THE ARITHMETICITY OF A KODAIRA FIBRATION IS DETERMINED BY ITS UNIVERSAL COVER

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Abstract. Let $S \to C$ be a Kodaira fibration. We show that whether or not the algebraic surface S is defined over a number field depends only on the biholomorphic class of its universal cover.

1. Introduction and statement of results

Let $X \subset \mathbb{P}^n$ be a complex projective variety and k a subfield of the field of the complex numbers $\mathbb{C}$. We shall say that X is defined over k or that k is a *field of definition* for X if there exists a collection of homogenous polynomials $f_0, \ldots, f_m$ with coefficients in k so that the variety they define is isomorphic to X. We will say that X is *arithmetic* if it is defined over $\overline{\mathbb{Q}}$ or equivalently over a number field.

While it is classically known that there are only three simply connected Riemann surfaces, there is a huge amount of possibilities for the holomorphic universal cover of a complex surface S. It would be interesting to understand the extent to which the arithmeticity of a projective surface can be read off from its holomorphic universal cover. In this short note we study this question for a very important class of complex surfaces known in the literature as Kodaira fibrations.

A *Kodaira fibration* consists of a non-singular compact complex surface S, a compact Riemann surface C and a surjective holomorphic map $S \to C$ everywhere of maximal rank such that the fibers are connected and not mutually isomorphic Riemann surfaces. The genera g of the fibre and b of C are called the genus of the fibration and of the base respectively. It is known that such a surface S must be an algebraic surface of general type and that necessarily $g \geq 3$ and $b \geq 2$. We notice that an important theorem by Arakelov [1] implies that, up to isomorphism, there are only finitely many Kodaira fibrations over a given algebraic curve C.

In 1967, Kodaira [13] used fibrations of this kind to show that the signature of a differentiable fiber bundle need not be multiplicative. Soon after Kas [12] studied the deformation space of the surfaces constructed by Kodaira, and two years later Atiyah [2] and Hirzebruch [10] studied further

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properties concerning the signature of Kodaira fibrations in a volume dedicated to Kodaira himself.

Explicit constructions of Kodaira surfaces have been made by González-Diez and Harvey [8], Bryan and Donagi [4], Zaal [16] and Catanese and Rollenske [6].

We now state the main results of the paper

Theorem 1. *Let k be an algebraically closed subfield of the complex numbers and $S_1 \to C_1$ and $S_2 \to C_2$ two Kodaira fibrations so that their respective holomorphic universal covers are biholomorphically equivalent. Then S_1 is defined over k if and only if S_2 is defined over k. In particular, S_1 is arithmetic if and only if S_2 is arithmetic.*

To prove this theorem we will have to show first the following result which is interesting in its own right

Theorem 2. *Let k be an algebraically closed subfield of the complex numbers and $S \to C$ a Kodaira fibration. Then S is defined over k if and only if C is defined over k. In particular, S is an arithmetic surface if and only if C is an arithmetic curve.*

Theorem 1 states that the arithmeticity of a Kodaira fibration can be recognized in its holomorphic universal cover. We anticipate that the holomorphic universal cover of S is a contractible bounded domain $\mathscr{B} \subset \mathbb{C}^2$ (see Section 2). Clearly, Theorem 1 implies that the biholomorphism class of $\mathscr{B}$ varies together with the variation of S in moduli space. We note that in general Kodaira surfaces are not rigid ([12], [6]).

2. Uniformization of Kodaira Surfaces

It is well-known that the universal cover of a Riemann surface is biholomorphically equivalent to the projective line $\mathbb{P}^1$, the complex plane $\mathbb{C}$ or the upper half-plane $\mathbb{H}$. Understanding universal covers of complex manifolds of higher dimension seems to be a very complicated task. However, thanks to the work of Bers [3] and Griffiths [9] on uniformization of algebraic varieties, it is possible to describe the universal cover of a Kodaira fibration $f : S \to C$ in a very explicit way.

Let $\pi : \mathbb{H} \to C$ be the universal covering map of C and Γ the covering group so that $C \cong \mathbb{H}/\Gamma$. By considering the pull-back $h : \pi^* S \to \mathbb{H}$ of f by π, we obtain a new fibration in which, for each $t \in \mathbb{H}$, the fiber $h^{-1}(t)$ agrees with the Riemann surface $f^{-1}(\pi(t))$. Teichmüller theory enables us to choose uniformizations $h^{-1}(t) = D_t/K_t$ possessing the following properties:

(a) K_t is a Kleinian group acting on a bounded domain D_t of $\mathbb{C}$ which is biholomorphically equivalent to a disk.

(b) The union of all these disks $\mathscr{B} := \cup_{t\in\mathbb{H}} D_t$ is a contractible bounded domain of $\mathbb{C}^2$ which is biholomorphic to the universal cover of S, that is, $S \cong \mathscr{B}/\mathbb{G}$, where $\mathbb{G} < \mathrm{Aut}(\mathscr{B})$ is the covering group.

