On complex curves and complex surfaces defined over number fields

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Abstract

Belyi's theorem states that a compact Riemann surface C can be defined over a number field if and only if there is on it a meromorphic function with three critical values. Such functions (resp. Riemann surfaces) are called *Belyi functions* (resp. *Belyi surfaces*). Alternatively Belyi surfaces can be characterized as those which contain a proper Zariski open subset uniformised by a *torsion free* subgroup of the classical modular group $\mathbb{P}\mathrm{SL}_2(\mathbb{Z})$. In the first part of this survey article we discuss this result and the companion theory of Grothendieck's *dessins d'enfants*. In the second part we establish a result analogous to Belyi's theorem in complex dimension two. It turns out that the role of Belyi functions is now played by (*composed*) *Lefschetz pencils* with three critical values while the analogous to torsion free subgroups of the modular group will be certain extensions of them acting on a *Bergman domain* of $\mathbb{C}^2$. These groups were first introduced by Bers and Griffiths.

1 Introduction

This article is an extended version of the talk given by the second author at the International Workshop on *Teichmüller Theory and Moduli Problems* held at the Harish-Chandra Research Institute (HRI), Allahabad (India), on January 2006. It is intended to be of expository nature and for this reason proofs are merely outlined or explained via examples.

The paper is divided in two parts. The first part starts with Belyi's theorem, which states that a compact Riemann surface (= complex algebraic curve) C can be defined over a number field if and only if there is a meromorphic function $f : C \to \mathbb{P}^1$, called a *Belyi function*, with only three critical values, say $0, 1, \infty$ ([4]). From the uniformization point of view, this is readily seen to be equivalent to saying that C contains a finite set Σ such that $C \setminus \Sigma$ can be uniformized by a torsion free subgroup of the classical modular group $\mathbb{P}\mathrm{SL}_2(\mathbb{Z})$.

Belyi's theorem has attracted much attention ever since Grothendieck noticed in his *Esquisse d'un Programme* ([14]) that it implies amazing interrela-

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tions between algebraic curves defined over number fields and a certain class of graphs embedded in a topological surface, which he named *dessins d'enfants*.

The proof of the "only if" part of this theorem results from a surprisingly simple construction of Belyi (in Grothendieck's words "*jamais sans doute un résultat profond et déroutant ne fut démontré en si peu de lignes!*", [14]). The construction is illustrated in an explicit example in Section 4.

The "if" part can be deduced from an old criterion of rationality due to Weil ([30]). The proof we present here is based on a criterion of our own which, although less powerful than Weil's, we find easier to handle. And, in fact, it will be used in a different setting in the second part of the article.

The absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}})$, i.e. the group of field automorphisms of the algebraic closure of $\mathbb{Q}$, is a basic object of interest in algebraic number theory. The astonishing fact about Belyi's theorem is that it permits to define an action of $\mathrm{Gal}(\overline{\mathbb{Q}})$ on these merely topological, or combinatorial, objects called dessins. The hope to gain understanding of this still mysterious group via this action seems to have been the main reason that led Grothendieck to the introducion of the theory of dessins. In the first part of the paper this action is described in detail in several explicit examples.

In the second part the goal is to establish a result analogous to Belyi's theorem in dimension 2, that is for complex surfaces. It will turn out that in this case the role of Belyi functions is going to be played by (composed) Lefschetz pencils with three critical values (*Lefschetz functions*). As for the point of view of uniformization the analogous result goes as follows. It is known that any complex projective surface S possesses a closed subvariety Σ such that the Zariski open set $S \smallsetminus \Sigma$ can be uniformized in the form $S \smallsetminus \Sigma = \mathbb{B}/G$ where $\mathbb{B}$ is a Bergman domain in $\mathbb{C}^2$ and G is a group of biholomorphic transformations of $\mathbb{B}$ which, roughly speaking, is an extension of a Fuchsian group Γ of type $(0, r)$ by a Kleinian group K (Bers-Griffiths Uniformization, [5], [13]). Now, except for an exceptional case, the complex surfaces that can be defined over a number field are those for which Γ can be chosen to be a torsion free subgroup of $\mathbb{P}\mathrm{SL}_2(\mathbb{Z})$.

The proof we outline here uses, mainly, tools that belong to Teichmüller theory.

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Part I: Complex Curves

2 Riemann surfaces defined over $\overline{\mathbb{Q}}$

Probably, the feature that makes the theory of (compact) Riemann Surfaces so intensively atractive is the equivalence between the following three classes of objects:

$$\begin{tikzcd}
\{\text{Compact Riemann Surfaces}\} \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\{\text{Algebraic Curves}\} \arrow[r, leftrightarrow] & \{\text{Fuchsian Groups}\}
\end{tikzcd}$$

According to this equivalence a given Riemann surface C may be described as $C_F = \{F(x,y) = 0\}$ for an irreducible polynomial $F \in \mathbb{C}[x,y]$, but also as $C \simeq \mathbb{H}/\Gamma$ where $\mathbb{H}$ is the upper half plane and $\Gamma \subset \mathbb{P}\mathrm{SL}_2(\mathbb{R})$ is a Fuchsian group.

A natural problem from the point of view of arithmetic is to determine which Riemann surfaces are definable over the field of algebraic numbers $\overline{\mathbb{Q}}$, i.e. which Riemann surfaces correspond to curves C_F with $F \in \overline{\mathbb{Q}}[x,y]$.

One difficulty comes from the fact that different algebraic curves may correspond to the same Riemann surface. The curve $y^2 = x^3 - \pi$, for example, is not directly defined over $\overline{\mathbb{Q}}$, but it is isomorphic to $\{y^2 = x^3 - 1\}$ via the isomorphism

$$\begin{array}{rcl} \{y^2 = x^3 - \pi\} & \longrightarrow & \{y^2 = x^3 - 1\} \\ (x,y) & \longmapsto & (x/\sqrt[3]{\pi}, y/\sqrt{\pi}) \end{array}$$

Recall that the moduli space $\mathcal{M}_g$, whose points are in one to one correspondence with isomorphy classes of compact Riemann surfaces of genus g, can be obtained as a quotient

$$\mathcal{M}_g = \frac{\mathcal{T}_g}{\mathrm{Mod}_g}$$

where $\mathcal{T}_g$ is the Teichmüller space of genus g, of complex dimension $(3g-3)$, and Mod_g is the mapping class group. The mapping class group is known to act properly and discontinuously on $\mathcal{T}_g$ as a group of biholomorphic transformations. Therefore $\mathcal{M}_g$ is an analytic space of complex dimension $3g-3$ (see [24]).

Now the subset representing Riemann surfaces definable over $\overline{\mathbb{Q}}$ is clearly countable. Hence most of the Riemann surfaces are not arithmetic.

In the following result we summarize several characterizations of the arithmeticity of a Riemann surface.

Theorem 1 *The following conditions are equivalent:*

1. *C is defined over $\overline{\mathbb{Q}}$.*
2. *There exists a covering $f: C \to \widehat{\mathbb{C}}$ ramified over $\{0, 1, \infty\}$, where $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$ is the Riemann sphere.*
3. *C can be described as $C = \mathbb{H}/K$, where K is a finite index subgroup of a Fuchsian triangle group.*
4. *C is isomorphic to the compactification of $\mathbb{H}/\Gamma$, where Γ is a finite index subgroup of the modular group $\mathbb{P}\mathrm{SL}_2(\mathbb{Z})$.*

The equivalence between 1 and 2 is nowadays known as *Belyi's theorem*, since Belyi provided an algorithm showing $1 \Rightarrow 2$ (the implication $2 \Rightarrow 1$ was already known by that time). This is the reason why Riemann surfaces defined over $\overline{\mathbb{Q}}$ or, equivalently, over a number field, are often called *Belyi surfaces*. Similarly, the corresponding coverings $f : C \to \widehat{\mathbb{C}}$ are usually referred to as *Belyi functions*.

Part of the importance of Belyi surfaces comes from the fact that $\mathrm{Gal}(\overline{\mathbb{Q}})$, the *absolute Galois group*, acts on the set of *Belyi pairs*. A Belyi pair (C, f) consists of a Belyi surface with a Belyi function defined on it. Two such pairs (C_1, f_1) and (C_2, f_2) are considered equivalent when they are so as ramified coverings, that is, when there exists an isomorphism of Riemann surfaces $F : C_1 \to C_2$ such that the diagram

$$\begin{tikzcd} C_1 \arrow[r, "F"] \arrow[d, "f_1"] & C_2 \arrow[dl, "f_2"] \\ \widehat{\mathbb{C}} & \end{tikzcd}$$

commutes.

2.1 The action of $\mathrm{Gal}(\overline{\mathbb{Q}})$ on Belyi pairs

It is well known that in the dictionary

$$\begin{array}{rcl} \{\text{Compact Riemann Surfaces}\} & \longleftrightarrow & \{\text{Algebraic Curves}\} \\ C & \longleftrightarrow & C_F \end{array}$$

meromorphic functions correspond to rational functions, that is

$$\{\text{Meromorphic Functions } f \text{ in } C\} \quad \longleftrightarrow \quad \{\text{Rational Functions } R \text{ in } C_F\}$$

Therefore a given Belyi pair (C, f) may also be represented as (C_F, R), where C_F is the algebraic curve corresponding to C and R is the rational function corresponding to f. According to Theorem 1 the polynomial $F(x, y)$ can be chosen so as to have coefficients in a number field. Furthermore, the following stronger version of Belyi's theorem holds.

Theorem 2 *Every Belyi pair (C, f) can be represented by a pair (C_F, R) in which both the algebraic curve $F(x, y)$ and the rational function $R(x, y)$ have coefficients in $\overline{\mathbb{Q}}$.*

Now let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}})$. Given a polynomial $P(x, y) = \sum_{i,j} a_{i,j} x^i y^j \in \overline{\mathbb{Q}}[x, y]$, we denote by P^σ the polynomial obtained after applying σ to the coefficients, that is $P^\sigma(x, y) = \sum_{i,j} \sigma(a_{i,j}) x^i y^j$.

This action on the coefficients of polynomials induces an action of $\mathrm{Gal}(\overline{\mathbb{Q}})$ on Belyi pairs. If (C, f) is represented as (C_F, R), then we put

$$\sigma(C, f) = (C_{F^\sigma}, R^\sigma).$$

3 Grothendieck's dessins

Belyi functions can be understood in a surprisingly simple combinatorial way, by means of certain graphs which Grothendieck named dessins d'enfants (child's drawings).

Definition 3 A *dessin d'enfant* is a pair $(X, \mathcal{D})$ where X is an oriented compact topological surface, and $\mathcal{D} \subset X$ is a graph such that:

1. $\mathcal{D}$ is connected.
2. $\mathcal{D}$ is bicolored, i.e. the vertices have been given either white or black color and two vertices connected by an edge have always distinct colors.
3. $X \setminus \mathcal{D}$ is the union of finitely many topological discs, which we call *faces* of $\mathcal{D}$.

The *genus* of $(X, \mathcal{D})$ is simply the genus of the topological surface X.

Two dessins $(X_1, \mathcal{D}_1)$ and $(X_2, \mathcal{D}_2)$ are considered equivalent if there is an orientation-preserving homeomorphism from X_1 to X_2 whose restriction to $\mathcal{D}_1$ induces a homeomorphism of the colored graphs $\mathcal{D}_1$ and $\mathcal{D}_2$.

