On the fundamental groups at infinity of the moduli spaces of compact Riemann surfaces

Gabino González Diez and Pierre Lochak

1 Introduction and statement of the result.

In his seminal manuscript *Esquisse d'un programme* (1984; now available in [GGA]) A.Grothendieck explains that the structure of the tower of all moduli spaces of curves is somehow governed by its "first two levels" ("deux premiers étages"), *i.e.* the moduli spaces of dimensions 1 and 2. We refer to the *Esquisse* and to [L] for more context and details about this statement. Let us only mention that speaking in terms of topology, Grothendieck was concerned more precisely with the orbifold fundamental groups of the moduli spaces of curves and he explains that the above "principle" is essentially equivalent to the fact that the orbifold fundamental group of any moduli space of dimension > 2 is equal to its fundamental group at infinity. We do not recall here the notion of orbifold fundamental group, which is due to Thurston in a topological context, because we will be concerned only with the ordinary topological fundamental group, *i.e.* the fundamental groups of the moduli spaces of curves viewed as manifolds, forgetting about their orbifold structure. In terms of analytic or algebraic geometry, this amounts to viewing them as coarse and not as fine moduli spaces for curves.

Before we consider moduli spaces of curves in detail, let us make precise the notion of fundamental group at infinity in a topological context. Note that it is less easy (although feasible) to do it in terms of algebraic geometry because a quasiprojective variety cannot usually be exhausted by an increasing sequence of projective subvarieties, nor is it easy to define tubular neighborhoods of closed subvarieties. So let M be a paracompact differentiable manifold and partially order the compact submanifolds (possibly with boudary) of M by inclusion. Their complements define an obvious inverse

[1]Partially supported by Grant PB96–0036 of the CICYT.MEC

system: if $K \subset K'$, we simply consider the inclusion $M \setminus K' \subset M \setminus K$. We need a base point for our fundamental group and exploit the fact that a fundamental group need not be based at a point but in fact at any simply connected subset of the ambient manifold. Here a *base point at infinity*, simply denoted by $*$, is given by an open part $U \subset M$ such that for any compact set K, there exists a compact set K' with $K \subset K'$ and $U \setminus K'$ nonempty and *simply* connected. Let π_1 denote as usual the topological fundamental group (functor).

Definition (The fundamental group at infinity.) Let M be a paracompact differentiable manifold and assume there exists a base point at infinity $*$ for M, defined by an open set U. We define the topological fundamental group at infinity of M based at $*$ as:

$$\pi_1^\infty(M, *) = \varprojlim \pi_1(M \setminus K, U \setminus K),$$

where the inverse limit is over the cofinal family of compact subsets K of M such that $U \setminus K$ is simply connected, partially ordered by inclusion and using the natural induced maps on the fundamental groups. From now on we will often lightheartedly ignore the base points in the notation, having done with the problem of base points at infinity as above. Let us very briefly recall a few notions from the theory of Teichmüller and moduli spaces of curves, essentially in order to fix notation. We refer to any standard textbook on the subject (*e.g.* [IM]) for the necessary background information. For simplicity we will mainly consider in this note the case of Riemann surfaces without marked or deleted points, postponing to the closing remarks some observations on the more general case. Yet in the intermediary steps, we will have to consider surfaces with marked points anyway, so let us introduce the more general objects right away. We will denote by $\mathcal{T}_{g,n}$ the Teichmüller space of compact Riemann surfaces of type (g, n), that is those which are obtained from surfaces of genus g by marking n points. One may also consider, taking a hyperbolic rather than conformal viewpoint, that the points are deleted, giving rise to surfaces with cusps. Let $\mathcal{M}_{g,n}$ be the fine moduli space of surfaces of type (g, n), which is obtained as the quotient of $\mathcal{T}_{g,n}$ by the (Teichmüller) modular group (alias mapping class group) $\Gamma_{g,n}$. More precisely $\Gamma_{g,n}$ acts properly and discontinuously on $\mathcal{T}_{g,n}$ with quotient $\mathcal{M}_{g,n}$. Since the action is not free the latter space naturally inherits an orbifold structure (a structure of stack in the algebraic context). One can also "forget" about the orbifold structure, thus getting a bona fide (in general normal but singular) manifold $M_{g,n}$, which is a coarse moduli space for the surfaces of type (g,n). In all this the points or punctures were labeled and $\Gamma_{g,n}$ denotes the pure modular group, preserving the marked points individually. One can now allow permutations of these points and get the full group $\Gamma_{g,[n]}$, which is an extension of the permutation group S_n by the pure group $\Gamma_{g,n}$. We also get the corresponding spaces $\mathcal{M}_{g,[n]} = \mathcal{T}_{g,n}/\Gamma_{g,[n]}$ and $M_{g,[n]}$. Note that there is no notion of permuting the punctures at the level of the Teichmüller space and that this operation is not a mere ornament when one is interested in torsion elements, as we will be. The effect may be dramatic: typically $\Gamma_{0,n}$ is torsionfree, whereas $\Gamma_{0,[n]}$ is generated by its torsion elements. Finally we mention that we drop n from the notation when it is 0: we write Γ_g for $\Gamma_{g,0}$ etc.

