GALOIS ACTION ON UNIVERSAL COVERS OF KODAIRA FIBRATIONS

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Abstract. F. Catanese has recently asked if there exists an element of the absolute Galois group $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}})$ for which there is a Kodaira fibration $f : S \to B$ defined over a number field such that the universal covers of S and its Galois conjugate surface S^σ are not isomorphic. The main result of this article is that every element $\sigma \neq \mathrm{Id}.$ has this property.

1. Introduction and statement of results

A *Kodaira fibration* (of genus g) consists of a non-singular compact complex surface S, a compact Riemann surface (or, equivalently, a complex algebraic curve) B of genus q and a surjective holomorphic map $f : S \to B$ everywhere of maximal rank such that the fibers $F_b, b \in B$ are connected and not mutually isomorphic Riemann surfaces of genus g. It is known that necessarily $g \geq 3$ and $q \geq 2$. It is also known that such a surface S must be a minimal algebraic surface of general type and that its Euler characteristic (or Euler number) is

$$e(S) = e(F)e(B) = (2-2g)(2-2q).$$

With the term *Kodaira surface* we will refer to the total space S of a Kodaira fibration (although we warn the reader that this expression is sometimes used to refer to a different kind of surfaces, see e.g. [4]).

Kodaira fibrations were introduced by Kodaira in [24] and studied from different points of view by several authors including Atiyah [2], Hirzebruch [22] Catanese [7] and Catanese–Rollenske [8].

Due to results of Griffiths [20] and Bers [5] it is known that the (holomorphic) universal cover of any Kodaira surface is a bounded contractible domain $\mathscr{B} \subset \mathbb{C}^2$. In [18] S. Reyes-Carocca and the author

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proved the following result relative to universal covers and fields of definition of Kodaira surfaces.

Theorem (GD-RC, [18]**)** Kodaira surfaces with isomorphic universal covers can be defined over exactly the same set of algebraically closed subfields of $\mathbb{C}$.

Note that in particular this implies that the *arithmeticity* of a Kodaira surface S, that is the property of being definible over the field of complex algebraic numbers $\overline{\mathbb{Q}}$, depends only on its universal cover.

One may wonder if the converse holds (i.e. one can ask if all arithmetic Kodaira surfaces share a same universal cover). This looks at first sight (and indeed it will be shown here to be) too strong to be true. However F. Catanese has posed the following more subtle question

Question 33 (Catanese, [7]**):** Does there exist a Kodaira surface S defined over $\overline{\mathbb{Q}}$ and an automorphism $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}})$ such that the universal coverings of S and (its Galois conjugate) S^σ are not isomorphic?

The main result of this paper is that there are such Kodaira surfaces for all $\sigma \neq Id$. More precisely, we will prove the following results

Theorem 19. The absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}})$ acts faithfully on the set of biholomorphy classes of bounded contractible domains of $\mathbb{C}^2$ that arise as universal covers of Kodaira surfaces defined over $\overline{\mathbb{Q}}$.

In fact this will be a particular case of the following more general result. Let $\overline{k}$ be an algebraically closed subfield of $\mathbb{C}$. Let us denote by $\mathscr{U}_{BG}(\overline{k})$ the set of biholomorphy classes of bounded contractible domains of $\mathbb{C}^2$ which arise as universal covers of Kodaira surfaces definable over $\overline{k}$. (Here the subindex BG stands for Bers-Griffiths). We will prove the following

Theorem 18. The Galois group $\mathrm{Gal}(\overline{k}/\mathbb{Q})$ acts faithfully on $\mathscr{U}_{BG}(\overline{k})$.

Theorem 18 hints to the abundance of bounded contractible domains in $\mathbb{C}^2$. This is in contrast with the situation in the 1-dimensional case in which the unit disc serves as universal cover of all compact Riemann surfaces of negative Euler characteristic. Concerning this issue we will prove the following

Corollary 5. A bounded contractible domain $\mathscr{B} \subset \mathbb{C}^2$ can be the universal cover of only finitely many Kodaira surfaces of given Euler number e.

This result implies that the number of isomorphy classes of bounded contractible domains of $\mathbb{C}^2$ is uncountable. In addition, it may be worth mentioning that not all such domains arise as universal covers of Kodaira surfaces (see [19], Example 11).

The results in this article depend heavily on the following important result by G. Shabat

Theorem (Shabat, [29], [30]**)** Let $\mathscr{B}$ be the holomorphic universal cover of a Kodaira surface S. Then the covering group $\mathbb{G} \cong \pi_1(S)$ has finite index in $\mathrm{Aut}(\mathscr{B})$ (hence, $\mathrm{Aut}(\mathscr{B})$ is a discrete group).

A key result towards the proof of our main Theorem 18 will be the following improvement of Shabat's theorem

Theorem 3. The index $[\mathrm{Aut}(\mathscr{B}) : \pi_1(S)]$ is bounded by a constant that depends only on the Euler number of the Kodaira surface S (and not on its universal cover $\mathscr{B}$).

The paper is organized as follows:

In Section 2 we describe the universal cover of a Kodaira surface following the work of Griffiths and Bers. We also make the useful observation that finite index subgroups of groups that uniformise Kodaira surfaces do uniformise Kodaira surfaces themselves (**Remark 1**).

In Section 3 we focus on the fact that, due to Shabat's theorem, Kodaira surfaces with the same universal cover are commensurable. We use this result together with the Bogomolov-Miyaoka-Yau inequality and a theorem of Xiao on the automorphism groups of complex surfaces of general type to prove Theorem 3 and Corollary 5 mentioned above.

In Section 4 we produce the Kodaira fibrations $f_\mu : S_\mu \to B_\mu$ whose universal covers $\mathscr{B}_\mu$ will be our candidates to get non-trivially transformed by elements of $\mathrm{Gal}(\overline{k}) := \mathrm{Gal}(\overline{k}/\mathbb{Q})$, as required in Theorem 18. These will be explicit in the following sense:

(i) There is one for each genus 2 curve $D_\mu : y^2 = \prod_{d=1}^{6}(x - \mu_d)$.
(ii) The base curve B_μ will be an unramified cover of the genus 3 curve $C_\mu : y^2 = \prod_{k=3}^{6}(x^2 - \frac{\mu_k - \mu_2}{\mu_k - \mu_1})$ which itself is the unramified double cover of D_μ defined as the quotient map by the fixed point free involution $\alpha(x, y) = (-x, -y)$.
(iii) The fibre F_b over a point $b \in B_\mu$ mapping into a point $(x_b, y_b) \in C_\mu$ will be a double cover of C_μ ramified over the points (x_b, y_b) and $(-x_b, -y_b)$. Hence they will be genus 6 Kodaira fibrations.

The construction of these fibrations will be carried out within the framework of Teichmüller and moduli theories.

In Section 5 we define the action of $\mathrm{Gal}(\overline{k})$ on $\mathscr{U}_{BG}(\overline{k})$ and prove our main Theorem 18. A key point will be the construction of a Kodaira surface $S = S^\mu$ enjoying the following properties:

(i) S covers S_μ (hence it carries a surjective morphism onto D_μ).
(ii) $S^\sigma \cong S$ whenever $\sigma \in \mathrm{Gal}(\overline{k})_{\mathscr{B}_\mu}$, the stabiliser of $\mathrm{Gal}(\overline{k})$ at $\mathscr{B}_\mu$.

Therefore conjugation by any $\sigma \in \mathrm{Gal}(\overline{k})_{\mathscr{B}_\mu}$ (say of infinite order) will yield a collection of maps $F^{\sigma^n} : S^{\sigma^n} = S \to D_\mu^{\sigma^n}$ from a fixed complex surface S onto a collection of curves $D_\mu^{\sigma^n}$. Then the idea will be that if D_μ is suitably chosen (with respect to σ) the number of mutually non-isomorphic curves $D_{\mu^{\sigma^n}}$ will exceed Howard-Sommese's bound [23] for the number of Riemann surfaces that can arise as targets of a fixed complex surface such as S, thus contradicting the fact that $\sigma \in \mathrm{Gal}(\overline{k})_{\mathscr{B}_\mu}$, in other words $\mathscr{B}_\mu^\sigma$ must be different from $\mathscr{B}_\mu$.