There is a very convenient way of encoding all the information describing a dessin by means of two permutations of its edges. This is done as follows.

Suppose $\mathcal{D}$ has N edges, and label them with integer numbers from 1 to N. Consider one of the white vertices, and look at the edges incident to it by turning around the vertex in positive sense, according to the orientation of the surface. This procedure gives a cycle, and the same strategy applied to every white vertex produces a permutation $\sigma_0 \in \mathbb{S}_N$, namely the product of all these cycles. The same construction applied to the black vertices defines a second permutation σ_1.

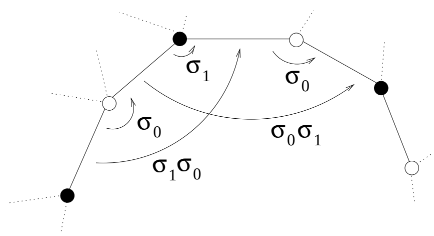

Figure 1: The monodromy of a dessin.

Note that $\sigma_1 \cdot \sigma_0$ sends the edge labeled k to another edge that belongs to one of the two faces meeting at edge k (see Figure 1). In particular, every cycle of $\sigma_1 \cdot \sigma_0$ corresponds to a face of $\mathcal{D}$, the length of the cycle being half the number of edges of the corresponding face (an edge incident at both sides with the same face counts twice). We thus see that the dessin can be reconstructed from the cycle structure of σ_0, σ_1, and $\sigma_1 \cdot \sigma_0$.

Let us describe how σ_0 and σ_1 determine the dessin in a concrete example.

Suppose $\sigma_0 = (1,2)(3,4)(5,6)$ and $\sigma_1 = (4,1,5)(3,2,6)$. The corresponding dessin $\mathcal{D}$ has six edges, three white vertices of degree two (since σ_0 has three cycles of length two), and two black vertices of degree three. Moreover $\sigma_1 \cdot \sigma_0 = (1,6,4,2,5,3)$, hence $\mathcal{D}$ has only one face. As for the genus of the surface where $\mathcal{D}$ is embedded, the Euler-Poincaré formula gives

$$2 - 2g = \text{faces} - \text{edges} + \text{vertices} = 1 - 6 + 5 = 0,$$

thus $\mathcal{D}$ is embedded in a topological torus $\mathbb{T}$ (see Figure 2, where $\mathbb{T}$ is also depicted as a rectangle whose opposite sides are identified).

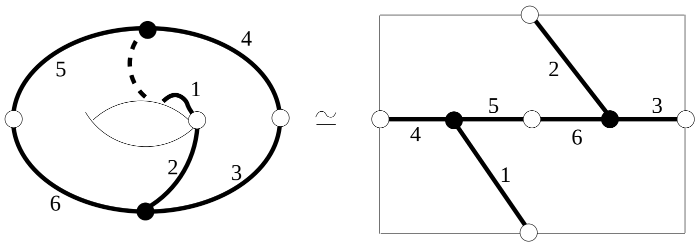

Figure 2: A dessin in a topological torus.

This construction entitles us to speak of the *permutation representation* of $\mathcal{D}$ and to write

$$\mathcal{D} = (\sigma_0, \sigma_1)$$

or, equivalently, $\mathcal{D} = (\sigma_0, \sigma_1, \sigma_\infty)$, with $\sigma_\infty = (\sigma_0\sigma_1)^{-1}$. Conjugate representations will be regarded as equivalent.

The crucial point in the theory of dessins is the fact first noticed by Grothendieck that a dessin defines in a very precise way a Belyi pair. This is a most striking fact. In more precise terms it means that out of the combinatorics of the inclusion of the graph $\mathcal{D}$ in the topological surface X, a mere collection of purely topological data, one is able to endow X with a Riemann surface structure $X_{\mathcal{D}}$ and a meromorphic function $f : X_{\mathcal{D}} \to \mathbb{P}^1$ which, in fact, are defined over a number field.

Later we will show that the correspondence can be reversed, from Belyi pairs to dessins. But let us first describe the construction of the Belyi pair associated to the dessin in our particular example. It will transpire that the construction works for general dessins.

3.1 The Belyi pair associated to a dessin

Mark the center (meaning a point in the interior) of each of the faces (in our example, just one) of $\mathbb{T} \setminus \mathcal{D}$ with the symbol $\times$, and draw a simple path from it to each vertex (in our example, 2 black and 3 white). This way we get a triangulation $\mathcal{T}(\mathcal{D})$, i.e. a subdivision of the faces into triangles whose three vertices are of the three distinct types: $\circ$, $\bullet$ and $\times$. Now colour the triangles white or black, according to whether or not the circuit $\circ \to \bullet \to \times \to \circ$ occurs clockwise (see Figure 3).

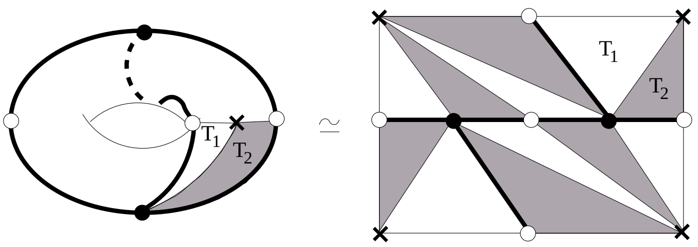

Figure 3: Triangulation associated to a dessin.

Choose a white triangle T_1, and a homeomorphism $f_1 : T_1 \to \overline{\mathbb{H}}^+ := \mathbb{H} \cup (\mathbb{R} \cup \{\infty\})$ such that

$$f_1 : \begin{cases} \partial T_1 & \to & \mathbb{R} \cup \{\infty\} \\ \circ & \to & 0 \\ \bullet & \to & 1 \\ \times & \to & \infty \end{cases} \tag{1}$$

Then take a triangle T_2 adjoint to T_1 (therefore a black triangle), and map it to the lower half-plane via a homeomorphism $f_2 : T_2 \to \overline{\mathbb{H}}^-$, which coincides with f_1 on their common edge and behaves on ∂T_2 according to (1). Going on with this process we end up with a continuous mapping $f_{\mathcal{D}} : \mathbb{T} = (\cup T_i) \to \widehat{\mathbb{C}}$, that is even a covering mapping if we puncture the points $\circ$,$\bullet$ and $\times$ (that is, the vertices of the triangles).

There is on $\mathbb{T}$ a unique Riemann surface structure $\mathbb{T}_{\mathcal{D}}$ for which $f_{\mathcal{D}} : \mathbb{T}_{\mathcal{D}} \to \widehat{\mathbb{C}}$ becomes a holomorphic function, in fact a Belyi function all whose ramification points are vertices of some triangle. The Belyi pair we obtain does not depend on the various choices made along the construction, such as the division of the faces or the different homeomorphisms. As we shall see in Section 3.3 it is completely determined by the representation $\mathcal{D} = (\sigma_0, \sigma_1, \sigma_\infty)$.

3.2 The dessin associated to a Belyi function

The description of the correspondence in the reverse direction is quite simple. Given a Belyi function $f : C \to \widehat{\mathbb{C}}$, we merely take $\mathcal{D}_f = f^{-1}([0, 1])$, the preimage of the real interval $[0,1]$. Then $\mathcal{D}_f$ is a dessin d'enfant, and the following properties hold.

Proposition 4 *Let* $\mathcal{D}_f = f^{-1}([0,1])$. *Then*

- *The degree of* f *gives the number of edges of* $\mathcal{D}_f$.
- *The inverse images* $f^{-1}([0,1])$, $f^{-1}([-\infty,0])$ *and* $f^{-1}([1,\infty])$ *consist of a union of segments that together form the triangulation* $\mathcal{T}(\mathcal{D}_f)$.
- *The set of white (resp. black) vertices of* $\mathcal{D}_f$ *is precisely* $f^{-1}(0)$ *(resp* $f^{-1}(1)$*). The degree of a vertex equals the ramification index of* f *at that vertex. Similarly, the centers of faces can be chosen to be the preimages of infinity, and the ramification index at each of them is half the number of edges of the corresponding face.*
- *The inverse images* $f^{-1}(\overline{\mathbb{H}}^+)$ *and* $f^{-1}(\overline{\mathbb{H}}^-)$ *are respectively the black and white triangles of the triangulation* $\mathcal{T}(\mathcal{D}_f)$.

Thus, we have a clear description of the correspondence

$$\boxed{\begin{array}{ccc} \{\text{ Dessins d'enfants }\} & \longleftrightarrow & \{\text{ Belyi pairs }\} \\ \mathcal{D}\subset X & \longrightarrow & (X_{\mathcal{D}}, f_{\mathcal{D}}) \\ \mathcal{D}_f = f^{-1}([0,1]) & \longleftarrow & (C,f) \end{array}}$$

3.3 The monodromy of a dessin

3.3.1 Monodromy of a covering of the Riemann sphere

Recall that the monodromy of a degree d covering $f : C \longrightarrow \mathbb{P}^1$ ramified over $\{q_1,\dots,q_r\}\subset\mathbb{P}^1$, is a group homomorphism

$$\begin{array}{rcl} \mathrm{Mon} : \pi_1\left(\mathbb{P}^1 \smallsetminus \{q_i\}, p\right) & \longrightarrow & \mathrm{Bij}\left\{f^{-1}(p)\right\} = \mathbb{S}_d \\ \gamma & \longmapsto & \mathrm{Mon}(\gamma) \end{array}$$

where $p\in\mathbb{P}^1\smallsetminus\{q_i\}$ is the base point with fibre $f^{-1}(p)=\{x_1,\dots,x_d\}$ and $\mathrm{Mon}(\gamma)(x_i)$ is the end point of the path in C obtained by lifting the loop γ with initial point $x_i\in C$.

It is well known that once the branching values $\{q_i\}$ are fixed, the homomorphism Mon characterizes the covering. We will sometimes represent the monodromy as

$$\mathrm{Mon} \equiv (\sigma_1,\dots,\sigma_r) \tag{2}$$

where $\sigma_i = \mathrm{Mon}(\gamma_i)$ and γ_i is a loop surrounding q_i as in Figure 4. This is because the loops γ_i generate $\pi_1\left(\mathbb{P}^1\smallsetminus\{q_i\}\right)$ subject to the single relation $\prod\gamma_i = 1$. In particular, in the above representation we must also have $\prod\sigma_i=1$.

The connectedness of C implies that the group generated by the permutations σ_i is a transitive subgroup of $\mathbb{S}_d$, known as the *monodromy group* of f.

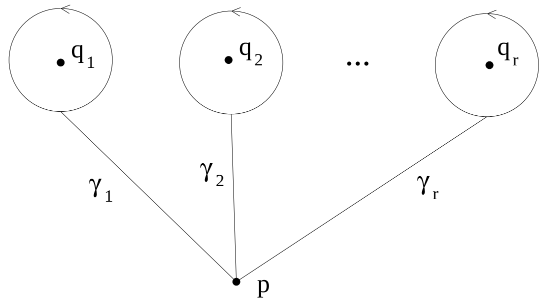

Figure 4: Generators of $\pi_1\left(\mathbb{P}^1 \smallsetminus \{q_i\}\right)$.