Let us now stick to the unmarked (or compact) case $n = 0$ for stating the results. As mentioned above marked surfaces reappear as intermediate objects in the course of the proof (§2 below) and are briefly commented on again in the closing section. By definition, one has $\pi_1(\mathcal{M}_g) = \Gamma_g$ where one considers the orbifold fundamental group and one assumes $g > 1$ in order to deal with hyperbolic surfaces. One can actually also include $g = 0$ because $\mathcal{M}_0$ is a point (this is Schönfliess theorem), and $\mathcal{M}_1 \simeq \mathcal{M}_{1,1}$. For the coarse moduli spaces, one has the result of Maclachlan in [Ml] who proved that they are simply connected: $\pi_1(M_g) = \{1\}$ for $g > 1$ (one can again in fact include the cases $g = 0, 1$). This deals with the ordinary topological fundamental group, that is considering M_g as a manifold. In view of [A] this amounts to saying that Γ_g is generated by its torsion elements. Note that both results hold in the algebraic context, that is if one considers $\mathcal{M}_g$ as a Deligne-Mumford stack (see [DM]) over $\mathbf{C}$ (or over $\overline{\mathbf{Q}}$), its fundamental group is the profinite completion of Γ_g, whereas the coarse moduli space M_g, which is nothing else as $\mathcal{M}_g$ viewed as a scheme, is simply connected. In [L] it was shown, confirming Grothendieck's prediction, that $\pi_1^\infty(\mathcal{M}_g) \simeq \Gamma_g$ for $g \geq 2$. Note that the assertion is contentfree for $g = 0$ and does not hold for $g = 1$ (even identifying $\mathcal{M}_1$ with $\mathcal{M}_{1,1}$). To put this result in context, it is best to state the result with marked points: recall that $\mathcal{M}_{g,n}$ has dimension $d(g,n) = 3g-3+n$ (assume $2g-2+n > 0$, *i.e.* the surfaces are hyperbolic). With this in mind, it is shown in [L] that $\pi_1^\infty(\mathcal{M}_{g,n}) \simeq \Gamma_{g,n}(= \pi_1(\mathcal{M}_{g,n}))$ if and only $d(g,n) > 2$, which is indeed Grothendieck's prediction. This result is also valid if one does not label the marked points (replace n by $[n]$ everywhere) and can be immediately transposed (as above) in the algebraic context.

In this note we prove, at least for surfaces without marked points, a statement which was left as a "plausible assertion" in [L]. In fact we show the following

Theorem $\pi_1^\infty(M_g) = \{1\}(= \pi_1(M_g))$ *for* $g > 2$ *and* $\pi_1^\infty(M_2) \simeq \mathbf{Z}/5\mathbf{Z}$.