2. Uniformisation of Kodaira fibrations

Thanks to the work of Bers [5] and Griffiths [20] on uniformization of algebraic varieties it is possible to describe the universal cover of a Kodaira fibration $f : S \to B$ in a very explicit way.

As usual, we shall denote by $\mathbb{H}$ the upper-half plane of the complex plane. Let $\pi : \mathbb{H} \to B$ be the universal covering map of B and let $\Gamma < PSL(2, \mathbb{R})$ be the corresponding covering group, so that $B \cong \mathbb{H}/\Gamma$. By considering the pull-back of f by π we obtain a new fibration $\hat{f} : \pi^*S \to \mathbb{H}$ over $\mathbb{H}$. For each $t \in \mathbb{H}$, the fiber $(\hat{f})^{-1}(t)$ agrees with the Riemann surface $f^{-1}(\pi(t))$. The results of Bers and Griffiths show that one can choose uniformizations $(\hat{f})^{-1}(t) = D_t/K_t$ possessing the following properties:

(a) K_t is a *Kleinian* group acting on a bounded domain D_t of $\mathbb{C}$ which is biholomorphically equivalent to a disc (in fact a *quasidisc*).
(b) The union of all these (disjoint) discs $\mathscr{B} := \cup_{t\in\mathbb{H}} D_t$ is a bounded contractible domain of $\mathbb{C}^2$ biholomorphy equivalent to $\tilde{S}$, the universal cover of S, so that $S \cong \mathscr{B}/\mathbb{G}$, where $\mathbb{G} < \mathrm{Aut}(\mathscr{B})$ is the covering group.
(c) The group $\mathbb{G}$ is endowed with an epimorphism $\rho : \mathbb{G} \to \Gamma$ defined by $\rho(\varphi) = \gamma$ if and only if $\varphi(D_t) = D_{\gamma(t)}$. This induces an exact sequence of groups

$$1 \longrightarrow \mathbb{K} \longrightarrow \mathbb{G} \xrightarrow{\rho} \Gamma \longrightarrow 1 \tag{2.1}$$

where the group $\mathbb{K} = Ker\rho$ happens to be isomorphic to its restriction to each quasidisc D_t. This restriction is precisely the group K_t mentioned above [19].

Bounded contractible domains of $\mathbb{C}^2$ obtained in this way shall be referred to as *Bers-Griffiths* domains and the set of biholomorphy classes of them will be denoted by $\mathscr{U}_{BG} = \mathscr{U}_{BG}(\mathbb{C})$. We note that a Bers-Griffiths domain $\mathscr{B}$ carries itself a fibration structure $\tilde{f} : \mathscr{B} \to \mathbb{H}$ whose fiber over $t \in \mathbb{H}$ is D_t. The situation is summarized in the following commutative diagram

$$\begin{array}{ccc} D_t \subset \mathscr{B} & \longrightarrow & S \cong \mathscr{B}/\mathbb{G} \\ \big\downarrow \tilde{f} & & \big\downarrow f \\ t \in \mathbb{H} & \xrightarrow{\pi} & B \cong \mathbb{H}/\Gamma \end{array}$$

The following observation will be used repeatedly throughout the paper

Remark 1. Let $\mathbb{G}_1$ be a finite index subgroup of $\mathbb{G}$. Then the restriction of the sequence (2.1) to $\mathbb{G}_1$ defines a new Kodaira fibration $f_1 : S_1 \to B_1$, where $S_1 = \mathscr{B}/\mathbb{G}_1$, $B_1 = \mathbb{H}/\Gamma_1$, with $\Gamma_1 := \rho(\mathbb{G}_1)$, and f_1 is induced by $\tilde{f}$ in the obvious way. Clearly S_1 is a smooth cover of S of degree $[\mathbb{G} : \mathbb{G}_1] = \frac{e(S_1)}{e(S_2)}$.

3. The automorphism groups of Bers-Griffiths domains

By Shabat's theorem, there is a canonical complex surface associated to any Bers-Griffiths domain $\mathscr{B}$, namely $S_{\mathscr{B}} = \mathscr{B}/\mathrm{Aut}(\mathscr{B})$. This may be a singular surface but, by Cartan's theorem, it will always be normal.

Let S be any Kodaira surface with universal cover $\tilde{S} \cong \mathscr{B}$ and uniformising group $\mathbb{G} \cong \pi_1(S)$. Let us denote by $N(\mathbb{G})$ the normaliser of $\mathbb{G}$ in $\mathrm{Aut}(\mathscr{B})$ so that $\mathscr{B}/\mathbb{G} \cong S$ and $\mathscr{B}/N(\mathbb{G}) \cong S/\mathrm{Aut}(S)$. The inclusions $\mathbb{G} < N(\mathbb{G}) < \mathrm{Aut}(\mathscr{B})$ give rise to the following commutative diagram of (possibly ramified) covers.

$$\begin{tikzcd} & S \arrow[dl, "p_S"'] \arrow[dr, "\pi_S"] & \\ S/\mathrm{Aut}(S) \arrow[rr, "\overline{\pi}_S"] & & S_{\mathscr{B}} \end{tikzcd} \tag{3.1}$$

3.1. Commensurability.

We recall that two non-singular compact complex surfaces X_1, X_2 are said to be commensurable if they admit a common finite smooth cover, that is, if there is a non-singular compact complex surface Y admitting surjective morphisms $\pi_i : Y \to X_i$. Clearly, commensurable surfaces have isomorphic universal covers. For Kodaira surfaces the converse statement holds because if $S_i = \mathscr{B}/\mathbb{G}_i$ $(i = 1, 2)$ are two Kodaira surfaces, Shabat's theorem shows that the surface $S_{12} := \mathscr{B}/\mathbb{G}_1 \cap \mathbb{G}_2$ is a smooth finite cover of both of them. Moreover, one has

Lemma 2. (1) *Let S_1, S_2 be two commensurable Kodaira surfaces with universal cover $\mathscr{B}$ and uniformising groups $\mathbb{G}_1$ and $\mathbb{G}_2$. Then*
$$\frac{e(S_2)}{e(S_1)} = \frac{[Aut(\mathscr{B}) : \mathbb{G}_2]}{[Aut(\mathscr{B}) : \mathbb{G}_1]}$$
In particular two subgroups of $Aut(\mathscr{B})$ uniformise Kodaira surfaces with the same Euler number if and only if they have the same index.