Conversely, it is also well known that any r-tuple of elements of $\mathbb{S}_d$, $(\sigma_1, \ldots, \sigma_r)$, which generates a transitive subgroup of $\mathbb{S}_d$ and satisfies the relation $\prod \sigma_i = 1$, arises as the monodromy of a degree d ramified covering of $\mathbb{P}^1$.

3.3.2 Monodromy of a dessin

It is interesting to observe that in this language the permutation representation of a dessin $\mathcal{D}$ is nothing but the monodromy of $f_{\mathcal{D}}$. This can be easily seen by choosing $p = 1/2$ as base point of the fundamental group of $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$. Then σ_0 and σ_1 are simply the images of loops around 0 and 1 (as in Figure 4) under the monodromy homomorphism of $f_{\mathcal{D}}$.

This is the reason why the permutation representation of a dessin is often called the *monodromy* of a dessin. Similarly the group generated by σ_0 and σ_1 will be called the *monodromy group* of $\mathcal{D}$.

It follows that there is a correspondence between the following three classes of objects:

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{Belyi pairs } (C,f) \\ \text{with } \deg(f) = n\end{array}\right\} \arrow[r, "f^{-1}([0,1])"] \arrow[d, "\mathrm{Mon}"'] & \left\{\begin{array}{c}\text{Dessins d'enfants} \\ \text{with } n \text{ edges}\end{array}\right\} \arrow[dl, "\text{Perm.rep.}"'] \\
\left\{\begin{array}{c}\text{Ordered pairs of} \\ \text{permutations generating} \\ \text{transitive subgroups of } \mathbb{S}_n\end{array}\right\} &
\end{tikzcd}$$

It is clear from what has gone above that these are bijective correspondences.

4 Some examples

We shall now show some explicit examples of Belyi pairs and the corresponding dessins.

The first examples will be of genus 0, the simplest case since a half of the Belyi pair, namely the Riemann surface, is already known (the Riemann sphere $\widehat{\mathbb{C}}$).

Example 5 Consider $\mathcal{D} = (\sigma_0, \sigma_1)$, where $\sigma_0 = (1,2,3,4), \sigma_1 = \mathrm{Id} \in \mathbb{S}_4$. It can be easily checked that it corresponds to the dessin in Figure 5, where the first two pictures represent two possible concrete aspects of $\mathcal{D}$ (remember the equivalence up to homeomorphisms). It is in fact more convenient to depict $\mathcal{D}$ as in the right hand side of Figure 5, where the sphere has been mapped to the plane by stereographic projection.

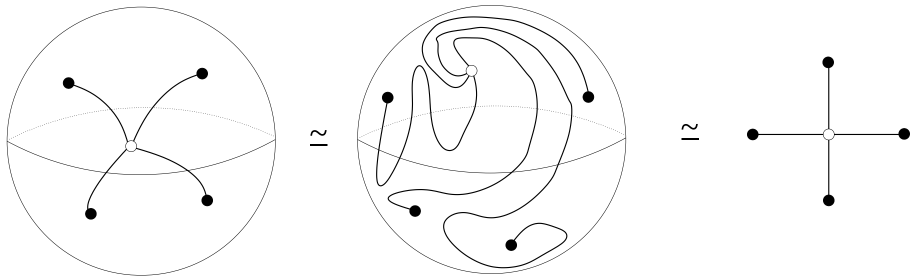

Figure 5: The dessin on the sphere corresponding to $\beta(z) = z^4$.

We would like to describe explicitly the Belyi pair $\left(\mathbb{S}^2_{\mathcal{D}}, f_{\mathcal{D}}\right)$ associated to this dessin. Since the only Riemann surface of genus zero is $\widehat{\mathbb{C}}$, we must have a commutative diagram of the form

$$\begin{tikzcd} \mathbb{S}^2_{\mathcal{D}} \arrow[r, "g"] \arrow[d, "f_{\mathcal{D}}"'] & \widehat{\mathbb{C}} \arrow[dl, "\beta"] \\ \widehat{\mathbb{C}} & \end{tikzcd}$$

where g is an isomorphism of Riemann surfaces and β is a rational function. Thus, our goal is to find the rational function β such that $\beta^{-1}([0,1]) = g(\mathcal{D})$.

The following information about β follows from Proposition 4.

- β has degree 4.
- $\beta^{-1}(\infty)$ has only one point.
- $\beta^{-1}(0)$ has only one point, whose ramification index is four.
- $\beta^{-1}(1)$ has four non-ramified points.

Note that by composing g with a Möbius transformation we can assume that the center of the face of $\mathcal{D}$, the white vertex, and a chosen black vertex correspond via g to ∞, 0 and 1. It follows that the Belyi function corresponding to the dessin in Figure 5 is $\beta(z) = z^4$.

In the same way, the dessin in the sphere given by a star with n arms (monodromy $\sigma_0 = (1, \ldots, n)$, $\sigma_1 = \mathrm{Id}$) has $\beta(z) = z^n$ as associated Belyi function.

Example 6 ([22]) Let us consider a slightly more interesting case: the dessin in the left hand side of Figure 6.

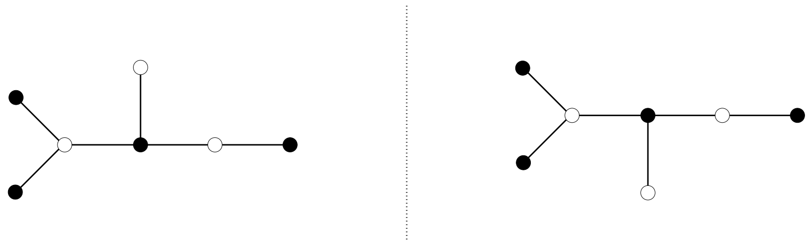

Figure 6: Two conjugate trees.

Choose the white vertices of degrees 3 and 2 to be $z = 0$ and $z = 1$ respectively, and let $z = a$ be the third white vertex. Then, according to Proposition 4

$$\beta(z) = z^3(z-1)^2(z-a),$$

and the cuestion is whether or not it is possible to determine the value of a.

Let us compute the derivative of β:

$$\begin{aligned} \beta'(z) &= 3z^2(z-1)^2(z-a) + 2z^3(z-1)(z-a) + z^3(z-1)^2 \\ &= z^2(z-1)(6z^2 + (-5a-4)z + 3a) \end{aligned}$$

Now, again by Proposition 4, β must have a ramification point of order 3 (different from 0 or 1), therefore that point must occur as double zero of β'. Then the discriminant of the polynomial $P(z) = 6z^2 + (-5a-4)z + 3a$ must vanish, that is

$$25a^2 - 32a + 16 = 0,$$

hence $a = \frac{4}{25}(4+3i)$ or $a = \frac{4}{25}(4-3i)$.

If we draw the preimage of the interval $[0,1]$ by the two Belyi functions corresponding to the two possible choices of a, we find the two graphs of Figure 6. Because these two values of a are Galois-conjugate of each other, the two dessins will be said to be *conjugate dessins* (see Section 5).

See [22] for more examples of this kind.

Example 7 Let us study in detail an example of a dessin in a genus 1 topological surface. Consider again the dessin $\mathcal{D}$ in Figure 2, whose monodromy is $\sigma_0 = (1,6)(2,4)(3,5)$ and $\sigma_1 = (1,2,3)(4,5,6)$.

Since the construction of the Belyi pair depends only on the topology, we can replace each of the triangles in Figure 3 by an Euclidean triangle of angles $\dfrac{\pi}{2}$, $\dfrac{\pi}{3}$ and $\dfrac{\pi}{6}$ at the vertices $\circ, \bullet$ and $\times$ respectively. This way we obtain a quadrangle with identified opposite sides (see the left part of Figure 7). We can now cut the torus along the dessin, and paste the pieces together to form a regular hexagon with opposite sides identified (right part of Figure 7). We think of the hexagon as centered at the origin of the Euclidean plane, and $\mathcal{D}$ can be seen now at the boundary.

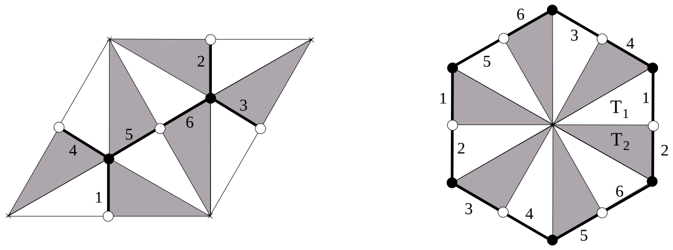

Figure 7: Further topological representations of the dessin in Figure 2.

With this highly symmetric representation of $\mathcal{D}$ at our disposal, we can be more precise about the construction, explained in Section 3, of the associated function $f_{\mathcal{D}}$. Recall that to define the function $f_{\mathcal{D}}$, one begins with a homeomorphism $f_1 : T_1 \to \overline{\mathbb{H}}^+$. Let T_1 be as in Figure 7. We choose a homeomorphism f_1, and extend it to $f_2 : T_2 \to \overline{\mathbb{H}}^-$ by $f_2(z) = \overline{f_1(\overline{z})}$, where T_2 is also shown in Figure 7.

Then we can define $f_{\mathcal{D}}$ on each triangle as follows: $f_{\mathcal{D}} = f_1$ in T_1, $f_{\mathcal{D}} = f_2$ in T_2, and $f_{\mathcal{D}} = f_j \circ R^k_{\pi/3}$ in the remaining triangles, where $R_{\pi/3}$ is the rotation through angle $\pi/3$ around the origin, $j = 1$ or 2 for the white and black triangles respectively, and k is the necessary exponent for the composition to make sense.

Now consider the mapping $\gamma : \mathbb{T}_{\mathcal{D}} \to \mathbb{T}_{\mathcal{D}}$ induced by $R_{\pi/3}$ in the Belyi surface $\mathbb{T}_{\mathcal{D}}$. It is clear that γ is a homeomorphism. Moreover, since $f_{\mathcal{D}} \circ \gamma = f_{\mathcal{D}}$, it follows that γ is an (order 6) automorphism of $\mathbb{T}_{\mathcal{D}}$. Furthermore γ has at least one fixed point, namely the center of our hexagon. Now, it is well known, by the classical theory of elliptic curves, that there is a unique torus with such an automorphism, namely

$$\begin{array}{rcl} \{y^2 = x^3 - 1\} & \longrightarrow & \{y^2 = x^3 - 1\} \\ (x, y) & \longmapsto & (\xi_3 x, -y) \end{array}$$

where $\xi = \exp 2\pi i/3$. This ensures that $\mathbb{T}_{\mathcal{D}}$ corresponds to the algebraic curve $y^2 = x^3 - 1$. Accordingly, $f_{\mathcal{D}}$ can be described by $(x, y) \mapsto x^3 - 1$.

All the previous examples show in concrete cases the correspondence from dessins to Belyi pairs. Let us focus now in the reverse construction, from Belyi pairs to dessins, by studying in detail an example in genus 1.