(c) The group $\mathbb{G}$ is endowed with a surjective homomorphism of groups $\Theta : \mathbb{G} \to \Gamma$ which induces an exact sequence of groups

$$1 \longrightarrow \mathbb{K} \longrightarrow \mathbb{G} \xrightarrow{\Theta} \Gamma \longrightarrow 1$$

where, for each $t \in \mathbb{H}$, the subgroup $\mathbb{K}$ preserves D_t and acts on it as the Kleinian group K_t.

We note that $\mathscr{B}$ carries itself a fibration structure $\mathscr{B} \to \mathbb{H}$ whose fiber over $t \in \mathbb{H}$ is D_t (i.e. $\mathscr{B}$ is a *Bergman domain* in Bers' terminology).

In [14] and [15] Shabat studied the automorphism groups of universal covers of families of Riemann surfaces and proved a deep result which in the case of Kodaira fibrations amounts to the following theorem.

Theorem (Shabat) Let $f : S \to C$ be a Kodaira fibration and $\mathscr{B}$ the holomorphic universal cover of S. Then:

(a) the covering group $\mathbb{G}$ of S has finite index in $\mathrm{Aut}(\mathscr{B})$.
(b) $\mathrm{Aut}(\mathscr{B})$ is a discrete group.

3. Proof of Theorems 1 and 2

We denote by $\mathrm{Gal}(\mathbb{C})$ the group of field automorphisms of $\mathbb{C}$. The natural action of $\mathrm{Gal}(\mathbb{C})$ on the ring of polynomials $\mathbb{C}[x_0, \ldots, x_n]$ induces a well-defined action $(\sigma, X) \to X^\sigma$ on the set of isomorphism classes of algebraic varieties. From now on k will denote an algebraically closed subfield of $\mathbb{C}$ and $\mathrm{Gal}(\mathbb{C}/k)$ the subgroup of $\mathrm{Gal}(\mathbb{C})$ consisting of all automorphisms σ fixing the elements of k.

3.1. **Proof of Theorem 2.** Let $f : S \to C$ be a Kodaira fibration. Let us assume that the curve C is defined over k. Then $C^\sigma = C$ for all $\sigma \in \mathrm{Gal}(\mathbb{C}/k)$, and so, by Arakelov's finiteness Theorem, there are only finitely many pairwise non-isomorphic Kodaira fibrations $f^\sigma : S^\sigma \to C^\sigma$ with $\sigma \in \mathrm{Gal}(\mathbb{C}/k)$. This implies that S is defined over k [7, Crit. 2.1].

In order to prove the converse, we begin by recalling that a complex manifold X is named *hyperbolic* if every holomorphic map $\mathbb{C} \to X$ is a constant map. We claim that Kodaira fibrations are hyperbolic. In fact, let $f : S \to C$ be a Kodaira fibration and $\varphi : \mathbb{C} \to S$ a non-constant holomorphic map. As C has genus greater than one, the map $f \circ \varphi : \mathbb{C} \to C$ must be constant and therefore the image of φ has to be contained in a fiber $f^{-1}(t)$ for some $t \in C$. Since the fibers are also hyperbolic, φ must be constant too.

Let us now assume that S is defined over k, so that $S^\sigma = S$ for any $\sigma \in \mathrm{Gal}(\mathbb{C}/k)$. Now as S is a Kähler hyperbolic manifold, the canonical divisor K_S is ample [5] and this implies that only finitely many curves R of genus greater than one can arise as the image of a surjective morphism $S \to R$ [11]. In particular the family $\{C^\sigma : \sigma \in \mathrm{Gal}(\mathbb{C}/k)\}$ itself contains only finitely many isomorphism classes of curves. It then follows that C is defined over k [7, Crit. 2.1], as required.

3.2. **Proof of Theorem 1.** Let $f_2 : S_2 \to C_2$ be a Kodaira fibration and S_1 an arbitrary non-singular complex surface. Let us denote by $\mathscr{B}_i$ the universal cover of S_i and suppose that there exists an isomorphism $\alpha : \mathscr{B}_1 \to \mathscr{B}_2$ between them. Let $\mathbb{G}_i$ be the uniformizing group of S_i so that $\mathscr{B}_i/\mathbb{G}_i \cong S_i$. By Shabat's Theorem $\mathbb{G}_2$ has finite index in $\mathrm{Aut}(\mathscr{B}_2)$. We claim that $\mathbb{G}_1$ has finite index in $\mathrm{Aut}(\mathscr{B}_1)$ too. In fact, as $\mathscr{B}_1/\mathrm{Aut}(\mathscr{B}_1) \cong \mathscr{B}_2/\mathrm{Aut}(\mathscr{B}_2)$ and as $\mathrm{Aut}(\mathscr{B}_2)$ is a discrete group, the projection map $S_1 = \mathscr{B}_1/\mathbb{G}_1 \to \mathscr{B}_1/\mathrm{Aut}(\mathscr{B}_1)$ is a holomorphic map between (normal) compact complex surfaces; from here the claim follows.