(2) *The uniformising group $\mathbb{G}$ of a Kodaira surface S with universal cover $\mathscr{B}$ is a normal subgroup of $Aut(\mathscr{B})$ if and only if $|Aut(S)| = [Aut(\mathscr{B}) : \mathbb{G}]$. In that case the subgroup $\mathbb{G} \triangleleft Aut(\mathscr{B})$ is uniquely determined by S and the map $\overline{\pi}_S : S/Aut(S) \to S_{\mathscr{B}}$ is an isomorphism.*

Proof. The proof will easily follow from consideration of the following commutative diagram:

$$\begin{tikzcd}[column sep=small]
 & S_{12} := \mathscr{B}/\mathbb{G}_1 \cap \mathbb{G}_2 \arrow[dl] \arrow[dr] & \\
S_1 = \mathscr{B}/\mathbb{G}_1 \arrow[dr] & & \mathscr{B}/\mathbb{G}_2 = S_2 \arrow[dl] \\
 & S_{\mathscr{B}} = \mathscr{B}/\mathrm{Aut}(\mathscr{B}) &
\end{tikzcd}$$

(1) follows from the observation that $\frac{e(S_{12})}{e(S_i)} = [\mathbb{G}_i : \mathbb{G}_1 \cap \mathbb{G}_2]$ and the multiplicativity of degrees.
(2) Since $\mathrm{Aut}(S) \equiv N(\mathbb{G})/\mathbb{G}$ the equality $N(\mathbb{G}) = \mathrm{Aut}(\mathscr{B})$ holds if and only if $|\mathrm{Aut}(S)| = [\mathrm{Aut}(\mathscr{B}) : \mathbb{G}]$. Clearly $\overline{\pi}_S$ is an isomorphism in this case. The uniqueness of $\mathbb{G}$ when $\mathbb{G} \triangleleft \mathrm{Aut}(\mathscr{B})$ follows from the fact that subgroups of $\mathrm{Aut}(\mathscr{B})$ uniformising isomorphic Kodaira surfaces are conjugate. □

3.2. **Universal bounds.** The following result which can be seen as an improvement of Shabat's theorem (in the compact case) will be crucial in this paper.

Theorem 3. *Let e be a given positive integer. There is a constant M_e such that, for any Kodaira surface S with Euler number $e(S) = e$, the index $[Aut(\mathscr{B}) : \pi_1(S)]$ is bounded by M_e, where $\mathscr{B}$ denotes the universal cover of S and the fundamental group $\pi_1(S)$ is identified with the corresponding covering group $\mathbb{G} < Aut(\mathscr{B})$.*

Proof. Let us denote by S^{nor} the normalisation of the natural projection $\pi_S : S \to S_{\mathscr{B}}$. By this we mean the Kodaira surface $S^{nor} = \mathscr{B}/\mathbb{G}^{cor}$, where $\mathbb{G}^{cor}$ is the *core* normal subgroup defined by

$$\mathbb{G}^{cor} = \bigcap_{\varphi \in \mathrm{Aut}(\mathscr{B})} \varphi \mathbb{G} \varphi^{-1}. \tag{3.2}$$

Notice that $\mathbb{G}^{cor}$ can also be obtained by intersecting only the conjugates of $\mathbb{G}$ by a set of representatives of $\mathrm{Aut}(\mathscr{B})/\mathbb{G}$ so, by Shabat's theorem, $\mathbb{G}^{cor}$ is a finite index subgroup of $\mathbb{G}$. Therefore corresponding to the inclusions $\mathbb{G}^{cor} < \mathbb{G} < \mathrm{Aut}(\mathscr{B})$ there are obvious surjective morphisms of compact complex surfaces

$$S^{nor} \to S \xrightarrow{\pi_S} S_{\mathscr{B}} \cong S^{nor}/\mathrm{Aut}(S^{nor}),$$

were the isomorphism $S_{\mathscr{B}} \cong S^{nor}/\mathrm{Aut}(S^{nor})$ holds by the second part of Lemma 2.

Note that what we want to prove is that the degree of π_S is bounded by a constant depending only on $e(S)$. Our first ingredient is a theorem by Xiao [31] according to which there is a universal constant c such that the order of the automorphism group of a minimal surface of general type X satisfies the inequality $|\mathrm{Aut}(X)| \leq cK_X^2$, where as usual K_X^2 stands for the self-intersection of the canonical divisor K_X. This applied to our surface S^{nor} gives

$$deg(\pi_S) = \frac{e(S)}{e(S^{nor})} \, |\mathrm{Aut}(S^{nor})| \leq \frac{e(S) c K^2_{S^{nor}}}{e(S^{nor})}$$

Now, by the Bogomolov-Miyaoka-Yau inequality (see [32]) the *slope* $K_X^2/e(X)$ of a complex surface of general type X is bounded by 3 (in fact, for the particular case of Kodaira surfaces we are dealing with here, the surfaces with largest slope have been constructed by Catanese and Rollenske [8] and attain the value $2 + 2/3$). Letting X be S^{nor} we conclude that $deg(\pi_S) \leq 3ce$, as desired. □

Corollary 4. *Let $\mathscr{B}$ a Bers-Griffiths domain and e the Euler number of any Kodaira surface which has $\mathscr{B}$ as universal cover. Then the following statements hold:*

1. *$Aut(\mathscr{B})$ can be generated by a set of cardinality $16eM_e$.*
2. *The number of subgroups of index n of $Aut(\mathscr{B})$ is bounded by $(16eM_e)^{n!}$*

Proof. Let $S = \mathscr{B}/\mathbb{G}$ be a Kodaira surface with $e(S) = e$, which exists by hypothesis. To prove our two claims we argue as follows.

(1) The group $\mathbb{G}$ fits in the middle of a short exact sequence (2.1) whose kernel is a surface group of genus g (the genus of the fiber) and its image is a surface group of genus q (the genus of the base). Therefore $\mathbb{G}$ can be generated by $4gq$ elements. From here, using Theorem 3,

we infer that $\mathrm{Aut}(\mathscr{B})$ can be generated by $4gqM_e$ elements. Since $e = e(S) = 4(g-1)(q-1)$, the result follows.

(2) It is well-known that index n subgroups of a group G arise as point stabilisers of transitive representations of G in the symmetric group S_n. But, by part (1), the number of repesentations of $\mathrm{Aut}(\mathscr{B})$ in S_n is bound by $(16eM_e)^{n!}$. □

Corollary 5. *The number of Kodaira surfaces with given universal cover $\mathscr{B}$ and Euler number e is finite.*

Proof. By the part (1) of Lemma 2 fixing the Euler number of a Kodaira surface is tantamount to fixing the index of its covering group in $\mathrm{Aut}(\mathscr{B})$. Therefore the result follows from part (2) of Corollary 4. □

Remark 6. Theorem 3 and its corollaries extend to surfaces S with ample canonical bundle and slope $K_S^2/e(S) \neq 2, 3$. This is because by a theorem of Nadel [27] the index $[\mathrm{Aut}(\tilde{S}) : \pi_1(S)]$ is finite unless $\tilde{S}$ is biholomorphic to the bi-disc $\mathbb{H} \times \mathbb{H}$ or the 2-ball $\mathbb{B}^2$. But it is well-known that in these cases the slope equals 2 and 3 respectively (see e.g. [7], 2.1.2).

4. Construction of the Kodaira fibrations

We will start by considering unramified double covers of Riemann surfaces of genus 2.

Any algebraic curve of genus 2 is isomorphic to one of the form

$$D_\mu : y^2 = \prod_{d=1}^{6}(x - \mu_d)$$

with $\mu = (\mu_1, \cdots, \mu_6) \in \mathbb{C}^6 \setminus \Delta$, where Δ is the multidiagonal set of sextuples satisfying $\mu_i = \mu_j$ for some $i \neq j$. We recall that any curve D_μ admits 15 such double covers (necessarily) of genus 3. Here, for the sake of explicitness, we will choose one them, namely

$$C_\mu : y^2 = \prod_{k=3}^{6}(x^2 - \frac{\mu_k - \mu_2}{\mu_k - \mu_1});$$

the covering map $G : C_\mu \to D_\mu$ being defined by the formula

$$G(x, y) = \left(\frac{\mu_2 - \mu_1 x^2}{1 - x^2}, xy(\mu_1 - \mu_2)\frac{\sqrt{\prod_{d=3}^{d=6}(\mu_d - \mu_1)}}{(1 - x^2)^3} \right)$$

which (see [10]) is nothing but the quotient map corresponding to the fixed point free involution

$$\alpha = \alpha_\mu : (x, y) \to (-x, -y).$$

As a first attempt to construct explicit Kodaira fibrations one can consider the family of Riemann surfaces obtained by associating to each point $x \in C_\mu$ a double cover F_x^μ of C_μ ramified over the (distinct!) points x and $\alpha(x)$. Unfortunately, this procedure does not provide a well-defined Kodaira fibration with base curve C_μ because the Riemann surface F_x^μ is not uniquely determined. However, it will be possible to take a further unramified cover $p_\mu : B_\mu \to C_\mu$ such that for each point $b \in B_\mu$ a double cover F_b^μ of C_μ ramified over $p_\mu(b)$ and $\alpha p_\mu(b)$ can be coherently chosen so as to produce a Kodaira fibration with base B_μ and fibers F_b^μ. We will work within the framework of Teichmüller and moduli theories. Some of the ideas we use go back to the articles [14], [15] and [16] by Harvey and the author.