Section 2 is devoted to some geometrical properties which provide, much as in [L], the crucial ingredients for the proof of the theorem. We recall and make precise in passing some properties of the loci of curves with nontrivial automorphisms which may have some independent interest. In section 3 we give the proof of the theorem, including the determination of the fundamental group at infinity in the case of genus 2 and at the end we briefly comment on the cases with marked points.

2 Some geometry at infinity on the moduli spaces of curves.

We consider hyperbolic surfaces of fixed type (g, n) (with $2g-2+n > 0$) and first recall some observations from [L]. Note that these purely geometric features do not depend on whether we view the moduli spaces as fine or coarse. First for $\varepsilon > 0$ we define the set $\mathcal{M}_{g,n}^{\varepsilon} \subset \mathcal{M}_{g,n}$ of points $[X] \in \mathcal{M}_{g,n}$ representing surfaces X such that it has at least one geodesic γ with length $l(\gamma) < \varepsilon$ in the Poincaré metric. For $\varepsilon \leq \varepsilon'$ there is an obvious inclusion $\mathcal{M}_{g,n}^{\varepsilon} \subset \mathcal{M}_{g,n}^{\varepsilon'}$ and thus a natural map between fundamental groups: $\pi_1(\mathcal{M}_{g,n}^{\varepsilon}) \to \pi_1(\mathcal{M}_{g,n}^{\varepsilon'})$. As mentioned above one can replace the fine moduli ($\mathcal{M}$) by the coarse moduli (M), which will be used below.

By [M] (see also [B1]) the complements of $\mathcal{M}_{g,n}^{\varepsilon}$ in $\mathcal{M}_{g,n}$ are compact and they form a cofinal sequence, *i.e.* they exhaust the space $\mathcal{M}_{g,n}$. So one has $\pi_1^\infty(\mathcal{M}_{g,n}) = \lim_{\varepsilon \to 0} \pi_1(\mathcal{M}_{g,n}^{\varepsilon})$. The same statement holds true for coarse moduli spaces. Let $p : \mathcal{T}_{g,n} \to \mathcal{M}_{g,n}$ denote the canonical projection, and set $\mathcal{T}_{g,n}^{\varepsilon} = p^{-1}(\mathcal{M}_{g,n}^{\varepsilon})$. One goes on to show (see [L]) that:

i) $\mathcal{T}_{g,n}^{\varepsilon}$ is simply connected for any $\varepsilon > 0$;

ii) There exists an absolute constant ε_0 such that if $\varepsilon \leq \varepsilon' < \varepsilon_0$, then $\mathcal{T}_{g,n}^{\varepsilon}$ is a deformation retract of $\mathcal{T}_{g,n}^{\varepsilon'}$ and the retraction can be chosen to be $\Gamma_{g,n}$ equivariant, thus producing in particular a diffeomorphism between $\mathcal{M}^{\varepsilon}_{g,n}$ and $\mathcal{M}^{\varepsilon'}_{g,n}$. From this one concludes that the sequence of groups $(\pi_1(\mathcal{M}^{\varepsilon}_{g,n}))_{\varepsilon>0}$ stabilizes for $\varepsilon < \varepsilon_0$. One can in fact pick $\varepsilon_0 = \frac{1}{3}\ln(1+\sqrt{2})$. Again one may replace fine with coarse moduli spaces.

We now turn more specifically to the case of coarse moduli spaces, which are the subject of this note. By an elementary topological result ([A]), assertion i) above implies (see again [L] for details) that $\pi_1(M^{\varepsilon}_{g,n}) = \Gamma_{g,n}/G^{\varepsilon}$ where G^{ε} is the normal subgroup of $\Gamma_{g,n}$ generated by the set T^{ε} (T stands for "torsion") of elements which have a nonempty fixed point set when acting on $\mathcal{T}^{\varepsilon}_{g,n}$. So we get a decreasing sequence of subgroups $(G^{\varepsilon})_{\varepsilon>0}$. By ii) above it is in fact stationary and we let G be the limit, so that in fact $G = G^{\varepsilon}$ for any $\varepsilon < \varepsilon_0$. We record what we have got in the following statement:

Proposition 1 *$\pi_1^{\infty}(M_{g,n}) = \Gamma_{g,n}/G$ where $G = G^{\varepsilon}$ for any $\varepsilon < \varepsilon_0$.*

In order to vindicate the description of G given in [L] (see Proposition 2 below) we bring in the relative Teichmüller and moduli spaces from [MlH] and [GH]. Let $h \in \Gamma_{g,n}$ be a nontrivial torsion element; we denote by $\mathcal{T}_{g,n}(h)$ the fixed point set of h, or equivalently of the finite cyclic group $\langle h\rangle$ it generates, acting on $\mathcal{T}_{g,n}$. The elements of the modular group permute these relative Teichmüller spaces according to the familiar rule: $f(\mathcal{T}_{g,n}(h)) = \mathcal{T}_{g,n}(fhf^{-1})$ for $f \in \Gamma_{g,n}$. The stabiliser of $\mathcal{T}_{g,n}(h)$ is thus the normalizer of the cyclic group $\langle h\rangle$ in $\Gamma_{g,n}$, which we denote by $\Gamma_{g,n}(h)$. The quotient $\mathcal{T}_{g,n}(h)/\Gamma_{g,n}(h) = \widetilde{\mathcal{M}}_{g,n}(h)$ is called the relative moduli space. Again there is a fine and a coarse version, and by a classical theorem of Cartan the underlying variety is normal. As a matter of fact if $\mathcal{M}_{g,n}(h) = p(\mathcal{T}_{g,n}(h))$ denotes the image of the relative Teichmüller space in the moduli space, one shows ([GH]) that $\widetilde{\mathcal{M}}_{g,n}(h)$ is precisely the normalization of $\mathcal{M}_{g,n}(h)$. In this setting the generating set T^{ε} introduced before Proposition 1 can be described as:

$$T^{\varepsilon} = \{h \in \Gamma_{g,n}, \quad \mathcal{M}_{g,n}(h) \cap \mathcal{M}^{\varepsilon}_{g,n} \neq \emptyset\}.$$

By a classical result of Nielsen, any element $h \in \Gamma_{g,n}$ of finite order can be realized as an automorphism of some Riemann surface X of type (g,n). Let $X' = (X/\langle h\rangle)^*$ denote the surface obtained by puncturing the quotient $X/\langle h\rangle$ at the ramification points of the covering $X \to X/\langle h\rangle$, and let X' be of type (p,ν). Then (see [MlH] and [GH]) one has $\mathcal{T}_{g,n}(h) \simeq \mathcal{T}_{p,\nu}$ and $\widetilde{\mathcal{M}}_{g,n}(h)$ is a finite cover of $\mathcal{M}_{p,\nu}$. As in [L] we call a torsion element *maximal* if its fixed point set is just a point, that is if $\mathcal{T}_{g,n}(h)$ has dimension zero, which is equivalent to saying that X' is a thrice punctured sphere ($(p,\nu) = (0,3)$).

Now since the order of an automorphism of a Riemann surface of given type is bounded, one finds, using the elementary theory of coverings of Riemann surfaces, that there is only a finite number of surfaces having maximal automorphisms, *i.e.* whose underlying mapping classes are maximal. The lengths of their geodesics are bounded from below by some ε_{min}, so for ε small enough, indeed for $\varepsilon < \varepsilon_{min}$, the set T^ε does *not* contain any maximal element. The keypoint of this section is that this accounts for the set of torsion elements which are excluded from T^ε for ε small enough. In other words our next goal is to prove

Proposition 2 *For ε sufficiently small, T^ε consists of the set of* non *maximal torsion elements. In particular G is the group generated by these elements.*

In order to prove this we need two more geometric lemmas, leading to a geometric version (Proposition 3 below) of Proposition 2. For simplicity we omit the subscript (g,n) from the notation until the end of this section.

