On extremal discs inside compact hyperbolic surfaces

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Abstract

It has been proved by C. Bavard that the radius of a disc isometrically embedded in a compact hyperbolic surface of genus g is bounded by $R_g = \cosh^{-1}\left(\dfrac{1}{2\sin\frac{\pi}{12g-6}}\right)$, and that surfaces containing discs of such extremal radius are found in every genus. By constructing explicit surfaces of genus 2, he has also shown that extremal discs may or may not be unique.

Here we show that a compact surface of genus $g > 3$ has at most one embedded extremal disc.

Sur les disques extrémaux dans des surfaces hyperboliques compactes

Résumé

C. Bavard a prouvé que le rayon d'un disque métrique plongé dans une surface hyperbolique compacte de genre g est

majorée par $R_g = \cosh^{-1}\left(\dfrac{1}{2\sin\frac{\pi}{12g-6}}\right)$, et il a construit pour chaque genre un exemple d'un tel disque.

Il a construit aussi des exemples de surfaces de genre 2 contenant un seul disque de rayon R_g et de surfaces en contenant plusieurs.

Dans cette note nous montrons qu'une surface de genre $g > 3$ ne peut contenir au plus qu'un seul disque.

Version française abrégée

On se propose de montrer:

Théorème *Une surface hyperbolique compacte de genre $g > 3$ ne peut contenir qu'un disque de rayon* $R_g = \cosh^{-1}\left(\dfrac{1}{2\sin\frac{\pi}{12g-6}}\right)$.

Idée de la preuve:

Soit $S = \dfrac{\mathbb{H}}{K}$ une surface de Riemann compacte de genre $g \geq 2$ munie de la métrique de Poincaré (ici $\mathbb{H}$ est le demi-plan superieur et K le sous-groupe de $\mathbb{PSL}(2,\mathbb{R})$ qui uniformise S).

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On note $N = 12g - 6$. D'après Bavard ([1]) dire que S possède deux disques extrémaux D_1, D_2 équivaut à dire que K est contenu avec indice N dans deux groupes triangulaires Γ_1 et Γ_2 de signature $(2, 3, N)$; ce ci veut dire que le groupe Γ_i $(i = 1, 2)$ est engendré par les rotations d'angles respectifs $\pi, \frac{2\pi}{3}, \frac{2\pi}{N}$ autour des sommets d'un triangle T_i d'angles $\frac{\pi}{2}, \frac{\pi}{3}$ et $\frac{\pi}{N}$. Le disque D_i étant le disque inscrit dans le domain fondamental de K fourni par le N-gone régulier obtenu comme la réunion de $2N$ copies du triangle T_i autour du sommet d'angle $\frac{\pi}{N}$.

Notons que, puisque les groupes Γ_1 et Γ_2 ont la même signature, ils doivent être conjugués, c'est-à-dire il doit exister une isométrie $\alpha \in \mathbb{P}\mathrm{SL}(2, \mathbb{R})$ tel que $\Gamma_1 = \alpha^{-1} \circ \Gamma_2 \circ \alpha$.

A ce point on invoque le théorème de Margulis ([5]) qui'etablit que si le groupe Γ_2 n'est pas arithmétique, Γ_2 doit avoir un index fini dans le groupe

$$\mathrm{Com}(\Gamma_2) := \left\{\gamma \in \mathbb{P}\mathrm{SL}(2, \mathbb{R}) : \ \Gamma_2 \cap \left(\gamma^{-1} \circ \Gamma_2 \circ \gamma\right) \text{ a un index fini dans } \Gamma_2 \text{ et } \gamma^{-1} \circ \Gamma_2 \circ \gamma\right\}$$

(ce groupe est appelé le *commensurateur* de Γ_2).

Ce-ci implique que quand le groupe Γ_2 est non-arithmétique on a $\mathrm{Com}(\Gamma_2) = \Gamma_2$, car les groupes triangulaires de signature $(2, 3, N)$ sont maximaux ([6]).

Mais on a une liste des groupes triangulaires arithmetiques ([7]): les seules groupes qui nous concernent sont les groupes obtenus pour $N = 18, 30$ et correspondant aux genres $g = 2, 3$.