4.1. **The (Teichmüller theoretical) construction of our Kodaira fibrations.** This will be done in six steps of which the first two summarize well-known facts of Teichmüller and moduli theories.

1) Let R be a Riemann surface with n distinguished points and let us denote by $T(R)$ the corresponding *Teichmüller space*. We recall that points in $T(R)$ are (Teichmüller) classes of pairs $[\psi, D]$ where $\psi : R \to D$ is a homeomorphism of n-pointed Riemann surfaces; the point $[Id, R]$ is called the *base point*. If R' is another such Riemann surface and $\theta : R \to R'$ is a homeomorphism of n-pointed Riemann surfaces then the rule $[\psi, D] \to [\psi\theta^{-1}, D]$ identifies $T(R')$ with $T(R)$, so the Teichmüller space is independent of the base point and one usually writes $T_{g,n}$, or simply T_g when $n = 0$. The *moduli space* of compact Riemann sufaces of genus g with n distinguished points, usually denoted $\mathscr{M}_{g,n}$, can be presented as the quotient $\mathscr{M}_{g,n} = T_{g,n}/\mathrm{Mod}_{g,n}$, where $\mathrm{Mod}_{g,n} = \mathrm{Mod}(R)$ is the modular (or *mapping class*) group consisting of mapping classes (of R) that preserve the set of distinguished points and the action is defined by the same rule as that of the homeomorphism θ above. Its points represent isomorphy classes $[F]$ of Riemann surfaces F of genus g with n distinguished points. Again when $n = 0$ one simply writes $\mathscr{M}_g$ and Mod_g respectively. (For all this see e.g. [5] and [28]).

2) The moduli space $\mathscr{M}_g$ possesses the property that for any Kodaira fibration $f : S \to B$ of genus g the map $\Phi_f : B \to \mathscr{M}_g$ defined by sending each point $b \in B$ to the point in $\mathscr{M}_g$ representing the Riemann surface $f^{-1}(b)$ is a holomorphic map (called the *classifying map*). And

the converse almost holds, for $\mathscr{M}_g$ comes equipped with a fibration $\mathscr{C}_g \to \mathscr{M}_g$ (called the *universal curve*), whose fiber above a point $[F] \in \mathscr{M}_g$ is a Riemann surface isomorphic to $F/\mathrm{Aut}(F)$. Thus, for any non-constant holomorphic map $\Phi : B \to \mathscr{M}_g$ the pullback $\Phi^*(\mathscr{C}_g)$ is a family of Riemann surfaces parametrised by B. When $\Phi = \Phi_f$ this family coincides with the initial fibration $f : S \to B$ except at the exceptional points $b \in B$ for which $\mathrm{Aut}(f^{-1}(b))$ is not trivial.

The way to deal with these inconvenient exceptions is to consider the *moduli space with level-*3 *structure* $\mathscr{M}_g[3] = T_g/\mathrm{Mod}_g[3]$ where, denoting by F_0 the base point of moduli space, $\mathrm{Mod}_g[3]$ can be regarded as the subgroup of $\mathrm{Mod}_g = \mathrm{Mod}(F_0)$ consisting of those mapping classes $\theta : F_0 \to F_0$ whose action on the homology is trivial (mod 3). Its points correspond to isomorphism classes $[F, \{\lambda_i\}]$ of pairs $(F, \{\lambda_i\})$ in which $\{\lambda_i\}_{i=1}^{i=2g}$ is a basis of the homology group $H_1(F, \mathbb{Z}/3\mathbb{Z})$. Now, since no non-trivial automorphism of a compact Riemann surface can leave invariant the homology (mod 3), the corresponding (level 3) universal curve $\mathscr{C}_g[3] \to \mathscr{M}_g[3]$ has the property that the fiber over a point $[F, \{\lambda_i\}]$ is exactly isomorphic to F. And the analogous statement holds for the general moduli spaces $\mathscr{M}_{g,n}$ with respect to fibrations endowed with n disjoint sections. (For all this we refer to [1], [16] and [28]).

3) From what has gone above it follows that in order to construct a Kodaira fibration it is enough to define a non-constant holomorphic map $\Phi : B \to \mathscr{M}_g[3]$, where B is a compact Riemann surface, for then the pull-back $\Phi^*\mathscr{C}_g[3]$ will provide a genus g Kodaira fibration with base B.

Moreover, as $\mathrm{Mod}_g[3]$ acts freely on T_g, covering space theory shows that this is equivalent to define a holomorphic map $\tilde{\Phi} : \mathbb{H} \to T_g$ (the lift of Φ) equivariant with respect to the actions of a Fuchsian surface group Γ (the uniformising group of B) on $\mathbb{H}$ and the group $\mathrm{Mod}_g[3]$ on T_g, thereby inducing a group homomorphism $\Phi_* : \Gamma \to \mathrm{Mod}_g[3] < \mathrm{Mod}_g$.

Our Kodaira surfaces S_μ will arise from maps $\Phi^\mu : \mathbb{H} \to T_6$ such as $\tilde{\Phi}$. The construction will be achieved in next three steps. In Step 4 we construct a map $\Phi^\mu : \mathbb{H} \to T_{3,2}$. In Step 5 we show that $T_{3,2}$ can be embedded in T_6, thus allowing us to regard Φ^μ as a map $\Phi^\mu : \mathbb{H} \to T_6$. Finally in Step 6 we identify a Fuchsian group Γ_μ with respect to which Φ^μ is equivariant, hence obtaining a map $\Phi_\mu : B_\mu \to \mathscr{M}_6[3]$.

4) (*The map* $\mathbb{H} \to T_{3,2}$). The obvious projection $C_\mu \times C_\mu \to C_\mu$ of the surface $C_\mu \times C_\mu$ into the first coordinate has constant fibres isomorphic to C_μ, hence it is not a Kodaira fibration. But endowed with the disjoint sections $s_1(x) = (x, x)$ and $s_2(x) = (x, \alpha(x))$ this projection

becomes a non isotrivial family of 2-pointed curves of genus 3, the fiber over x being the pointed curve $F_x = (C_\mu, \{x, \alpha x\})$. Therefore there is a corresponding classifying map defined as

$$\begin{array}{rccl} \Phi_\mu : & C_\mu & \to & \mathscr{M}_{3,2} \\ & x & \to & [C_\mu, \{x, \alpha x\}] \end{array}$$

Let us denote by $\pi_\mu : \mathbb{H} \to C_\mu$ the universal cover of C_μ and by G_μ its covering group so that $C_\mu \cong \mathbb{H}/G_\mu$. Let us choose a base point x_μ in C_μ - e.g. $x_\mu = \left(0, \sqrt{\prod_{k=3}^{6} \frac{\mu_k - \mu_2}{\mu_k - \mu_1}}\right)$ - and set $T^\mu_{3,2} := T(C_\mu; \{x_\mu, \alpha x_\mu\})$. In this case the group $\mathrm{Mod}_{3,2}$ may have torsion but *Grothendieck-Teichmüller's universal property* still implies that Φ_μ lifts to a map

$$\begin{array}{rccl} \Phi^\mu : & \mathbb{H} & \to & T^\mu_{3,2} \\ & t & \to & \Phi^\mu(t) \end{array} \tag{4.1}$$

defined up to composition with an element of $\mathrm{Mod}\left(C_\mu; \{x_\mu, \alpha x_\mu\}\right)$. This is the map announced at the beginning of this point.