Lemma 1 *For any given h, the family of subvarieties $(f(\mathcal{T}(h))_{f\in\Gamma}$ is* locally finite*, that is for any point $t \in \mathcal{T}$ there is a neighborhood of t which meets only finitely many of these subvarieties.*

Proof. Since Γ acts properly discontinuously on $\mathcal{T}$, there is an open set U containing t such that U and $f(U)$ are disjoint except for finitely many $f \in \Gamma$, say $h_1, \ldots, h_r$. Now for any $f, h \in \Gamma$ (h of finite order), $f(\mathcal{T}(h)) = \mathcal{T}(fhf^{-1})$, in other words $f(\mathcal{T}(h))$ is just the fixed point set of fhf^{-1}. This implies that $U \cap f(\mathcal{T}(h)) \subset U \cap fhf^{-1}(U)$, from which we infer that if this intersection is nonempty, then $fhf^{-1} = h_i$ for some $i \in (1,r)$, and hence $f(\mathcal{T}(h)) = \mathcal{T}(h_i)$. □

Lemma 2 *For any $h \in \Gamma$ of finite order, $\mathcal{M}(h)$ is a closed subvariety of $\mathcal{M}$.*

Proof. We have to show that $p^{-1}(\mathcal{M}(h)) = \cup_{f\in\Gamma} f(\mathcal{T}(h))$ is closed in $\mathcal{T}$ ($p : \mathcal{T} \to \mathcal{M}$ the canonical projection). Let $t \in \mathcal{T}$ be in the closure of $p^{-1}(\mathcal{M}(h))$ and let $(t_m)_m$ be a sequence of points in $p^{-1}(\mathcal{M}(h))$ converging to t. Applying lemma 1 and passing to a subsequence, we may assume that all the t_m's belong to the same subvariety $f(\mathcal{T}(h)) = \mathcal{T}(fhf^{-1})$). But this last subvariety is well-known to be closed, which implies that actually $t \in f(\mathcal{T}(h)) \subset p^{-1}(\mathcal{M}(h))$ as was to be proved. □

We are now prepared to deduce:

Proposition 3 *If h is not a maximal torsion element in Γ, then $\mathcal{M}(h) \cap \mathcal{M}^\varepsilon \neq \emptyset$ for all $\varepsilon > 0$.*

Proof. The sets $\mathcal{M} \setminus \mathcal{M}^\varepsilon$ are compact as noted earlier (see [M] or [B1, **?**]). If the proposition did not hold then by lemma 2, $\mathcal{M}(h)$ would be a compact subvariety of $\mathcal{M}$. But the normalization $\widetilde{\mathcal{M}}(h)$ of $\mathcal{M}$ is a finite cover of the moduli space $\mathcal{M}_{p,\nu}$ where (p,ν) is the type of the quotient surface (see above) and this moduli space is not compact, except if $(p,\nu) = (0,3)$, that is if h is maximal. □

The result above,which can be rephrased by saying that $\mathcal{M}(h)$ "extends to infinity" except if it has dimension 0, is the content of statement i) in [L] p.152. It immediately implies the validity of proposition 2 and so, using propositions 1 and 2, in order to determine the fundamental groups at infinity of the coarse moduli spaces, it only remains to determine the subgroups of the modular groups generated by the nonmaximal torsion elements.

3 Nonmaximal torsion elements in the Teichmüller modular groups.

We turn to the proof of the theorem stated in section 1. The case $g > 2$ is now easy, using results in [MP]. Recall that we are reduced to showing that for $g > 2$ the modular group Γ_g is generated by non maximal torsion elements (this is statement ii) in [L] p. 152). Now in [MP] it is shown that Γ_g is generated by involutions (elements of order 2) which fix 2 points if g is even and 4 if g is odd. By the Riemann-Hurwitz formula the genera of the respective quotients are thus $g/2$ and $(g-1)/2$. This is enough to ensure that these generators are not maximal, which finishes the proof of the fact that the coarse moduli space M_g is simply connected for $g > 2$.