On conclut que quand $g > 3$ on a $\alpha \in \Gamma_2$, car $\alpha \in \mathrm{Com}(\Gamma_2)$, et alors $\Gamma_1 = \Gamma_2$. C'est-à-dire $D_1 = D_2$.

En ce qui concerne le genre 3, nous pensons qu'il existe des surfaces avec au moins deux disques extrémaux (voir la note a la fin de la version en anglais).

1 Introduction

Let S be a compact hyperbolic surface of genus g, and let $D(R)$ be a disc of radius R isometrically embedded in S (it goes without saying that S is understood to be equipped with the metric induced by the Poincaré metric of the upper half plane $\mathbb{H}$ via uniformization). By applying classical results on sphere packings, Bavard ([1]) has shown that R is necessarily bounded by a positive constant R_g, determined by

$$\cosh R_g = \frac{1}{2 \sin \frac{\pi}{12g - 6}}.$$

We shall refer to discs of such radius R_g as *extremal discs* and, accordingly, surfaces containing extremal discs will be called *extremal surfaces.*

In [1] the following results about extremal surfaces are proved:

i) extremal surfaces arise, precisely, as regular hyperbolic N-gons, $N = 12g - 6$, via compatible side pairing identifications; the extremal disc being the inscribed circle centered at the center of the N-gon.

ii) they occur in every genus (but, once the genus is fixed, there are only finitely many of them).

iii) they may contain one or several extremal discs (but, once the surface is fixed, there can only be finitely many of them).

The last statement is proved by construction of explicit examples in genus 2. The main purpose of this note is to prove that if we exclude the cases $g = 2$ and (possibly) $g = 3$, then extremal discs (within a given surface) are unique.

It should be noted that this uniqueness result implies that any isometry of an extremal surface S of genus $g > 3$ must fix the center of the extremal disc and hence is realized by a rotation of the corresponding regular N-gon. In particular it is a simple matter to construct families of explicit surfaces with no automorphisms at all (cf. [2], [8]).

This is a rather surprising fact since "special" points in moduli space tend to correspond to surfaces with "many" automorphisms.

2 Triangle groups: arithmeticity and maximality

A *Fuchsian group* is a discrete subgroup of the group of conformal isometries of the hyperbolic plane $\mathbb{P}\mathrm{SL}(2,\mathbb{R})$.

A Fuchsian group Γ is called a *triangle group* of type (p,q,n) if it consists of the conformal elements of a group G generated by the reflections across the sides of a triangle $T(p,q,n)$ with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{n}$. This triangle provides a fundamental domain for G. The group Γ has index 2 in G and therefore a fundamental domain for Γ can be obtained by gluing two suitable copies of $T(p,q,n)$.

In this terminology the characterization of extremal surfaces given in 1. i) can be restated as follows:

Let $S = \frac{\mathbb{H}}{K}$ be a compact hyperbolic surface of genus g uniformized by a Fuchsian group K which acts freely on the hyperbolic plane $\mathbb{H}$. Then, S is extremal if and only if K is a subgroup of a triangle group Γ of type $(2,3,N)$ with index $[\Gamma : K] = N = 12g - 6$.

To see that this statement is truly equivalent to 1.i) it is enough to observe that a fundamental domain for K can be obtained by reunion of $2N$ copies of the triangle $T(2,3,N)$ defining Γ around the vertex of angle $\frac{\pi}{N}$. Such a fundamental domain is a regular N-gon, because the result of gluing two copies of $T(2,3,N)$ along the edge joining the vertices of angles $\frac{\pi}{2}$ and $\frac{\pi}{N}$ is again a triangle (with angles $\frac{2\pi}{N}$, $\frac{\pi}{3}$, and $\frac{\pi}{3}$).

Two Fuchsian groups Γ_1 and Γ_2 are *commensurable* if their intersection $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2.