Example 8 Let C be the Riemann surface $\left\{y^2 = x(x-1)(x-\sqrt{2})\right\} \cup \{\infty\}$, and construct a Belyi function f on it by means of the following composition of maps

$$\begin{tikzcd}
\left\{y^2 = x(x-1)(x-\sqrt{2})\right\} \cup \{\infty\} \arrow[d, "(x,y)\mapsto x"] \arrow[dddd, bend right, "f"'] \\
\widehat{\mathbb{C}} \arrow[d, "t\mapsto t^2-2 \text{ (minimal polinomial of } \sqrt{2})"] \\
\widehat{\mathbb{C}} \arrow[d, "u\mapsto -1/u"] \\
\widehat{\mathbb{C}} \arrow[d, "z\mapsto 4z(1-z) \text{ (Belyi polynomial } P_{1/2}\text{, see Section 6.1)}"] \\
\widehat{\mathbb{C}}
\end{tikzcd}$$

that is

$$f(x,y) = \frac{-4\left(x^2-1\right)}{\left(x^2-2\right)^2}.$$

We can track back the ramification indices over the three ramification values $0, 1$, and ∞ from the bottom to the top in the previous diagram, and we obtain

$$\begin{tikzcd}
\left\{\begin{array}{c} \infty^{(4)} \\ (1,0)^{(2)} \\ \left(-1, \sqrt{-2-\sqrt{8}}\right)^{(1)} \\ \left(-1, -\sqrt{-2-\sqrt{8}}\right)^{(1)} \end{array}\right\} \arrow[d] & \left\{(0,0)^{(8)}\right\} \arrow[d] & \left\{\begin{array}{c} (\sqrt{2},0)^{(4)} \\ (-\sqrt{2}, \sqrt{-4-\sqrt{32}})^{(2)} \\ (-\sqrt{2}, -\sqrt{-4-\sqrt{32}})^{(2)} \end{array}\right\} \arrow[d] \\
\left\{\infty^{(2)}, 1^{(1)}, -1^{(1)}\right\} \arrow[d] & \left\{0^{(4)}\right\} \arrow[d] & \left\{\sqrt{2}^{(2)}, -\sqrt{2}^{(2)}\right\} \arrow[d] \\
\left\{\infty^{(1)}, -1^{(1)}\right\} \arrow[d] & \left\{-2^{(2)}\right\} \arrow[d] & \left\{0^{(2)}\right\} \arrow[d] \\
\left\{0^{(1)}, 1^{(1)}\right\} \arrow[d] & \left\{1/2^{(2)}\right\} \arrow[d] & \left\{\infty^{(2)}\right\} \arrow[d] \\
\{0\} & \{1\} & \{\infty\}
\end{tikzcd}$$

where the superindices denote branching indices.

Now we find the preimage of the real interval $[0,1]$ by f, performing inverse images step by step in the composition of maps that defines f. The result is shown in Figure 8, where the algebraic curve C is constructed by gluing two copies of the complex plane along cuts from 0 to 1 and from $\sqrt{2}$ to ∞.

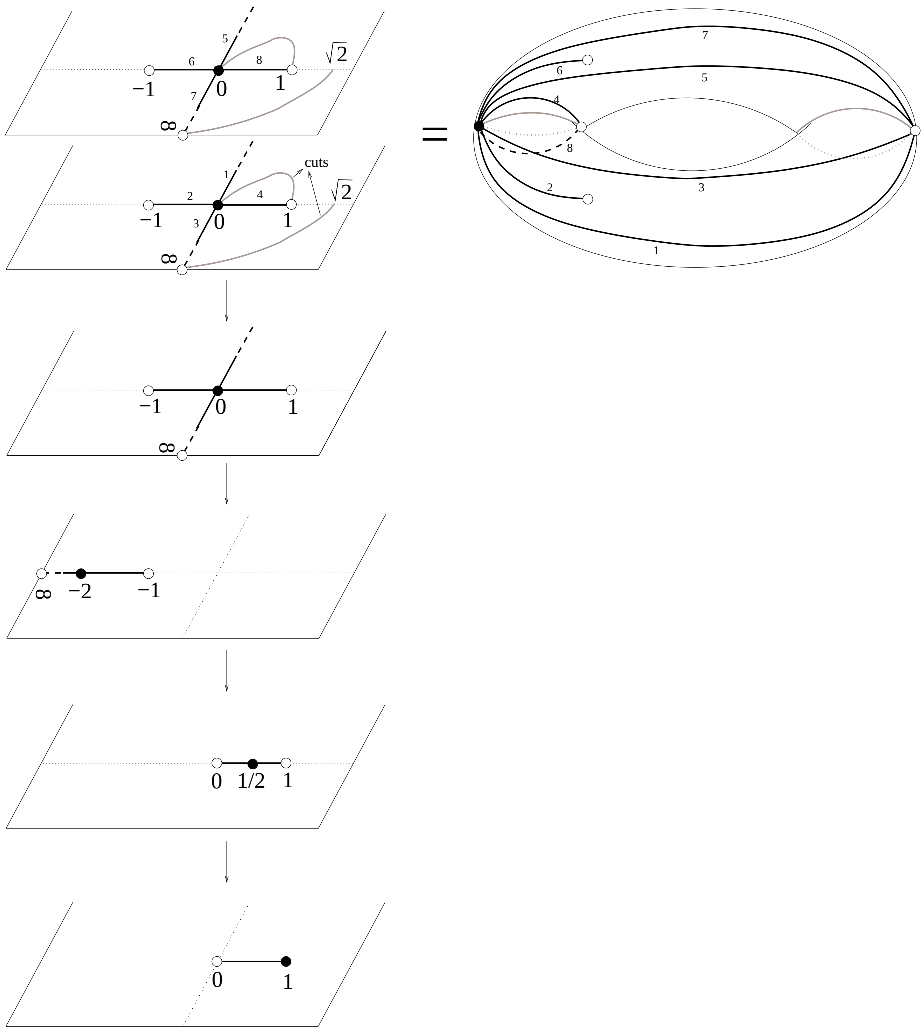

Figure 8: $\mathcal{D} = f^{-1}([0,1])$; f as in Example 8.

5 The action of $\mathrm{Gal}(\overline{\mathbb{Q}})$ on dessins

We already described how the absolute Galois group acts on Belyi pairs. Now, having established the equivalence between Belyi pairs and dessins, we can make the Galois elements $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}})$ act on the dessins themselves, by means of the rule

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d, leftrightarrow, dotted] & \sigma(\mathcal{D}) \arrow[d, leftrightarrow, dotted] \\ (X_{\mathcal{D}}, f_{\mathcal{D}}) \arrow[r, leftrightarrow, dotted, "\sigma"] & (X_{\mathcal{D}}^{\sigma}, f_{\mathcal{D}}^{\sigma}) \end{tikzcd}$$

Let us have a second look at our examples in Section 4.

In the genus 0 case, Galois conjugation affects only the Belyi function and not the Riemann surface (there is only one Riemann surface of genus 0). As the trees in Example 5 come from polynomials defined over the rationals, all of them remain fixed by the whole action of $\mathrm{Gal}\left(\overline{\mathbb{Q}}\right)$. In other words, we have $\sigma(\mathcal{D}) = \mathcal{D}$ for every $\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}\right)$.

Something different happens in Example 6. Every σ such that $\sigma(i) = -i$ transforms the parameter a into $\overline{a}$, hence sends the tree in the left hand side of Figure 6 to the tree in the right hand side of the same Figure and viceversa. On the contrary, both trees remain invariant by Galois elements fixing i. It can be seen that these two trees form a *complete orbit* of the action of $\mathrm{Gal}\left(\overline{\mathbb{Q}}\right)$ on dessins. This is done by computing certain invariants of the action of $\mathrm{Gal}\left(\overline{\mathbb{Q}}\right)$ and showing that no other graph has the same invariants (see Proposition 10).

The action can be more involved in genus 1 since the conjugation affects also the Riemann surface (nevertheless the conjugate Riemann surface will still have genus 1, see Proposition 10). But again this is not the case of the dessin $\mathcal{D}$ in Example 7. As both $X_{\mathcal{D}}$ and $f_{\mathcal{D}}$ are defined over the rationals, every Galois element fixes $\mathcal{D}$.

Example 8 is much more interesting. Let $\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}\right)$ such that $\sigma(\sqrt{2}) = -\sqrt{2}$. Consider the conjugate Belyi pair (C^{σ}, f^{σ}). As f was defined over $\mathbb{Q}$, f^{σ} is given by the same formula as f, namely

$$f^{\sigma}(x, y) = \frac{4(x^2 - 1)}{x^4},$$

but now f^{σ} must be viewed as a function on the conjugate Riemann surface $C^{\sigma} = \left\{y^2 = x(x-1)(x+\sqrt{2})\right\} \cup \{\infty\}$, which is not isomorphic to C since $J\left(\sqrt{2}\right) \neq J\left(-\sqrt{2}\right)$ where $J(\lambda) = \dfrac{\lambda^2 - \lambda + 1}{\lambda^2(\lambda-1)^2}$ is the classical J-invariant that classifies elliptic curves. To depict $\sigma(\mathcal{D})$ we can follow the same strategy as the one we used for $\mathcal{D}$, that is, we follow the track of the ramification through the four maps in which f^σ has been decomposed. We obtain

$$
\begin{array}{ccc}
\left\{\begin{array}{c}\infty^{(4)}\\ (1,0)^{(2)}\\ \left(-1,\sqrt{-2+\sqrt{8}}\right)^{(1)}\\ \left(-1,-\sqrt{-2+\sqrt{8}}\right)^{(1)}\end{array}\right\} & \left\{(0,0)^{(8)}\right\} & \left\{\begin{array}{c}(-\sqrt{2},0)^{(4)}\\ (\sqrt{2},\sqrt{-4+\sqrt{32}})^{(2)}\\ (\sqrt{2},-\sqrt{-4+\sqrt{32}})^{(2)}\end{array}\right\}\\
\downarrow & \downarrow & \downarrow\\
\left\{\infty^{(2)},1^{(1)},-1^{(1)}\right\} & \left\{0^{(4)}\right\} & \left\{\sqrt{2}^{(2)},-\sqrt{2}^{(2)}\right\}\\
\downarrow & \downarrow & \downarrow\\
\left\{\infty^{(1)},-1^{(1)}\right\} & \left\{-2^{(2)}\right\} & \left\{0^{(2)}\right\}\\
\downarrow & \downarrow & \downarrow\\
\left\{0^{(1)},1^{(1)}\right\} & \left\{1/2^{(2)}\right\} & \left\{\infty^{(2)}\right\}\\
\downarrow & \downarrow & \downarrow\\
\{0\} & \{1\} & \{\infty\}
\end{array}
$$

where the superindices denote branching indices. Now compute step by step the preimage of the real interval $[0,1]$. See Figure 9, where the final picture of $\sigma(\mathcal{D})$ is drawn on the topological torus. For another example of this kind, see [32].

Remark 9 An alternative way to obtain the branching table corresponding to (C^σ, f^σ) is to observe that $f(p) = 0$ (resp. 1 or ∞) if and only if $f^\sigma(p^\sigma) = \sigma(0) = 0$ (resp. $\sigma(1) = 1$ or $\sigma(\infty) = \infty$) and that the branching orders are also preserved (although this is a little more difficult). Therefore the branching table corresponding to (C^σ, f^σ) can be obtained by simply transforming the branching table for (C, f) under σ.