We now turn to the case $g = 2$, more precisely to showing that $\pi_1(M_2) = \mathbf{Z}/5\mathbf{Z}$. The keypoint is that one can realize M_2 as the quotient of $\mathbf{C}^3$ by the action of the cyclic group $C_5 = \mathbf{Z}/5\mathbf{Z}$ acting via a (nontrivial) 5-th root of unity ζ by: $\zeta \cdot (z_1, z_2, z_3) = (\zeta z_1, \zeta^2 z_2, \zeta^3 z_3)$ (see [I], p. 638). This representation comes of course from the fact that any genus two curve is hyperelliptic and the coordinates z_i play a role which is analogous to the classical j (or λ) modular function in the genus 1 case.

The origin in $\mathbf{C}^3$ is the only fixed point of the C_5-action, corresponding to the curve X_0 with equation $y^2 = x^5 - 1$, which has a cyclic automorphism group of order 10, generated by τ such that $\tau(x, y) = (\zeta x, -y)$. Both τ and τ^2 are maximal torsion elements of Γ_2 (see [S]). We now have the following

Lemma 3 *Let $\tilde{G} \subset \Gamma_2$ be the subgroup generated by the set of (maximal or not) torsion elements which are not conjugate to either τ or τ^2. Then $\tilde{G} = G$.*

Proof. We have that $G \subset \tilde{G}$ by definition. In the converse direction we first observe (see [S]) that apart from X_0, there are exactly two other curves with maximal automorphisms. The first one is the curve X_1 with equation $y^2 = x^6 - 1$ and automorphism group of order 24, generated by two elements γ and σ of order 6 and 4 respectively. Denoting by ζ_6 a primitive 6-th root of unity we can take $\gamma(x, y) = (\zeta_6 x, y)$ and $\sigma(x, y) = (x^{-1}, ix^{-3}y)$. The quotient of X_1 by σ is a sphere and there are 4 ramification values, so σ is not maximal; by the same token γ is found to be maximal, but $\gamma = \gamma^4\gamma^3$ and one finds that both γ^4 and γ^3 are not maximal so that γ and hence the full automorphism group of X_1 is contained in G.

The other curve is X_2 with equation $y^2 = x(x^4 - 1)$. It has automorphism group of order 48 generated by 3 elements ρ (order 8), β (order 3), α (order 2) with relation $\rho = \alpha\beta$ (see [K] or [C]). One can give again explicit equations (see [C] which contains a detailed study of the automorphism groups of curves of genus 2) but for our purpose it is enough to observe that because of their respective orders α and β cannot be maximal, which implies that the full automorphism group of X_2 is contained in G, thus also completing the proof of the lemma. $\square$

In order to prove the assertion in the theorem for the case of genus 2, let us introduce the punctured space M_2^* which is M_2 with the point corresponding to the curve X_0 removed, that is the origin in $\mathbf{C}^3$ in the representation of Igusa recalled above. So we have that $M_2^* \simeq (\mathbf{C}^3 \setminus \{0\})/C_5$ and also $M_2^* \simeq \mathcal{T}_2^*/\Gamma_2$, where of course $\mathcal{T}_2^*$ denotes the Teichmüller space of genus 2 with the fiber $p^{-1}(X_0)$ deleted. Let us use these two representations to compute the fundamental group of M_2^* in two ways. First, by standard covering theory applied to Igusa's representation, we find that $\pi_1(M_2^*) = C_5$. On the other hand since $p^{-1}(X_0)$ is simply connected, the result of [A] implies that $\pi_1(M_2^*) = \Gamma_2/\tilde{G}$. Since by lemma 2, $\tilde{G} = G$ we find indeed that $\Gamma_2/G = C_5$, thus finishing the proof of the assertion of the theorem: $\pi_1^\infty(M_2) \simeq \pi_1(M_2^*) = \mathbf{Z}/5\mathbf{Z}$. □

We note that the proof actually shows that $\pi_1^\infty(M_2)$ is generated by the mapping class τ^2. It also shows that any set of generators for Γ_2 consisting of elements of finite order must contain a conjugate of τ^2, which is a little more precise than saying that Γ_2 cannot be generated by non maximal torsion elements.