The *commensurator* of a Fuchsian group Γ is the group

$$\mathrm{Com}(\Gamma) = \{\gamma \in \mathbb{P}\mathrm{SL}(2,\mathbb{R}) : \ \Gamma \text{ and } \gamma^{-1} \circ \Gamma \circ \gamma \text{ are commensurable}\}.$$

According to **Margulis** ([5]) the commensutator of a triangle group Γ is Fuchsian if and only if Γ is *nonarithmetic*. By the term *arithmetic* (Fuchsian group) we refer to any finite index subgroup of a group G_ρ consisting of all elements of $\mathbb{PSL}(2,\mathbb{R})$ corresponding to matrices with integer coefficients via any finite dimensional representation ρ (see [4]).

Concerning arithmeticity of triangle groups of type $(2,3,k)$ we have the following result of **Takeuchi** ([7]):
The only arithmetic triangle groups of type $(2,3,k)$ *occur for*

$$k = 7, 8, 9, 10, 11, 12, 14, 16, 18, 24, 30.$$

A *maximal* Fuchsian group is a group which is not strictly contained in any other Fuchsian group.
As for maximality of triangle groups of type $(2,3,k)$ we have the following result of **Singerman** ([6], see also [3]):
Triangle groups of type $(2,3,k)$ *are all maximal.*

A combination of Margulis-Takeuchi-Singerman gives us the following

Corollary: *For a triangle group* Γ *of type* $(2,3,k)$ *with* $k > 30$*, we have* $\mathrm{Com}(\Gamma) = \Gamma$.

3 Uniqueness of discs

Theorem: *A hyperbolic surface of genus* $g > 3$ *contains at most one extremal disc.*
Proof
Let us represent our surface in the standard form

$$S \equiv \frac{\mathbb{H}}{K}.$$

Suppose that S contains two extremal discs D_1 and D_2. From the characterization of extremal discs given in the previous section one sees that each of these discs corresponds to an inclusion of K in a triangle group Γ_i $(i = 1, 2)$ of type $(2,3,N)$ such that

$$[\Gamma_i : K] = N = 12g - 6.$$

Let now $\alpha \in \mathbb{PSL}(2,\mathbb{R})$ be the unique isometry of $\mathbb{H}$ which conjugates the two triangle groups (of same type) Γ_1, Γ_2 so that we have $\Gamma_1 = \alpha^{-1} \circ \Gamma_2 \circ \alpha$.

We see that

$$K \subset (\Gamma_2 \cap \Gamma_1) = \Gamma_2 \cap \left(\alpha^{-1} \circ \Gamma_2 \circ \alpha\right),$$

which shows that $\alpha \in \mathrm{Com}(\Gamma_2)$.
Now, when $g > 3$ then $N > 30$, and hence, by the corollary above, $\mathrm{Com}(\Gamma_2)$ agrees with Γ_2; in other words, $\alpha \in \Gamma_2$, which means that we have $\Gamma_1 = \Gamma_2$.
We conclude that the discs D_1 and D_2 are one and the same.

□

Note: As for genus 3, we believe that the surface obtained from the regular polygon with 30 sides identified in the following way

$$(1,22)(2,11)(3,6)(4,8)(5,9)(7,10)(12,21)(13,16)$$
$$(14,18)(15,19)(17,20)(23,26)(24,28)(25,29)(27,30)$$

admits at least two extremal discs, one centered at the center of the polygon and another one at the vertex between sides 1 and 2.

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References

[1] C. Bavard: *Disques extrémaux et surfaces modulaires*, Ann. de la Fac. des Sciences de Toulouse V (1996), No. 2, 191-202.

[2] B. Everitt: *A family of conformally asymmetric Riemann Surfaces*, Glasgow Math. J., 39 (1997), 221-225.

[3] L. Greenberg: *Maximal Fuchsian groups*, Bull. Amer. Math. Soc., 69 (1963), 569-573.

[4] S. Katok: *Fuchsian Groups*, Chicago Lectures in Mathematics (1992).

[5] G. Margulis: *Discrete subgroups of semisimple Lie groups*, Springer-Verlag (1991).

[6] D. Singerman: *Finitely maximal Fuchsian groups*, J. London Math. Soc. (2), 6 (1972), 29-38.

[7] K. Takeuchi: *Arithmetic triangle groups*, J. Math. Soc. Japan, 29 (1977), 91-106.

[8] P. Turbek: *An explicit family of curves with trivial automorphism group*, Proc. Amer. Math. Soc. (3), 122 (1994), 657-664.

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