By construction Φ^μ gives rise to a group homomorphism

$$\Phi^\mu_* : G_\mu \to \mathrm{Mod}^\mu_{3,2} := \mathrm{Mod}\left(C_\mu; \{x_\mu, \alpha x_\mu\}\right) \tag{4.2}$$

(the *monodromy homomorphism*) characterised by the identity

$$\Phi^\mu_*(\gamma) \circ \Phi^\mu = \Phi^\mu \circ \gamma \quad \text{for every } \gamma \in G_\mu.$$

(For this point see [9], [14] and [28]).

5) (The *embedding* $T_{3,2} \subset T_6$). Let C_0 be a Riemann surface of genus $g' = 3$ with two distinguished points x and y and suppose that F_0 is a double cover of C_0 ramified over those points. This is equivalent to saying that F_0 admits an involution τ_0 with two fixed points such that $F_0/\langle \tau_0 \rangle \cong C_0$ in such a way that the two fixed points map to x and y. Note that the genera g of F_0 and g' of C_0 will be related by the formula $g = 2g'$, hence $g = 6$ in our case.

In the above situation one can consider the *relative moduli space* $\mathscr{M}_g(\tau_0)$ whose points are isomorphy classes $[F, \tau]$ of pairs (F, τ) where F is a Riemann surface of genus g and τ an automorphism topologically conjugate to τ_0. This moduli space can be realised as a quotient $\mathscr{M}_g(\tau_0) = T_g(\tau_0)/N\langle \tau_0 \rangle$ where $T_g(\tau_0)$ is the fixed locus of τ_0 in T_g, seen as an element of $\mathrm{Mod}_g = \mathrm{Mod}_g(F_0)$, and $N\langle \tau_0 \rangle$ stands for the normaliser of $\langle \tau_0 \rangle$ in Mod_g. What is relevant for us is that the natural map

$$\begin{array}{rcc} \mathscr{M}_g(\tau_0) & \to & \mathscr{M}_{g',2} \\ [F, \tau] & \to & \left[\overline{F}, \overline{\mathrm{Fix}(\tau)}\right], \end{array}$$

where $\overline{F}$ stands for the quotient $F/\langle\tau\rangle$ and $\overline{\mathrm{Fix}(\tau)}$ for the image on it of the two fixed points of τ, is induced by a holomorphic isomorphism between $T_g(\tau_0)$ and $T_{g',2}$. The corresponding inclusion $T_{g',2} \cong T_g(\tau_0) \subset T_g$ together with the map $\Phi^\mu : \mathbb{H} \to T_{3,2}$ defined in (4.1) of the previous step gives the desired holomorphic map to T_6 which we still denote $\Phi^\mu : \mathbb{H} \to T_6$. Now, the isomorphism $T_g(\tau_0) \cong T_{g',2}$ is equivariant with respect to the actions of $N\langle\tau_0\rangle$ on $T_g(\tau_0)$ and $\mathrm{Mod}_{g',2}$ on $T_{g',2}$ and the corresponding group homomorphism $N\langle\tau_0\rangle \to \mathrm{Mod}_{g',2}$ has kernel $\langle\tau_0\rangle$ and finite index image ([11], Proposition 1). Furthermore, since τ_0 does not lie in $\mathrm{Mod}_g[3]$, the above homomorphism restricts to a monomorphism on $N\langle\tau_0\rangle \cap \mathrm{Mod}_g[3]$ allowing us to identify this subgroup of Mod_g with its image $\overline{N\langle\tau_0\rangle \cap \mathrm{Mod}_g[3]}$ in $\mathrm{Mod}_{g',2}$. (For all this, see [15], [11], [21] and [25]).

6) (*The map* $\Phi_\mu : B_\mu \to \mathscr{M}_6[3]$). In the preceeding Step 5 let us consider the particular case in which $(C_0, \{x, y\})$ is the 2-pointed curve $(C_\mu, \{x_\mu, \alpha x_\mu\})$ introduced in Step 4) as the base point of $T_{3,2}$ and $F_0 = F_\mu$ is a double cover of C_μ induced by an involution τ_μ that fixes two points which project onto x_μ and αx_μ. Thus now $g' = 3$ and $g = 6$.

We define a subgroup Γ_μ of G_μ, the uniformising group of C_μ, by

$$\Gamma_\mu := (\Phi^\mu_*)^{-1}(\overline{N\langle\tau_\mu\rangle \cap \mathrm{Mod}_6[3]}) \tag{4.3}$$

with $\Phi^\mu_* : G_\mu \to \mathrm{Mod}_{3,2}$ as in (4.2).

This is a finite index subgroup of Γ_μ (a more precise statement is made within the proof of Lemma 8 below) and therefore there is a tower of unramified covers of compact Riemann surfaces as follows

$$B_\mu := \mathbb{H}/\Gamma_\mu \to C_\mu = \mathbb{H}/G_\mu \to D_\mu \tag{4.4}$$

By the definition of Γ_μ the map $\Phi^\mu : \mathbb{H} \to T_6$ is equivariant with respect to the action of the groups Γ_μ and $\mathrm{Mod}_6[3]$ and induces a map

$$\Phi_\mu : B_\mu \to \mathscr{M}_6[3]$$

resulting as the following composition of maps

$$B_\mu \to \frac{T_{3,2}}{\overline{N\langle\tau_\mu\rangle \cap \mathrm{Mod}_6[3]}} \equiv \frac{T_6(\tau_\mu)}{N\langle\tau_\mu\rangle \cap \mathrm{Mod}_6[3]} \longrightarrow T_6/\mathrm{Mod}[3] = \mathscr{M}_6[3]$$

As mentioned in 3) we can now define a Kodaira fibration by setting

$$f_\mu : S_\mu := \Phi^*_\mu(\mathscr{C}_6[3]) \to B_\mu$$

By construction this will be a genus 6 Kodaira fibration. We will denote by $\mathscr{B}_\mu$ and $\mathbb{G}_\mu$ the universal cover and the uniformising group of S_μ.

Remark 7. In the above construction we could have started with any algebraic curve D of arbitrary genus $p \geq 2$ and any unramified double cover C of genus $g' = 2p-1$ to obtain Kodaira fibrations of genus $2g'$.

4.2. **Some properties of the Kodaira fibrations $f_\mu : S_\mu \to B_\mu$.** Let us denote by $Sp(g,q)$ the *symplectic group* of genus g over the finite field $\mathbb{F}_q$. We recall that $Sp(g,q) \cong \mathrm{Mod}_g/\mathrm{Mod}_g[q]$ and that explicit formulas for the order of these classical groups are available. In the following proposition we will be interested in the particular case $g=6$ and $q=3$ for which we have $|Sp(6,3)| = \prod_{i=1}^{6}(3^{2i}-1)3^{2i-1}$.

Lemma 8. $-e(B_\mu) \leq 2^8\,|Sp(6,3)|$, *for every* $\mu \in \mathbb{C}^6 \setminus \Delta$.

Proof. Clearly, $e(B_\mu) = [G_\mu : \Gamma_\mu]\,e(C_\mu) = -4\,[G_\mu : \Gamma_\mu]$

In order to compute the index $[G_\mu : \Gamma_\mu]$ let us represent the ramified double cover $F_\mu \to C_\mu$ as the projection $\mathbb{H}/K_\mu \to \mathbb{H}/G_\mu^\circ$ induced by an index 2 inclusion $K_\mu < G_\mu^\circ$ where K_μ is a torsion free Fuchshian group uniformising F_μ and G_μ° is a *Fuchshian group of type* $(3,2)$ which uniformises the 2-pointed genus 3 Riemann surface $(C_\mu, \{x_\mu, \alpha x_\mu\})$. We recall that G_μ° has a standard presentation consisting of six hyperbolic generators $a_1, a_2, a_3; b_1, b_2, b_3$ and two order 2 elliptic generators γ_1, γ_2 subject to the single relation $[a_1b_1][a_2b_2][a_3b_3]\gamma_1, \gamma_2 = 1$. The Möbius transformations γ_1, γ_2 account for the distinguished points $x_\mu, \alpha x_\mu$ and there is an isomorphism $G_\mu^\circ/K_\mu \cong \langle \tau_\mu \rangle$ under which γ_1 and γ_2 map to the involution τ_μ.