5.1 Some invariants

Trying to get some insight of the group $\mathrm{Gal}\left(\overline{\mathbb{Q}}\right)$ by studying its action on dessins is probably the main goal that led Grothendieck to the introduction of the theory of dessins. One would like therefore to have a good collection of invariants at hand.

Proposition 10 *Let $\mathcal{D}$ be a dessin. The following properties of $\mathcal{D}$ remain invariant under the action of the absolute Galois group.*

- *The number of white vertices, black vertices, edges and faces.*
- *The degrees of the white vertices, the degrees of the black vertices, and the degrees of the faces.*

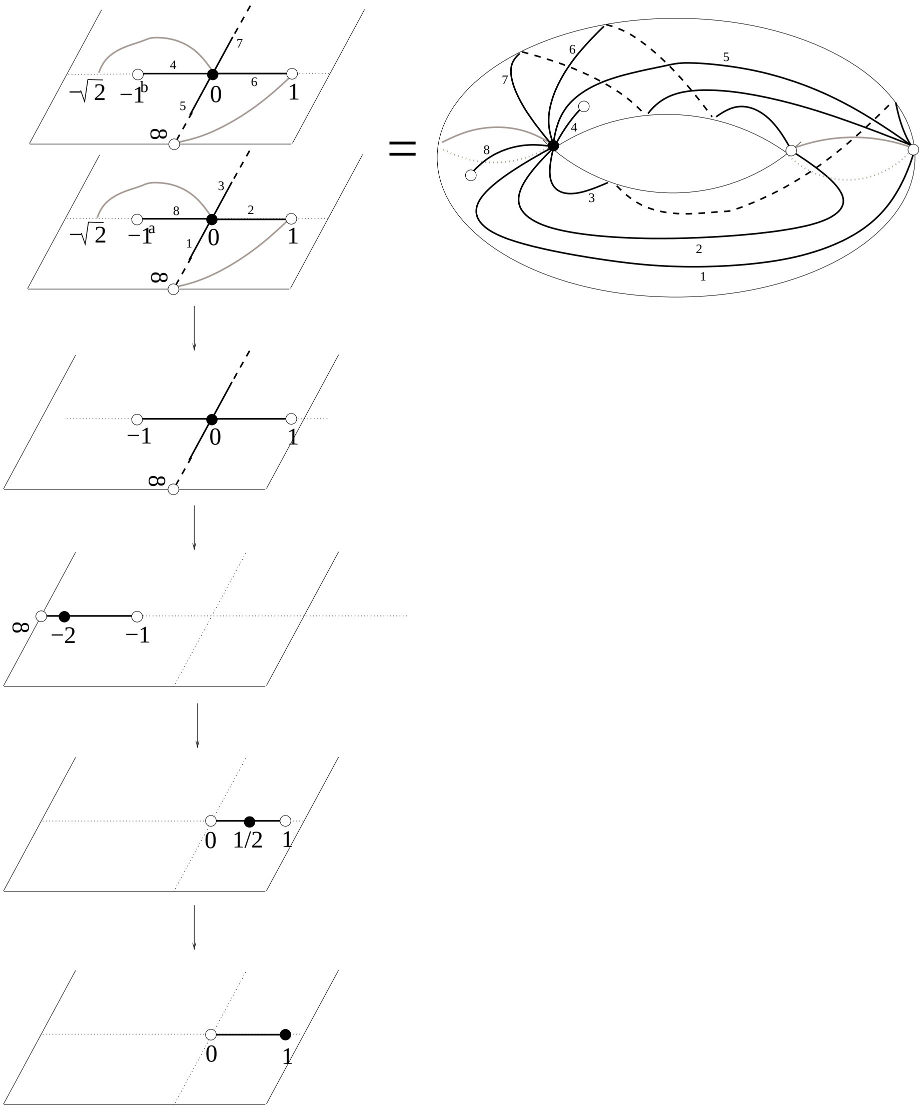

Figure 9: Description of $\sigma(\mathcal{D}) = (f^\sigma)^{-1}([0,1])$; f as in Example 8.

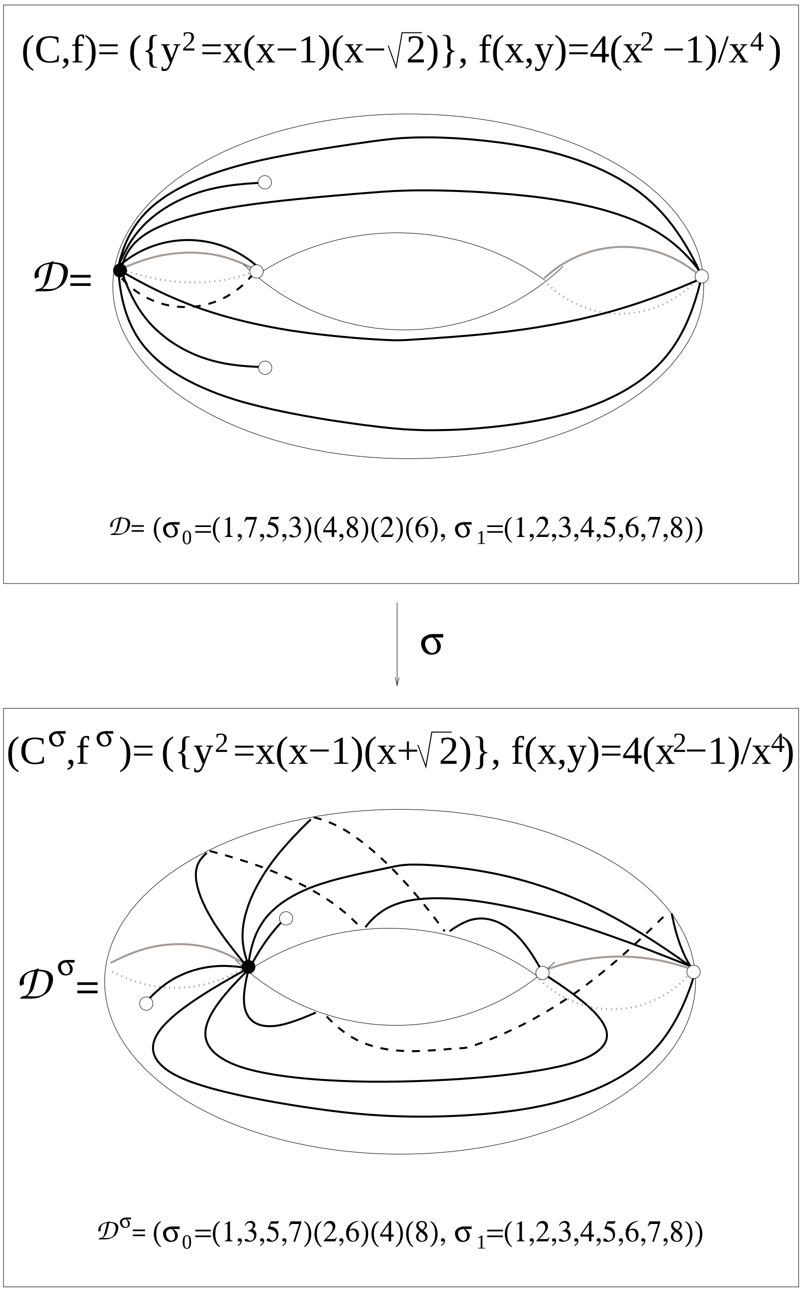

Figure 10: The accion of $\sigma(\sqrt{2}) = -\sqrt{2}$ on the dessin in Figure 8.

- *The genus.*
- *The monodromy group.*

Those in the previous list are the most simple invariants, but some others have also been studied. Anyway, there does not exist a *complete* list, in the sense that there exist explicit examples of non-conjugate dessins for which all known invariants can be computed and happen to agree.

We warn the reader that Proposition 10 is not completely trivial. Note that the conjugation by an element of the absolute Galois group is a highly discontinuous operation, hence it is not clear why any of the objects in the list should be preserved. The way to show it is by means of the correspondence between dessins and morphisms $f : C \to \mathbb{P}^1$, on which $\mathrm{Gal}\left(\overline{\mathbb{Q}}\right)$ acts nicely. Indeed it is not hard to see (Remark 9) that this action preserves degree, ramification, Hurwitz formula, hence genus, etc.

5.2 Faithfulness of the action of $\mathrm{Gal}\left(\overline{\mathbb{Q}}\right)$ on dessins

In order to obtain information about the absolute Galois group by making it act on dessins, it is natural to ask whether this action is faithful. In fact it is.

Theorem 11 *The restriction of the action of* $\mathrm{Gal}\left(\overline{\mathbb{Q}}\right)$ *to dessins of genus g is faithful for every g.*

For the case of genus zero, one can be even more restrictive, as the action is faithful even on trees. This was first shown by Lenstra (see Schneps article in [27]).

In genus 1 the result is quite obvious and follows from the classical theory of elliptic curves. Given $\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}\right)$ and an algebraic number j such that $\sigma(j) \neq j$, let $C_\lambda : y^2 = x(x-1)(x-\lambda)$ be the elliptic curve whose Jacobi invariant $J(\lambda) = \dfrac{\lambda^2 - \lambda + 1}{\lambda^2(\lambda-1)^2}$ is precisely j. Then $C_\lambda^\sigma = C_{\sigma(\lambda)}$ has Jacobi invariant $\sigma(j)$ and hence it is not isomorphic to C_λ.

For genus g greater than one, it can be shown that the action is already faithful when restricted to hyperelliptic Riemann surfaces of the form

$$\left\{y^2 = (x-1)(x-2)\cdots(x-(2g+1))(x-(a+n))\right\},$$

where $a \in \overline{\mathbb{Q}}$ (see [8]).

6 The proof of Belyi's theorem

We want to prove the following statement:

$$\boxed{C \text{ is defined over } \overline{\mathbb{Q}} \iff \text{There exists } f : C \to \widehat{\mathbb{C}} \text{ ramified over } \{0, 1, \infty\}}$$

6.1 The *only if* part [4]

The proof of the implication $\Rightarrow$, which is the real contribution of Belyi, is based on the clever algorithm illustrated in Example 8. Concerning the ramification values, it is rather obvious that

$$\{\text{Ram. values of } f \circ g\} = f\left(\{\text{Ram. values of } g\}\right) \cup \{\text{Ram. values of } f\},$$

and this is the reason why a suitable composition of maps may reduce the number of ramification values. Belyi's algorithm consists of the following steps:

1. Find a morphism to $\mathbb{P}^1$ that ramifies only above algebraic points. This is possible because C is defined over $\overline{\mathbb{Q}}$.

2. Compose with a rational function to push all the ramification values inside $\mathbb{Q} \cup \{\infty\}$. The minimal polynomial $m(x)$ of the ramification values of the first step is used at this point. Note that the new branch points created in this process will be roots of $m'(x)$. Since $\deg(m'(x)) < \deg(m(x))$ it is clear that after repeating this argument finitely many times one is left with only rational critical values.

3. The next step is to compose with a polynomial that sends one of the rational ramification values, $\lambda = \dfrac{m}{m+n}$, inside the set $\{0, 1, \infty\}$. This is precisely Belyi's polynomial P_λ, defined as follows.

   $$P_\lambda(x) = \frac{(m+n)^{m+n}}{m^m n^n} x^m (1-x)^n$$

   P_λ enjoys the following properties.