By replacing $\mathbb{G}_1$ by $\alpha\mathbb{G}_1\alpha^{-1}$ we can assume that $\mathscr{B}_1 = \mathscr{B}_2$, so we denote $\mathscr{B}_i$ simply by $\mathscr{B}$. As both $\mathbb{G}_1$ and $\mathbb{G}_2$ have finite index in $\mathrm{Aut}(\mathscr{B})$, so must do their intersection $\mathbb{G}_{12} = \mathbb{G}_1 \cap \mathbb{G}_2$. The complex surface $S_{12} := \mathscr{B}/\mathbb{G}_{12}$ is endowed with two finite degree covers $\pi_i' : S_{12} \to S_i$ with $i = 1, 2$; in particular, S_{12} is also a projective surface. Moreover, if we denote by Θ_{12} the restriction of the epimorphism $\Theta : \mathbb{G}_2 \to \Gamma_2$ introduced in the previous section to $\mathbb{G}_{12}$, then we obtain an exact sequence of groups

$$1 \longrightarrow \mathbb{K}_{12} \longrightarrow \mathbb{G}_{12} \xrightarrow{\Theta_{12}} \Gamma_{12} \longrightarrow 1$$

where $\Gamma_{12} = \Theta_{12}(\mathbb{G}_{12})$ and $\mathbb{K}_{12} = \ker(\Theta_{12}) = \mathbb{K} \cap \mathbb{G}_{12}$. As in Section 2, this sequence defines a Kodaira fibration $f_{12} : S_{12} \to C_{12} := \mathbb{H}/\Gamma_{12}$ whose fiber over $[t]_{\Gamma_{12}}$ is isomorphic to the Riemann surface D_t/K_t^{12} where K_t^{12} is the Kleinian group that realizes the action of $\mathbb{K}_{12}$ on D_t. We have the following commutative diagram

$$\begin{tikzcd} & \mathscr{B} \arrow[dl] \arrow[dr] & \\ S_{12} \arrow[rr, "\pi_2'"] \arrow[d, "f_{12}"'] & & S_2 \arrow[d, "f"] \\ C_{12} \arrow[rr, "p"] & & C_2 \end{tikzcd}$$

where p is the projection induced by the finite index inclusion $\Gamma_{12} < \Gamma_2$.

Let us now assume that S_2 is defined over k. Then Theorem 2 ensures that C_2 is also defined over k. Furthermore, being an unbranched cover of C_2, the curve C_{12} must also be defined over k [7, Th. 4.1]. Again, by Theorem 2 we conclude that S_{12} is defined over k. Now, as S_1 is a surface of general type arising as the image (by π_1') of a surface defined over k, it must be defined over k as well [7, Prop. 3.2]. This proves Theorem 1.

References

[1] S. Arakelov, *Families of curves with fixed degenerancies*, Math. USRR, Izvestija, **5**, 1277-1302, (1971).

[2] M. Atiyah, *The signature of fiber-bundles*, Global Analysis (Papers in Honor of K. Kodaira), University of Tokyo Press, (1969), 73-84.

[3] L. Bers, *Uniformization, moduli and kleinian groups*, London Math. Soc., **4,** (1972), 257-300.

[4] J. Bryan and R. Donagi, *Surface bundles over surfaces of small genus*, Geom. Topol., **6,** (2002), 59-67.

[5] F. Campana, *Twistor spaces and non-hyperbolicity of certain symplectic Kähler manifolds,* Complex analysis, Aspects of Mathematics, **1**, (1991), 64-69.

[6] F. Catanese and S. Rollenske, *Double Kodaira Fibrations*, J. Reine Angew. Math. **628** (2009), 205-233.

[7] G. González-Diez, *Variations on Belyi's Theorem*, Quart. J. Math. **57**, (2006), 339-354.

[8] G. González-Diez and W. Harvey, *On complete curves in moduli space I,* Math. Proc. Cambridge Philos. Soc., **110** (1991), no.3, 461-466.

[9] P. A. Griffiths, *Complex analytic properties of certain Zariski open sets on algebraic varieties,* Ann. of Maths., **94** (1971), 21-51.

[10] F. Hirzebruch, *The signature of ramified coverings*, Global Analysis (Papers in Honor of K. Kodaira), University of Tokyo Press, (1969), 253-265.

[11] A. Howard and A. J. Sommese, *On the theorem of de Franchis,* Annali della Scuola Normale Superiore di Pisa, 4 serie, tome 10, **3**, (1983), 429-436.

[12] A. Kas, *On deformations of a certain type of irregular algebraic surfaces,* J. Analyse Math., **19** (1967), 207-215.

[13] K. Kodaira, *A certain type of irregular algebraic surfaces,* Amer. J. Math., **90** (1968), 789-804.

[14] G. Shabat, *The complex structure of domains covering algebraic surfaces,* Funct. Anal. Appl. **11**, (1977), 135-142.

[15] G. Shabat, *Local reconstruction of complex algebraic surfaces from universal coverings,* Funktsional. Anal. i Prilozhen. **17** (1983), no.2, 90-91.

[16] C. Zaal, *Explicit complete curves in the moduli space of curves of genus three,* Geom. Dedicata, **56**, (1995), no.2, 185-196.

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