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On complete curves in moduli space I

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Introduction

Let $\mathcal{M}_g$ denote the moduli space of compact Riemann surfaces of genus $g > 3$. It is known that $\mathcal{M}_g$ is a non-complete quasi-projective variety that contains many complete curves. This is because the Satake compactification $\tilde{\mathcal{M}}_g$ of $\mathcal{M}_g$ is projective and the boundary $\tilde{\mathcal{M}}_g \backslash \mathcal{M}_g$ has co-dimension 2; thus by intersecting $\tilde{\mathcal{M}}_g \subset \mathbb{P}^N$ with hypersurfaces in sufficiently general position one obtains a complete curve in $\mathcal{M}_g$ passing through any given set of points [**8**].

Although such curves are clearly in plentiful supply, it is not entirely trivial to produce an explicit one in given $\mathcal{M}_g$: see [**8**]. Constructions have been given previously for infinitely many values of g ([**14**, **1**, **13**, **20**]), but all leave many gaps.

In this article we construct by standard methods a curve in $\mathcal{M}_g$ for each $g \geqslant 4$ (Theorem 1). By 'standard' we mean that, as in previous constructions, our curve lies inside the singular locus of $\mathcal{M}_g$; in other words, our curve parametrizes Riemann surfaces with automorphisms.

These methods do not work for $g = 3$: in fact they cannot by our Theorem 2. Consequently a second article under the same title is devoted to finding five hypersurfaces whose intersection is a curve in $\mathcal{M}_3$.

The techniques used in the two articles differ radically. For the present one, the reader should be familiar with the basic facts of Teichmüller theory and Bers' fibre spaces, while in the second, which is completely independent, the main tool will be the theory of theta functions and Siegel modular forms.

1. *Review of moduli theory*

1·1. We describe briefly the relevant facts from the theory of Teichmüller spaces. For a detailed account, the survey article [**2**] is recommended. A recent comprehensive text book [**18**] provides a more complete treatment.

Let G be a Fuchsian group acting on the upper half-plane U with compact orbit space U/G. Then G is known to have the following structure:

$$\left.\begin{array}{ll}\text{Generators} & \gamma_1, \ldots, \gamma_k; a_1, b_1, \ldots, a_g, b_g; \\ \text{Relations} & \begin{cases} \gamma_1^{m_1} = \gamma_2^{m_2} = \ldots = \gamma_k^{m_k} = 1, \\ \gamma_1 \gamma_2 \cdots \gamma_k \prod_{i=1}^{g} [a_i, b_i] = 1, \; [a_i, b_i] = a_i^{-1} b_i^{-1} a_i b_i. \end{cases}\end{array}\right\} \tag{1·1}$$

The integers $m_1, m_2, \ldots, m_k$ (all different from 1) are called *the periods* of the group.