   - P_λ ramifies only at $x = 0, 1$, ∞ and λ.
   - $P_\lambda(0) = 0$, $P_\lambda(1) = 0$, $P_\lambda(\infty) = \infty$ and $P_\lambda(\lambda) = 1$.
   - P_λ sends the remaining rational branching values to rational values.

4. Finally, compose with Belyi's polynomial to the remaining branching values, one by one.

Belyi's argument allows us to give a slightly more general, but equivalent, version of Belyi's theorem.

Theorem 12 *C is defined over a number field if and only if there exists a covering $f : C \longrightarrow \mathbb{P}^1$ ramified over algebraic values.*

6.2 A criterion for a variety to be defined over $\overline{\mathbb{Q}}$

The *if* part in Belyi's theorem was previously known, as a consequence of some results of Weil [30]. Instead of Weil's criterion we will apply a criterion given in [9] which will be also used in the second part of the paper (for other proofs, see [20], [31]).

Let $X \subset \mathbb{P}^n(\mathbb{C})$ be an irreducible projective variety of arbitrary dimension. Extending the terminology we have employed in the one dimensional case, we shall say that X *is defined over a field* $K \subset \mathbb{C}$ if there is a finite collection of homogeneous polynomials with coefficients in K

$$\left\{P_\alpha(X_0, ..., X_n) = \sum \alpha_\nu X_0^{\nu_0} ... X_n^{\nu_n}\right\}_\alpha$$

whose zero set $Z\left(P_\alpha\right)$ is X. We shall say that X *can be defined over* K if it is isomorphic to a variety defined over K.
Likewise, we shall say that a morphism $f : X \to Y$ between irreducible varieties $X \subset \mathbb{P}^n(\mathbb{C})$ and $Y \subset \mathbb{P}^r(\mathbb{C})$ *is defined over* K if X and Y are defined over K and there is an open cover $\{U_j\}$ of X such that $f_{|U_j} \equiv (F_{j,0}, ..., F_{j,r})$ for some homogeneous polynomials $F_{j,k} = F_{j,k}(X_0, ..., X_n)$ with coefficients in K ([28]). We will say that $f : X \to Y$ *can be defined over* K if it is equivalent to a morphism $f_0 : X \to Y$ defined over K. Here f and f_0 being equivalent means that there are automorphisms $h_1 : X \simeq X$ and $h_2 : Y \simeq Y$ such that the following diagram commutes

$$\begin{array}{ccc} X & \stackrel{f}{\to} & Y \\ h_1 \downarrow & & \downarrow h_2 \\ X & \stackrel{f_0}{\to} & Y \end{array}$$

We are interested in the question of whether a given variety X can be defined over a *number field*, or equivalently, over $\overline{\mathbb{Q}}$, the field of algebraic numbers. Let $\mathrm{Gal}(\mathbb{C}) = \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ denote the group of all field automorphisms of $\mathbb{C}$. The action of $\mathrm{Gal}(\mathbb{C})$ on complex varieties and morphisms will be denoted in the same way as we did in the first part of the paper.

In [9] the following criterion was established.

Criterion 13 ([9]) *The following conditions relative to an irreducible variety $X \subset \mathbb{P}^n(\mathbb{C})$ (resp. a morphism $f : X \to Y$ between irreducible projective varieties defined over a number field) are equivalent*
i) X (resp. $f : X \to Y$) can be defined over a number field.
ii) The family $\{X^\sigma\}_{\sigma \in Gal(\mathbb{C})}$ (resp. $\{f^\sigma : X^\sigma \to Y^\sigma\}_{\sigma \in Gal(\mathbb{C})}$) contains only finitely many isomorphism classes of complex projective varieties (resp. of morphisms).

The i) $\Rightarrow$ ii) part of this criterion is clear. By elementary Galois Theory an algebraic number gets transformed by $\mathrm{Gal}(\mathbb{C})$ into only finitely many complex numbers. The proof of the part ii) $\Rightarrow$ i) is more difficult but, at the same time, rather elementary, in the sense that it only uses trascendental field theory.

Example 14 For the curve $y^2 = x^3 - \pi$ considered in Section 2 we have

$$\mathrm{Gal}(\mathbb{C})\left(y^2 = x^3 - \pi\right) = \left\{y^2 = x^3 - e\right\}_{e \text{ transcendental}}$$

This family contains only one isomorphism class, since for any pair of curves in the family we have the following isomorphism

$$\begin{array}{ccc} y^2 = x^3 - \pi & \simeq & y^2 = x^3 - e \\ (x, y) & \to & (x\sqrt[3]{e/\pi}, y\sqrt{e/\pi}) \end{array}$$

6.3 The *if* part

We are now in position to prove the implication $\Leftarrow$) in Belyi's theorem (as stated in Theorem 12), as well as Theorem 2.

Let $f : C \to \mathbb{P}^1$ be a surjective morphism with branch values $y_1, \ldots, y_r \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$. For any $\sigma \in \mathrm{Gal}(\mathbb{C})$ the morphism $f^\sigma : C^\sigma \to \mathbb{P}^1$ has same degree as $f : C \to \mathbb{P}^1$. By hypothesis, the family $\{(y_1^\sigma, ..., y_r^\sigma)\}_{\sigma\in\mathrm{Gal}(\mathbb{C})}$ contains only finitely many distinct r-tuples. Therefore there are finitely many possible monodromy homomorphisms

$$\pi_1 : \mathbb{P}^1 \setminus \{y_i^\sigma\} \longrightarrow \mathbb{S}_d$$

and hence (Section 3.3) finitely many nonisomorphic covering curves C^σ and covering maps f^σ. Now, apply Criterion 13.

Part II: Belyi's theorem for complex surfaces

Let now S be a *complex surface*, that is a compact holomorphic manifold of complex dimension 2. Naturally, we shall say that S can be defined over a number field if it is biholomorphic to a projective surface defined over $\overline{\mathbb{Q}}$. Note that a complex surface need not be algebraic, that is, need not be isomorphic to a projective surface.

It is natural to ask whether a result similar to Belyi's theorem, say in the form given in Theorem 12, holds for complex surfaces. To be more explicit, we can ask the following

QUESTION 1 : Is it true that S can be defined over $\overline{\mathbb{Q}}$ if and only if there is a meromorphic function $f \in \mathcal{M}(S)$ such that its critical values lie all in $P^1(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}} \cup \{\infty\}$**?**

7 Minimal surfaces

The answer to Question 1, in the precise form it has been formulated, cannot be affirmative. To see this, consider first a surface S defined over $\overline{\mathbb{Q}}$ and a meromorphic function $f \in \mathcal{M}(S)$ whose critical values $\mathrm{crit}(f) = \{q_i\}$ lie in $\mathbb{P}^1(\overline{\mathbb{Q}})$. Then, choose $q \in \mathbb{P}^1(\overline{\mathbb{Q}})$, $q \neq q_i$, and blow up S at one (or several) points $P \in f^{-1}(q)$ (Figure 11). This gives us a new surface $\pi^P : S^P \to S$ and a new meromorphic function $f \circ \pi^P \in \mathcal{M}(S_P)$ whose critical value set $\{q_i, q\}$ is again contained in $\mathbb{P}^1(\overline{\mathbb{Q}})$. Now, one cannot expect S^P to be defined over a number field for an arbitrary choice of *transcental points* $P \in f^{-1}(q)$. This leads us to introduce the concept of minimal surfaces.

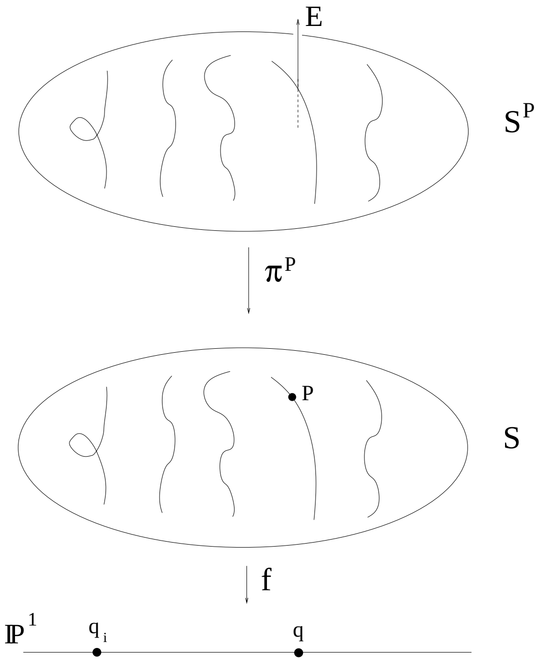

Figure 11: Blowing up S at the point P.

Recall that S is termed *minimal* when it does not contain genus zero Riemann surfaces with self-intersection -1. These are called *exceptional* or (-1)-*curves*. Exceptional curves arise when a surface is blown up. By this we mean that given a complex surface S and an exceptional curve E in it, there is a complex surface S_E and a holomorphic mapping $\pi_E : S \to S_E$, called *blowing down* or *contraction map*, so that E maps to a point $P \in S_E$ in such a way that π_E equals π^P, the blow up of S_E at the point P. For an arbitrary complex surface S, a minimal surface $S_{\min}$ can be obtained by a sequence of contractions

$$S = S_0 \to S_1 \to \cdots \to S_n = S_{\min}$$

A minimal surface obtained in this way is called *a minimal model of* S ([3], [2]).

Suppose that S is defined over $\overline{\mathbb{Q}}$ and let $E \subset S$ be an exceptional curve. Then, not surprisingly, E is also defined over $\overline{\mathbb{Q}}$ ([9]). Now, since blowing down an exceptional curve is an operation in Algebraic Geometry that works over any algebraic closed field, there can be no doubt that the contracted surface S_E and the point $P = \pi(E) \in S_E$ are both defined over $\overline{\mathbb{Q}}$. As the same argument can be applied at every next step, we see that if a complex surface S is defined over a number field then so must be any minimal model of S. Thus, we have

Proposition 15 *A complex surface S can be defined over a number field if and only if it can be obtained out of a minimal surface $S_{\min}$ defined over $\overline{\mathbb{Q}}$ by a finite sequence of blow-ups centered at points also defined over $\overline{\mathbb{Q}}$.*

Therefore in our search for a Belyi criterion for complex surfaces we can restrict ourselves to minimal ones.

8 Bertini-Lefschetz theory of pencils

Recall that by classical results of Bertini, given a projective surface $S \subset \mathbb{P}^n$ there is a *pencil of hyperplanes* $\{H_\lambda = \lambda_0 H_0 + \lambda_1 H_1\}_\lambda$ with $\lambda = (\lambda_0, \lambda_1) \in \mathbb{P}^1$, so that the hyperplane sections $S_\lambda = S \cap H_\lambda$ are *generically* non singular connected algebraic curves and $S = \cup_{\lambda \in \mathbb{P}^1} S_\lambda$. Moreover, the rule

$$f(x) = \lambda \in \mathbb{P}^1 \text{ if and only if } x \in S_\lambda$$

defines a *meromorphic function* $f \in \mathcal{M}(S)$ with nonempty *base locus* $B = \cap_{\lambda \in \mathbb{P}^1} S_\lambda = Z(H_0, H_1) \cap S$. By results of Lefschetz (see [21]), this meromorphic function $f : S \dashrightarrow \mathbb{P}^1$ can be chosen so as to satisfy the following requirements.

i) $f : S \setminus B \to \mathbb{P}^1$ is a holomorphic submersion outside a finite set of critical points $\mathrm{Crit}(f) = \{x_1, ..., x_d\}$, no two of them in a same fibre, which therefore correspond bijectively to the critical values $\mathrm{crit}(f) = \{q_1 = f(x_1), ..., q_d = f(x_d)\}$

ii) at each critical point x_i, f is locally of the form $(z_1, z_2) \to z_1^2 + z_2^2$, and

iii) at each base point b_k, f is locally of the form $(z_1, z_2) \to z_1/z_2$

Definition 16 A *Lefschetz pencil (L.P.) on a complex surface* S is a meromorphic function $f \in \mathcal{M}(S)$ which has a nonempty base locus $B = \{b_1, ..., b_r\}$ and satisfies conditions i) to iii) above.