In these terms the elements of $\mathrm{Mod}_{3,2}$ can be described as mapping classes of $\mathbb{H}/G_\mu^\circ$ induced by homeomorphisms $u : \mathbb{H} \to \mathbb{H}$ such that $uG_\mu^\circ u^{-1} = G_\mu^\circ$. Such elements belongs to the image $\overline{N\langle \tau_\mu \rangle}$ of $N\langle \tau_\mu \rangle$ exactly when $uK_\mu u^{-1} = K_\mu$ (see e.g. [15], [25]). Therefore the index of $\overline{N\langle \tau_\mu \rangle}$ in $\mathrm{Mod}_{3,2}$ is bounded by the number of torsion free index 2 subgroups of G_μ°. Now, this number is the same as the number of epimorphisms from G_μ° to $\mathbb{Z}/2\mathbb{Z}$ with torsion free kernel which, in view of the given prsentation, is equal to 2^6. Thus, we conclude that $\left[\mathrm{Mod}_{3,2} : \overline{N\langle \tau_\mu \rangle}\right] \leq 2^6$ (cf. [11], Proposition 1). From here we infer:

$$
\begin{aligned}
\left|\frac{G_\mu}{\Gamma_\mu}\right| \quad &\leq \quad \left|\frac{\mathrm{Mod}_{3,2}}{\overline{\mathrm{Mod}_6[3] \cap N\langle \tau_\mu \rangle}}\right| \quad \leq \quad 2^6 \left|\frac{\overline{N\langle \tau_\mu \rangle}}{\overline{\mathrm{Mod}_6[3] \cap N\langle \tau_\mu \rangle}}\right| \\
&\leq \quad 2^6 \left|\frac{N\langle \tau_\mu \rangle}{\mathrm{Mod}_6[3] \cap N\langle \tau_\mu \rangle}\right| \quad \leq \quad 2^6 \left|\frac{\mathrm{Mod}_6}{\mathrm{Mod}_6[3]}\right| \\
&= \quad 2^6\,|Sp(6,3)|\,.
\end{aligned}
$$

The proof follows. □

Proposition 9. *Let $f_\mu : S_\mu \to B_\mu$ the genus 6 Kodaira fibration introduced above, $\mathscr{B}_\mu$ its universal cover, $\mathbb{G}_\mu$ its covering group and $\mathbb{G}_\mu^{cor}$ the core subgroup of $\mathbb{G}_\mu$, as defined by the formula (3.2). Then*

1. *The Euler characteristic $e(S_\mu)$ is bounded by $5 \cdot 2^9\, |Sp(6,3)|$.*
2. *The number of subgroups of $Aut(\mathscr{B}_\mu)$ of given index n is bounded by a constant independent of μ.*
3. *There is a constant M such that $[Aut(\mathscr{B}_\mu) : \mathbb{G}_\mu] \leq M$ and $\left[Aut(\mathscr{B}_\mu) : \mathbb{G}_\mu^{cor}\right] \leq M^M$, for every μ.*
4. *S_μ is defined over a number field if and only if D_μ is.*

Proof. (1) $e(S_\mu) = e(F_\mu)e(B_\mu) = -10e(B_\mu)$. Now, apply Lemma 8.

(2) By Corollary 4, part 2, the number of subgroups of $\mathrm{Aut}(\mathscr{B}_\mu)$ of given index n is bounded by by a constant of the form $(16eM_e)^{n!}$ where e is the Euler number of any Kodaira surface S with $\tilde{S} = \mathscr{B}_\mu$ and M_e is a constant depending only on e. Making $S = S_\mu$ and applying the previous part (1) the result follows.

(3) The first bound is a consequence of part (1) and Theorem 3. The second one follows by considering the obvious injection

$$\mathrm{Aut}(\mathscr{B})/\mathbb{G}_\mu^{cor} \quad \longmapsto \prod_{\varphi \in \mathrm{Aut}(\mathscr{B})/\mathbb{G}_\mu} \mathrm{Aut}(\mathscr{B})/\varphi \mathbb{G}_\mu \varphi^{-1}.$$

(4) It will be sufficient to invoke the fact that a Kodaira fibration is defined over $\overline{\mathbb{Q}}$ if and only if the base is (see [18]) and that B_μ is defined over $\overline{\mathbb{Q}}$ if and only if D_μ is (see [13]). □

5. Galois action on Bers-Griffiths domains

Throughout this section $\overline{k}$ will denote an algebraically closed subfield of $\mathbb{C}$. Recall that $\mathscr{U}_{BG}(\overline{k})$ denotes the set of biholomorphy classes of bounded contractible domains of $\mathbb{C}^2$ which arise as universal covers of the Kodaira surfaces definable over $\overline{k}$. We will write $\mathscr{B} \in \mathscr{U}_{BG}(\overline{k})$ to mean that the biholomorphy class of the domain $\mathscr{B}$ lies in $\mathscr{U}_{BG}(\overline{k})$.

5.1. **Definition of the action.** Given $\mathscr{B} \in \mathscr{U}_{BG}(\overline{k})$ and $\sigma \in \mathrm{Gal}(\overline{k}) := \mathrm{Gal}(\overline{k}/\mathbb{Q})$ we define $\mathscr{B}^\sigma \in \mathscr{U}_{BG}(\overline{k})$ in the following manner. Choose a Kodaira surface S with universal cover $\tilde{S} \cong \mathscr{B}$ so that $S = \mathscr{B}/\mathbb{G}$. Then S^σ will be another Kodaira surface and we set

$$S^\sigma = \mathscr{B}^\sigma/\mathbb{G}^\sigma$$

that is, $\mathscr{B}^\sigma$ is going to denote the universal cover of S^σ and $\mathbb{G}^\sigma$ its uniformising group.

Proposition 10. *The isomorphy class of $\mathscr{B}^\sigma$ is independent of the choice of S.*

Proof. Let $S_1 = \mathscr{B}/\mathbb{G}_1$ and $S_2 = \mathscr{B}/\mathbb{G}_2$ be two Kodaira surfaces with the same universal cover $\mathscr{B}$, then $S_{12} = \mathscr{B}/\mathbb{G}_1 \cap \mathbb{G}_2$ is a smooth cover of both S_1 and S_2 and consequently the universal covers of S_1^σ and S_2^σ agree (with the universal cover of S_{12}^σ) □

Corollary 11. *For any algebraically closed subfield $\overline{k} \subset \mathbb{C}$, the rule*

$$\begin{array}{ccccc} Gal(\overline{k}) & \times & \mathscr{U}_{BG}(\overline{k}) & \to & \mathscr{U}_{BG}(\overline{k}) \\ (\sigma & , & \mathscr{B}) & \to & \mathscr{B}^\sigma \end{array}$$

defines an action of $Gal(\overline{k})$ on $\mathscr{U}_{BG}(\overline{k})$.

In order to state our next result we need to introduce two pieces of notation. One is $\mathrm{Gal}(\overline{k})_{\mathscr{B}}$, the stabiliser of $\mathrm{Gal}(\overline{k})$ at a given point $\mathscr{B} \in \mathscr{U}_{BG}(\overline{k})$. The second one is $\mathscr{N}(\mathscr{B})$ (resp. $\mathscr{N}_n(\mathscr{B})$) which will stand for the set of normal subgroups (resp. index n normal subgroups) of $\mathrm{Aut}(\mathscr{B})$ that uniformise Kodaira surfaces.

