 2006] Thomae type formula for Riemann surfaces of the form (1.3.3) have been invistigated. The non-special divisors used to obtain that formulae are among the ones described in Corollary 3(i) and Corollary 3(ii) with $k = 1$.

Remark 1 *An explicit description of the set of non-special divisors for arbitrary families, although doable, seems to be too much involved to be of any use. For instance, already in the case of the general curve with two arbitrary rotation numbers*

$$y^p = (x-a_1)...(x-a_{n_1})\left((x-a_{n_1+1})...(a_{n_1+n_2})\right)^m; \quad n_1+n_2m = np$$

the simple formulae given in (1.3.6) to express the relation between the r_i*'s and the* t_j*'s take now the following rather involved form*

$$\begin{cases} r_0 &= \sum_{r=m}^{p-1} t_r &+ \quad n-n_2+1 \\ r_k &= \sum_{r=m}^{p-1} t_{\overline{km+r}} &+ \quad n-n_2 \quad \textit{if} \quad k \neq 0, p-1 \\ r_{p-1} &= \sum_{r=m}^{p-1} t_{\overline{-m+r}} &+ \quad n-n_2-1 \end{cases}$$

Notes

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