We observe that $f : S \dashrightarrow \mathbb{P}^1$ is not well defined at the points in B, that is what the dashed arrow is meant to emphasize. However, associated to a L. P. there are two well defined holomorphic maps which will permit us to regard our complex surfaces either as

1. A family of Riemann surfaces of type (g, r) or
2. A family of compact Riemann surfaces of genus g with r sections.

Let us describe these two maps.

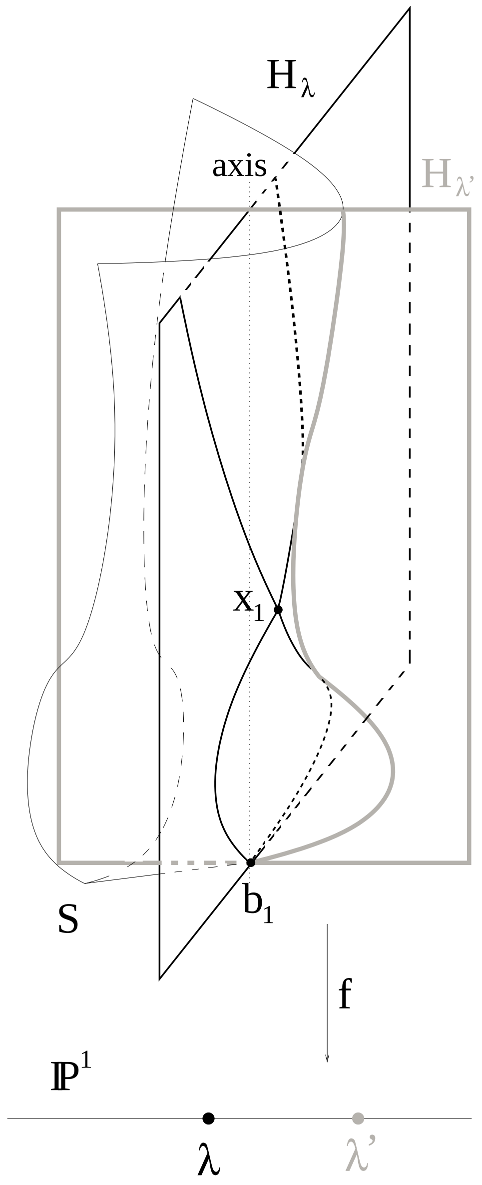

Figure 12: Bertini-Lefschetz theorem.

1. $f : S \setminus B \longrightarrow \mathbb{P}^1$.

   Away from the critical values $\lambda = q_i$ the fibres $f^{-1}(\lambda)$ are Riemann surfaces of finite type (g, r), that is compact Riemann surfaces of genus g with r punctures. The closure of each of the fibres in S is obtained by including the points $\{b_1, ..., b_r\}$.

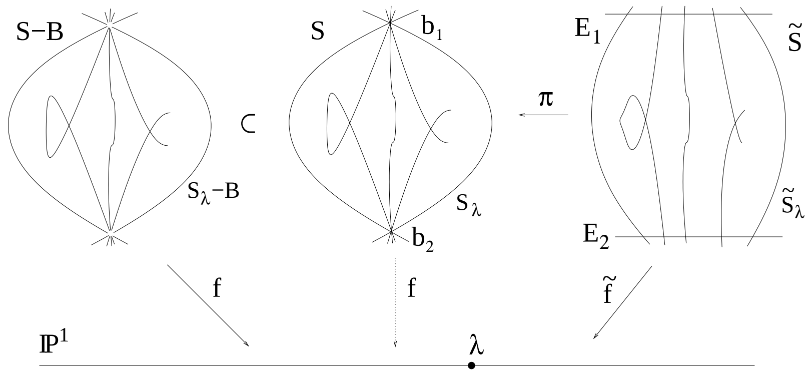

Figure 13: The two maps associated to a Lefschetz pencil.

2. $\widetilde{f} : \widetilde{S} \to \mathbb{P}^1$

   This is the *Lefschetz fibration* associated to the L. P. Here $\widetilde{S}$ is the surface obtained by blowing up S at the points in $B = \{b_1, ..., b_r\}$. Recall that, roughly, what the blowing up operation $\pi : \widetilde{S} \to S$ does is to replace each point $b_i \in B$ by a projective line $\mathbb{P}^1_{b_i}$ consisting of points b_i^λ, one for each Riemann surface S_λ so that one may coherently define $\widetilde{f}\left(b_I^\lambda\right) = \lambda$. As for the fibres of $\widetilde{f}$, the map π induces an isomorphism between $\widetilde{S}_\lambda = \widetilde{f}^{-1}(\lambda)$ and $f^{-1}(\lambda) \cup B$. It follows that the fibration $\widetilde{f} : \widetilde{S} \to \mathbb{P}^1$ comes equipped with r sections, one for each base point.

It is known that all nonsingular fibres $f^{-1}(\lambda)$ (resp. $\widetilde{f}^{-1}(\lambda)$) are connected Riemann surfaces of a constant genus g, which is usually referred to as the *genus of the pencil* (see e.g. [12], 8.1). Similarly the pair (g, r) will be called the type of the pencil. Accordingly a pencil will be said *of hyperbolic type* if $2g-2+r > 0$.

Having established the fact that algebraic complex surfaces admit Lefschetz pencils, the next fact to be observed is that, in the other direction, the results of Kodaira on complex surfaces (see [2]) show that the existence of a L.P. on an arbitrary complex surface forces that surface to be algebraic. Since for a complex surface S to be defined over $\overline{\mathbb{Q}}$ it has to be first of all algebraic, it is clear at this stage that Question 1 should be modified as follows.

QUESTION 2 : Is it true that S can be defined over $\overline{\mathbb{Q}}$ if and only if there is a L.P. $f \in M(S)$ such that $\mathrm{crit}(f) \subset \mathbb{P}^1(\overline{\mathbb{Q}})$**?**

It turns out that except for a very restricted class of complex surfaces the answer to this second question is going to be affirmative.

9 Monodromy of Lefschetz pencils

As in the case of morphisms between Riemann surfaces, Lefschetz pencils have also a monodromy attached to them.

9.1 Definition of the monodromy homomorphism

In analogy with Section 3.3 the monodromy of a L. P. $f : S \dashrightarrow \mathbb{P}^1$, with $\mathrm{crit}(f) = \{q_i\}$ is going to be a group homomorphism Mon from $\pi_1\left(\mathbb{P}^1 \setminus \{q_i\}\right)$ to the group of bijections of a nonsingular fibre $\mathrm{Bij}\left(\tilde{f}^{-1}(q)\right)$ (or $\mathrm{Bij}\left(f^{-1}(q)\right)$, depending on which of the two families of Riemann surfaces associated to our Lefschetz pencil we are working with).

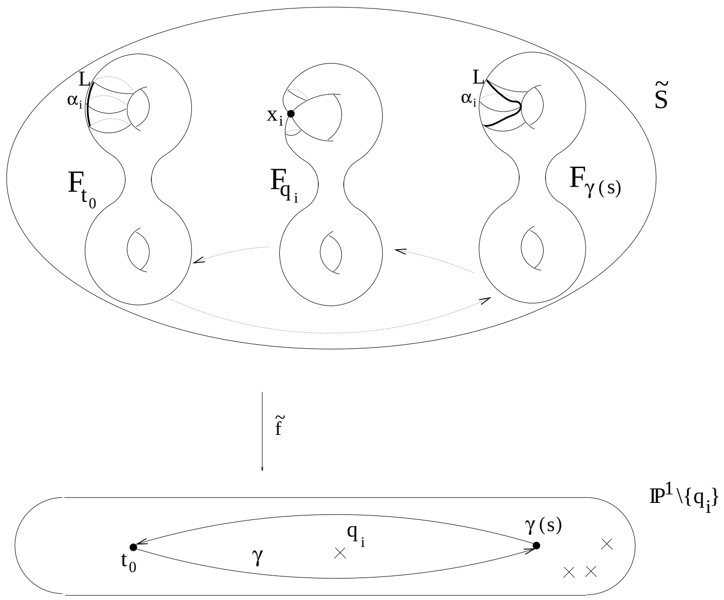

Figure 14: The monodromy of a Lefschetz pencil. As we walk along γ, the curve α_i rotates by 360 degrees.

Let us discuss the construction of this monodromy homomorphism for the family $\tilde{f} : \widetilde{S} \to \mathbb{P}^1 \setminus \{q_i\}$ of Riemann surfaces of genus g, the other case being similar. We choose first a base point $t_0 \in \mathbb{P}^1 \setminus \{q_i\}$ with fibre F_{t_0}. Since from the topological point of view this family of Riemann surfaces is locally trivial, it is clear that for any given loop $\gamma : I = [0,1] \to \mathbb{P}^1 \setminus \{q_i\}$ with base point $\gamma(0) = \gamma(1) = t_0$, one can define a continuous family of orientation preserving homeomorphisms $\left\{\varphi_s : F_{t_0} \to F_{\gamma(s)}\right\}_{s\in I}$ such that φ_0 is the identity map. For $t = 1$ we obtain a bijection $\varphi_\gamma : F_{t_0} \to F_{t_0}$ that defines our group element $\varphi_\gamma \in \mathrm{Bij}(F_{t_0})$.

Moreover, by definition, φ_γ is not a mere bijection of F_{t_0}. It is actually an orientation preserving homeomorphism of F_{t_0}, thereby giving rise to a mapping class $[\varphi_\gamma] \in \mathrm{Mod}_g$. Therefore we have a well defined group homomorphism

$$\begin{array}{rcl} \mathrm{Mon} : \pi_1\left(\mathbb{P}^1 \setminus \{q_i\}\right) & \longrightarrow & \mathrm{Mod}_g \\ \gamma & \longmapsto & [\varphi_\gamma] \end{array}$$

which will be referred to as the *monodromy of the Lefschetz pencil* ([7], [12], [6]). This same terminology will be used to refer to the companion homomorphism

$$\pi_1\left(\mathbb{P}^1 \setminus \{q_i\}\right) \longmapsto \mathrm{Mod}_{g,r}$$

arising in a parallel way from the family of Riemann surfaces of type (g, r) given by $f : S \setminus B \mapsto \mathbb{P}^1 \setminus \{q_i\}$.

It is well known that Mod_g is generated by *Dehn twists* along nontrivial loops of our reference Riemann surface F_{t_0} and in fact the homomorphism Mon can be described in terms of these generators in a very explicit manner.