Proposition 12. *For any Bers-Griffiths domain $\mathscr{B} \in \mathscr{U}_{BG}(\overline{k})$ the rule*

$$\begin{array}{ccccc} Gal(\overline{k})_{\mathscr{B}} & \times & \mathscr{N}(\mathscr{B}) & \longrightarrow & \mathscr{N}(\mathscr{B}) \\ (\sigma & , & \mathbb{G}) & \longrightarrow & \mathbb{G}^\sigma \end{array}$$

induces an action of $Gal(\overline{k})_{\mathscr{B}}$ on the set $\mathscr{N}(\mathscr{B})$ that permutes the elements of each subset $\mathscr{N}_n(\mathscr{B})$. Moreover, this action is inclusion and intersection preserving.

Proof. Let S be a Kodaira surface with universal cover $\tilde{S} = \mathscr{B}$ and uniformising group $\mathbb{G} \triangleleft \mathrm{Aut}(\mathscr{B})$ and let σ be an element of $\mathrm{Gal}(\overline{k})_{\mathscr{B}}$. Then, as $e(S^\sigma) = e(S)$, Lemma 2 shows that $[\mathrm{Aut}(\mathscr{B}) : \mathbb{G}] = [\mathrm{Aut}(\mathscr{B}) : \mathbb{G}^\sigma]$. Therefore we can write

$$|\mathrm{Aut}(S^\sigma)| = |\mathrm{Aut}(S)| = [\mathrm{Aut}(\mathscr{B}) : \mathbb{G}] = [\mathrm{Aut}(\mathscr{B}) : \mathbb{G}^\sigma]$$

By the second part of Lemma 2 we conclude that $\mathbb{G}^\sigma$ is a uniquely defined normal subgroup of $\mathrm{Aut}(\mathscr{B})$ and our first claim follows.

Now, let $\mathbb{G}_1, \mathbb{G}_2 \in \mathscr{N}(\mathscr{B})$ be such that $\mathbb{G}_1 < \mathbb{G}_2$. Then, setting $S_i = \mathscr{B}/\mathbb{G}_i$, this inclusion induces an obvious smooth covering map $S_1 \to S_2$. Transforming this covering by σ we get a new smooth covering of Kodaira surfaces $S_1^\sigma \to S_2^\sigma$. Therefore the uniformising group $\mathbb{G}_1^\sigma$ of S_1^σ is a subgroup of the uniformising group $\mathbb{G}_2^\sigma$ of S_2^σ, as claimed.

It remains to see that this action preserves intersections. Let $\mathbb{G}_1, \mathbb{G}_2 \in \mathscr{N}(\mathscr{B})$ and set $S_i = \mathscr{B}/\mathbb{G}_i$ and $S_{12} = \mathbb{G}_1 \cap \mathbb{G}_2$ as in Lemma 2. We have just shown that our action preserves inclusions, hence

$$(\mathbb{G}_1 \cap \mathbb{G}_2)^\sigma \leq \mathbb{G}_1^\sigma \cap \mathbb{G}_2^\sigma \tag{5.1}$$

Transforming this inclusion by σ^{-1} we obtain $\mathbb{G}_1 \cap \mathbb{G}_2 \leq (\mathbb{G}_1^\sigma \cap \mathbb{G}_2^\sigma)^{\sigma^{-1}}$. Now, using the relation (5.1) with σ^{-1} instead of σ and $\mathbb{G}_i^\sigma$ instead of $\mathbb{G}_i$ we can further write

$$\mathbb{G}_1 \cap \mathbb{G}_2 \leq (\mathbb{G}_1^\sigma \cap \mathbb{G}_2^\sigma)^{\sigma^{-1}} \leq (\mathbb{G}_1^\sigma)^{\sigma^{-1}} \cap (\mathbb{G}_2^\sigma)^{\sigma^{-1}} = \mathbb{G}_1 \cap \mathbb{G}_2$$

From here we infer that $\mathbb{G}_1 \cap \mathbb{G}_2 = (\mathbb{G}_1^\sigma \cap \mathbb{G}_2^\sigma)^{\sigma^{-1}}$ or equivalently that $(\mathbb{G}_1 \cap \mathbb{G}_2)^\sigma = \mathbb{G}_1^\sigma \cap \mathbb{G}_2^\sigma$, as claimed. □

5.2. **Faithfulness.** For any $n \in \mathbb{N}$ we shall consider the group

$$\mathbb{G}_n^{\mathscr{B}} = \bigcap_{\mathbb{G} \in \mathscr{N}_n(\mathscr{B})} \mathbb{G} \tag{5.2}$$

More specifically, for any $\mu = (\mu_1, \cdots, \mu_6) \in \mathbb{C}^6 \setminus \Delta$, we shall write

$$\mathbb{G}^\mu = \mathbb{G}_n^{\mathscr{B}}, \text{ where } \mathscr{B} = \mathscr{B}_\mu \text{ and } n = M^M \text{ are as in Proposition 9.}$$

These are finite index normal subgroups of $\mathrm{Aut}(\mathscr{B})$ (see Corollary 4, part 2). We shall consider the Kodaira surfaces

$$S_n^{\mathscr{B}} = \mathscr{B}/\mathbb{G}_n^{\mathscr{B}} \qquad \text{and, more specifically,} \quad S^\mu = \mathscr{B}_\mu/\mathbb{G}^\mu \tag{5.3}$$

Clearly, $\mathbb{G}^\mu = \mathbb{G}^\nu$, and therefore $S^\mu = S^\nu$, whenever $\mathscr{B}_\mu = \mathscr{B}_\nu$.

Proposition 13. (1) *For any Bers-Griffiths domain $\mathscr{B} \in \mathscr{U}_{BG}(\overline{k})$ and any $\sigma \in Gal(\overline{k})_{\mathscr{B}}$ one has $(S_n^{\mathscr{B}})^\sigma \cong S_n^{\mathscr{B}}$.*

(2) *For any μ and any algebraically closed field $\overline{k} \subset \mathbb{C}$ containing $\mathbb{Q}(\mu)$, the surface S^μ is invariant under the action of $Gal(\overline{k})_{\mathscr{B}_\mu}$. Moreover, S^μ is a smooth cover of S_μ.*

Proof. (1) By definition $S_n^{\mathscr{B}}$ and $(S_n^{\mathscr{B}})^\sigma$ are uniformised by the groups $\mathbb{G}_n^{\mathscr{B}} = \bigcap_{\mathbb{G} \in \mathscr{N}_n(\mathscr{B})} \mathbb{G}$ and $(\mathbb{G}_n^{\mathscr{B}})^\sigma = \left(\bigcap_{\mathbb{G} \in \mathscr{N}_n(\mathscr{B})} \mathbb{G}\right)^\sigma$. But, by Proposition 12, these two groups are the same.

(2) From Proposition 9, part (4), we infer that S_μ can be defined over such field $\overline{k}$ or, in other words, that $\mathscr{B}_\mu \in \mathscr{U}_{BG}(\overline{k})$. The fact that S^μ is a smooth cover of S_μ is clear by construction. □

Proposition 14. *There is a constant L such that for every $\mu \in \mathbb{C}^6 \setminus \Delta$ the inequality $e(S^\mu) < L$ holds.*

Proof. Clearly $\frac{e(S^\mu)}{e(S_\mu)} = [\mathbb{G}^\mu : \mathbb{G}_\mu]$, hence $e(S^\mu) \leq e(S_\mu)\,[\mathrm{Aut}(\mathscr{B}_\mu) : \mathbb{G}_\mu]$. Now the result follows from parts (1) and (3) of Proposition 9. □

Corollary 15. *There is a constant H such that for every μ the number of Riemann surfaces of genus ≥ 2 which arise as targets of surjective holomorphic maps from S^μ is bounded by H.*

Proof. By a theorem by Howard and Sommese ([23], Theorem 2) the number of surjective holomorphic maps whose domain is a complex projective non-singular surface S with ample canonical bundle (such as Kodaira surfaces, see e.g. [18], 3.1) and whose image is a Riemann surface of genus ≥ 2 is bounded by a finite number depending only on the Chern numbers $e(S)$ and K_S^2.