We close with a short comment on the case of surfaces with marked points. So let (g, n) be a given type and let $M_{g,[n]}$ be the coarse moduli space of surfaces of that type, where the points are *not* individually labeled, so giving rise to the full group $\Gamma_{g,[n]}$ $(= \pi_1(\mathcal{M}_{g,[n]}))$.

The following result seems to hold: $\pi_1^\infty(M_{g,[n]}) = \{1\}$ for $g > 2$. This would follow, just as the assertion of the theorem for $g > 2$ from the closing remark in [P] which states that for $g > 2$, $\Gamma_{g,[n]}$ is generated by involutions. These cannot be maximal; in fact if h is such an involution for a Riemann surface S, the quotient $S/\langle h\rangle$ either has genus > 0 or if it has genus 0, then S is hyperelliptic and h is the hyperelliptic involution, in which case it fixes $2g + 2 > 3$ points (with g the genus of S) and is thus again not maximal.

References

[A] M. A. Armgstrong, The fundamental group of the orbit space of a discontinuous group, Proc. Cambridge Phil. Soc. **64** (1968), 299-301.

[B1] L. Bers, On Mumford's compactness theorem. Israel J. Math. **12** (1979), 400-407.

[B2] L. Bers, Deformations and moduli of Riemann surfaces with nodes and signatures, Math. Scand. **36** (1975), 12-16.

[C] J. Cirre, *Curvas algebraicas de género 2*, Tesis doctoral, Universidad Complutense, Madrid, 1997.

[DM] P. Deligne and D .Mumford, The irreducibility of the space of curves of given genus, Publ. Math. IHES. **36** (1969), 75-109.

[GGA] *Geometric Galois Actions*, London Math. Soc. Lecture Notes **242**, L. Schneps and P. Lochak eds., Cambridge U. Press, 1997.

[GH] G. Gonzalez-Diez, W.J. Harvey. Moduli of Riemann sufaces with symmetry, in *Discrete Groups and Geometry*, London Math. Soc. Lecture Notes **173**, Cambridge U. Press, 1992, 75-94.

[H] W.J. Harvey, Spaces of discrete groups, in *Discrete Groups and Automorphic forms* Academic Press, 1977, 295-348.

[I] J. Igusa, Arithmetic variety of moduli for genus two, Ann. of Math. **72** (1960).

[IM] Y. Immayoshi, M. Taniguchi, *An introduction to Teichmüller spaces*, Springer Verlag, 1992.

[K] R. Kulkarni, Riemann surfaces admitting large automorphism groups, in Contemporary Maths **201**, AMS Publ., 1996.

[L] P. Lochak, The fundamental group at infinity of the moduli space of curves, in *Geometric Galois Actions I*, London Math. Soc. Lecture notes **242**, Cambridge U. Press, 1997, 139-158.

[M] D. Mumford. A remark on Mahler's compactness theorem, Proc. AMS **28** (1971), 289-294.

[Ml] C. Maclachlan, Modulus space is simply connected, Proc. AMS **29** (1971), 85-86.

[MlH] C. Maclachlan, W. J. Harvey, On mapping-class groups and Teichmüller spaces, Proc. London Math. Soc. **30** (1975), 495-512.

[MP] J. McCarthy, A. Papadopoulos, Involutions in surface mapping class groups, L'Enseignement Mathématique **33** (1987), 275-290.

[P] D.B. Patterson. the fundamental group of the modulus space, Michigan Math. J. **26,** (1979), 213-223.

[S] J. Schiller, Moduli for special Riemann surfaces of genus 2, Trans. AMS (1969), 95-113.

Gabino González Diez. Departamento de Matemáticas. Universidad Autónoma de Madrid. 28049 Madrid; gabino.gonzalez@uam.es

Pierre Lochak. Institut de Mathématiques, 175 rue du Chevaleret, 75013 Paris; lochak@math.jussieu.fr