Now, the Bogomolov-Miyaoka-Yau inequality mentioned already in the proof of Theorem 3 tells us that that $K_{S^\mu}^2 \leq 3e(S^\mu)$. Since, by Proposition 14, the Euler numbers $e(S^\mu)$ are simultaneously bounded there are only finitely many possibilities for the integers $e(S^\mu)$ and $K_{S^\mu}^2$. The result follows. $\square$

The following two lemmas will be useful (cf. Proposition 2.1 in [3] and Theorem 1 in [12])

Lemma 16. *For any $a \in \mathbb{C} \setminus \mathbb{Q}$ let us denote by D_a the genus 2 curve*

$$D_a : y^2 = (x-1)(x-2)(x-3)(x-4)(x-5)(x-a) \tag{5.4}$$

Let b another non-rational complex number. Then D_b is isomorphic to D_a if and only if $b = a$.

Proof. We have two show that if $a \neq b$ there is no Möbius transformation $M \in \mathbb{P}\mathrm{SL}_2(\mathbb{C})$ such that $M(\{1,2,3,4,5,a\}) = \{1,2,3,4,5,b\}$.

Such an M would have to satisfy the following conditions:

(1) $M \in \mathbb{P}\mathrm{SL}_2(\mathbb{Q})$, since it must map three rational points to three rational points.
(2) $M(\{1,2,3,4,5\}) = \{1,2,3,4,5\}$ and $M(a) = b$ since $a, b \notin \mathbb{Q}$.
(3) M induces an orientation preserving diffeomorphism of the circle $\mathbb{P}^1(\mathbb{R})$, since $M'(x) > 0$, for every $x \in \mathbb{R}$.
(4) Therefore, $(M(1), M(2), M(3), M(4), M(5))$ must be a positively ordered cycle in $\mathbb{P}^1(\mathbb{R})$

Let $M = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$, then for $i = 1, \cdots, 5$, one has the identities

$$si + t = M(i)ui + M(i)v \quad \text{with} \quad M(i) \in \{1,2,3,4,5\}$$

By (4) there must be some i such that $M(i), M(i+1)$ and $M(i+2)$ are consecutive integers, that is $M(i+2) = M(i+1)+1 = M(i)+2$. Now subtracting $s(i+1)+t$ from $si+t$ and $s(i+2)+t$ from $s(i+1)+t$ one finds that $u = 0$ and $s = v$, hence M is of the form $M = \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ which contradicts (2) unless $M = \mathrm{Id}$. $\square$

Lemma 17. *Let N be a given positive integer and let $\sigma \in \mathrm{Gal}(\overline{k}/\mathbb{Q})$ be an element of infinite order. Then there is an element $a \in \overline{k}$ such that the genus 2 algebraic curve D_a defined by the equation (5.4) enjoys the* *property that among all the Galois conjugate curves* $D_a^{\sigma^n} = D_{\sigma^n(a)}, n \in \mathbb{N}$, *there are al least* N *mutually non-isomorphic curves.*

Proof. Choose $a \in \overline{k}$ such that $\sigma^{N!}(a) \neq a$. Then the elements

$$a, \sigma(a), \sigma^2(a), \cdots, \sigma^N(a)$$

are mutually distinct, for otherwise we would have $\sigma^{N!}(a) = a$. Now the result follows from Lemma 16. $\square$

Theorem 18. $\mathrm{Gal}(\overline{k}/\mathbb{Q})$ *acts faithfully on* $\mathscr{U}_{BG}(\overline{k})$.

Proof. 1) Let first $\sigma \in \mathrm{Gal}(\overline{k}/\mathbb{Q})$ have infinite order.

Let N equal the constant H occurring in Corollary 15 and choose $a \in \overline{k}$ satisfying Lemma 17. Set

$$\mu_a = (1, 2, 3, 4, 5, a).$$

Since D_a is defined over $\overline{k}$, Proposition 9, part (4), implies that the universal cover $\mathscr{B}_{\mu_a}$ of the corresponding Kodaira fibration $f_a : S_{\mu_a} \to B_{\mu_a}$ lies in $\mathscr{U}_{BG}(\overline{k})$. We claim that $\mathscr{B}_{\mu_a}^{\sigma}$ is not isomorphic to $\mathscr{B}_{\mu_a}$.

Suppose that $\mathscr{B}_{\mu_a}^{\sigma} \cong \mathscr{B}_{\mu_a}$. Let $\pi_a : S^{\mu_a} \to S_{\mu_a}$ be a smooth covering map (Proposition 13). Let us denote by $p_a : B_{\mu_a} \to D_a$ the composition of the unramified covers $B_{\mu_a} \to C_{\mu_a}$ and $C_{\mu_a} \to D_a$ (see Section 4) and let us set $F = p_a \circ f_a \circ \pi_a : S^{\mu_a} \to D_{\mu_a}$. Since $(S^{\mu_a})^{\sigma} \cong S^{\mu_a}$ (Proposition 13), transforming this morphism by the powers of σ yields a collection of morphisms $F^{\sigma^n} : S^{\mu_a} \to D_{\sigma^n(a)}$ such that the number of non-isomorphic targets exceeds $N = H$. This contradicts Corollary 15.

2) Now let $\sigma \neq Id.$ have finite order.

In this case we use the following argument taken from [17]. By the Artin-Schreier theorem (see e.g. [26]) all non-trivial finite subgroups of $\mathrm{Gal}(\overline{k})$ have order 2. In particular σ is an involution. Therefore for any $\omega \in \overline{k}$ not fixed by σ this automorphism restricts to a non-trivial involution of the algebraically closed field $\overline{\mathbb{Q}(\omega, \sigma(\omega))} \subset \overline{k}$. Moreover, since this is an algebraically closed extension of finite transcendental degree the centralizer of σ in $\mathrm{Gal}(\overline{\mathbb{Q}(\omega, \sigma(\omega))}/\mathbb{Q})$ is a countable group. (See [6], VI, §2, Exercise 32). Therefore there are plenty of non-trivial elements $\beta \in \mathrm{Gal}(\overline{\mathbb{Q}(\omega, \sigma(\omega))}/\mathbb{Q})$ such that $\beta\sigma\beta^{-1} \neq \sigma$. Moreover, by extending β to $\overline{k}$ we see that the same statement holds in $\mathrm{Gal}(\overline{k}/\mathbb{Q})$.

Now if σ acted trivially on $\mathscr{U}_{BG}(\overline{k})$ then so would do its conjugate $\beta\sigma\beta^{-1}$ and the product of them $\tau := (\beta\sigma\beta^{-1})\sigma$. Then, by the first part, τ must be an element of finite order, hence of order 2. It would then follow that the subgroup $\langle \beta\sigma\beta^{-1}, \sigma \rangle$ is the Klein 4-group, thereby contradicting Artin-Schreier's theorem. This contradiction brings the proof to an end. $\square$

The most interesting instance occurs when $k = \overline{\mathbb{Q}}$, so we state this case separately.

Theorem 19. $\mathrm{Gal}(\overline{\mathbb{Q}})$ *acts faithfully on* $\mathscr{U}_{BG}(\overline{\mathbb{Q}})$.

We end the paper with the following observation.

Corollary 20. *The converse of Theorem* (*G-R,* [18]) *stated in Section 1 does not hold.*

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