To do that we associate to each critical value q_i, $i = 1, \ldots, d$, the following objects:

1. A simple loop γ_i in $\mathbb{P}^1 \setminus \{q_i, t_0\}$ as in Figure 4.
2. A curve $\alpha_i \subset F_{t_0}$ which shrinks to the critical point x_i as we *retract* F_{t_0} to the singular fibre F_{q_i} (α_i is the so called *vanishing cycle* corresponding to the node x_i).
3. The (right) Dehn twist D_{α_i} along the curve α_i. This is a homeomorphism of F_{t_0} which rotates 360 degrees the curve α_i, and keeps fixed the complement of a small tubular neighborhood of α_i.

It can be seen ([7]) that $\mathrm{Mon}(\gamma_i) = D_{\alpha_i}$. Of course, the identity $\prod \gamma_i = 1$ implies that $\prod D_{\alpha_i} = 1$.

We will sometimes write

$$\mathrm{Mon} = (D_{\alpha_1}, \cdots, D_{\alpha_d})$$

which is analogous to the representation (2) in Section 3.3.

From the point of view of Teichmüller theory the monodromy map can be interpreted in a very interesting way. As usual, let us denote by $\mathcal{M}_{g,r}$ the moduli space of Riemann surfaces of type (g, r). This is a complex analytic space whose points correspond bijectively to isomorphic classes of Riemann surfaces of type (g, r). It is a fundamental fact in analytic Teichmüller theory (see [24], [5]) that $\mathcal{M}_{g,r}$ can be obtained as the quotient of Teichmüller space $\mathcal{T}_{g,r}$ by the group of all its biholomorphic automorphisms which turns out to be $\mathrm{Mod}_{g,r}$. In other words we have $\mathcal{M}_{g,r} = \mathcal{T}_{g,r}/\mathrm{Mod}_{g,r}$. Of course when $r = 0$ we simply write $\mathcal{T}_{g,0} = \mathcal{T}_g$, $\mathrm{Mod}_{g,0} = \mathrm{Mod}_g$ and $\mathcal{M}_{g,0} = \mathcal{M}_g$. By the very nature of moduli space there is a holomorphic map $\phi : \mathbb{P}^1 \setminus \{q_i\} \to \mathcal{M}_{g,r}$, *the classifying map*, defined by sending an arbitrary point $q \in \mathbb{P}^1 \setminus \{q_i\}$ to the point $\phi(q) \in \mathcal{M}_{g,r}$ representing the Riemann surface F_q. Let now Γ be the Fuchsian group uniformizing $\mathbb{P}^1 \setminus \{q_i\}$ so that $\mathbb{P}^1 \setminus \{q_i\} \simeq \mathbb{H}/\Gamma$ and $\Gamma \simeq \pi_1(\mathbb{P}^1 \setminus \{q_i\})$. Then the map $\phi : \mathbb{P}^1 \setminus \{q_i\} = \mathbb{H}/\Gamma \to \mathcal{M}_{g,r} = \mathcal{T}_{g,r}/\mathrm{Mod}_{g,r}$ lifts to a holomorphic map $\widetilde{\phi} : \mathbb{H} \to \mathcal{T}_{g,r}$ and the monodromy morphism Mon is nothing but the homomorphism

$$\mathrm{Mon} = \phi_* : \Gamma \to \mathrm{Mod}_{g,r}$$

defined by the property

$$\widetilde{\phi} \circ \gamma(z) = \phi_*(\gamma)(\widetilde{\phi}(z)), \text{ for any } z \in \mathbb{H}. \tag{3}$$

Of course the choice of a different lift $\varphi \circ \widetilde{\phi}$, for some $\varphi \in \mathrm{Mod}_{g,r}$, would result on a conjugate homomorphism $(\varphi \circ \widetilde{\phi})_*(\gamma) = \varphi \circ \phi_*(\gamma) \circ \varphi^{-1}$.

9.2 The monodromy characterizes the pencil

As in the case of mappings between Riemann surfaces, Lefschetz pencils too are characterized by their monodromies. This is a very interesting result due to Imayoshi and Shiga. To make it precise we need some terminology.

A L.P. of type (g, r), $f : S \dashrightarrow \mathbb{P}^1$, is said to be *locally trivial* if the corresponding classifying map $\phi : \mathbb{P}^1 \setminus \{q_i\} \to \mathcal{M}_{g,r}$ is constant; that is, if the fibres are all isomorphic to each other.

Theorem 17 ([18]) *A non locally constant L.P. of hyperbolic type (g, r) is determined by its monodromy morphism* $\mathrm{Mon} : \pi_1(\mathbb{P}^1 \setminus \{q_i\}) \to \mathrm{Mod}_{g,r}$ *up to finitely many choices.*

The main idea of the proof is as follows. By construction, the classifying map ϕ determines the family of Riemann surfaces (perhaps up to a finite number of choices if the generic fibre has nontrivial automorphisms). Therefore, with the preceding notation, all one has to show is that $\mathrm{Mon} = \phi_*$ determines the map $\widetilde{\phi}$, hence the map ϕ.

The strategy to do this is rather standard. The map $\widetilde{\phi}$ is determined by its limit values at the points $\xi \in \partial\mathbb{H}$, so one has to show that these limit values are determined by ϕ_*. For that, one writes $\xi = \lim \gamma_n(z)$ for a suitable sequence of elements $\gamma_n \in \Gamma$ and, in view of equation (3), one sets $\widetilde{\phi}(\xi) = \lim \phi_*(\gamma_n)(\widetilde{\phi}(z))$.

9.3 Finiteness (Arakelov's theorem)

In the case of ramified coverings of $\mathbb{P}^1$, dealt with in the first part of the article, the finiteness of the number of monodromies

$$\mathrm{Mon} : \pi_1(\mathbb{P}^1 \setminus \{q_i\}) \to \mathbb{S}_d$$

was an obvious consequence of the finiteness of the symmetric group $\mathbb{S}_d$.

On the contrary, in the case of Lefschetz pencils the target group $\mathrm{Mod}_{g,r}$ is an infinite group and consequently there are infinititly many homomorphisms from $\pi_1(\mathbb{P}^1 \setminus \{q_i\}$ to $\mathrm{Mod}_{g,r}$; one for each r-tuple $(D_1, \cdots, D_d)$ of Dehn twists (or, for that matter, arbitrary elements of $\mathrm{Mod}_{g,r}$) satisfying $\prod D_i = 1$. However, only finitely many of them may arise as monodromy maps of L.P. with $\mathrm{crit}(f) = \{q_i\}$.

Theorem 18 *Let $2g - 2 + r > 0$. Then, up to conjugation, only finitely many group homomorphisms*

$$\chi : \pi_1(\mathbb{P}^1 \setminus \{q_i\}) \to \mathrm{Mod}_{g,r}$$

arise as the monodromy map of a L.P. $f : S \dashrightarrow \mathbb{P}^1$ with $\mathrm{crit}(f) = \{q_i\}$.

By Section 9.1 the above theorem is equivalent to the statement that there are only finitely many non constant holomorphic maps $\phi : \mathbb{P}^1 \setminus \{q_i\} \to \mathcal{M}_{g,r}$. This is actually a very deep result. What lies behind it is nothing less that Arakelov's theorem [1], which states the finiteness of families of compact Riemann surfaces over a a Riemann surface of finite type. This result is known in the literature by several other names: Parshin-Arakelov theorem, Mordell Conjecture for the function field case, Geometric Shafarevich Conjecture, etc. The particular presentation above comes from a proof of a more general version of Arakelov's theorem, which includes also the case of families of Riemann surfaces with punctures, due to Imayoshi and Shiga [18] (see also the nice account given by Macmullen in [23]). The special feature about this proof is that it is carried out entirely within the framework of Teichmüller theory. And in fact, most of the fundamental facts of the analytic theory of Teichmüller spaces are involved in the proof. To mention some: the boundedness of $\mathcal{T}_{g,r}$ inside $\mathbb{C}^{3g-3+r}$, pants decompositions of hyperbolic surfaces, Mumford's compactness theorem, the equality of Teichmüller and Kobayashi distances on $\mathcal{T}_{g,r}$, the fact that the group of isometries of $\mathcal{T}_{g,r}$ coincides with the mapping class group $\mathrm{Mod}_{g,r}$, etc.

10 Belyi's theorem for complex surfaces

In order to formulate our characterization of complex surfaces definable over $\overline{\mathbb{Q}}$ we need one more definition.

Definition 19 By a *Lefschetz function* we shall refer to a meromorphic function $h \in \mathcal{M}(S)$ obtained as composition of a Lefschetz pencil $f : S \dashrightarrow \mathbb{P}^1$ with a rational function $\beta : \mathbb{P}^1 \to \mathbb{P}^1$.

Theorem 20 ([10]) *Let S be a minimal complex surface S which is not a non-rational ruled surface. The following statements are equivalent.*

a) S can be defined over a number field.

b) S admits a Lefschetz pencil $f \in \mathcal{M}(S)$ with critical values $q_1, ..., q_r$ in $\mathbb{P}^1(\overline{\mathbb{Q}})$.

c) S admits a Lefschetz function $h \in \mathcal{M}(S)$ with three critical values, say $0, 1, \infty$.

For a nonrational ruled surface $p : S \to C$ the result is still true if in b) and c) we make the additional requirement that the curve C and the projection of the critical points of f on it are also defined over a number field.

We recall that S is called *rational* (resp. *ruled)* if it is bimeromorphically equivalent to $\mathbb{P}^2$ *(resp.* $C \times \mathbb{P}^1$, for some Riemann surface C). A ruled surface is rational if and only if $C \equiv \mathbb{P}^1$.

In view of Proposition 15, the above result can be reformulated as follows

Theorem 21 ([10]) *A complex surface S can be defined over a number field if and only if it is either a minimal surface admitting a Lefschetz function $h = \beta \circ f$ with only three critical values, or it is obtained from one such surface by a finite sequence of blow-ups centered at points with coordinates in $\overline{\mathbb{Q}}$. In the case of nonrational ruled surfaces $p : S \to C$, we must in addition require that the curve C and the image on it of the critical points of the Lefschetz pencil f are defined over a number field too.*

10.1 The proof of Theorem 20

$b \Leftrightarrow c$)

What this equivalence means is that if there exists a L.P. $f_1 \in \mathcal{M}(S)$ and a rational function $\beta : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\beta \circ f_1$ has only three critical values then there is a L.P. $f \in \mathcal{M}(S)$ such that $\mathrm{crit}(f) \subset \mathbb{P}^1(\overline{\mathbb{Q}})$. This can be seen by suitable application of the classical Belyi theorem.

$a \Rightarrow b$)

This part is easy. As we have recalled in Section 8, Bertini's theory provides S with a Lefschetz pencil structure. Moreover if S is defined over $\overline{\mathbb{Q}}$ and we take our pencil of hyperplanes $\{H_\lambda\}_\lambda$ defined also over $\overline{\mathbb{Q}}$, then the corresponding Lefschetz pencil will have critical points x_i and critical values q_i defined over $\overline{\mathbb{Q}}$ too.

$b \Rightarrow a$)

As it has been mentioned before, the existence of a L.P. $f \in \mathcal{M}(S)$ ensures that S is algebraic. What remains to be shown is that if, moreover, $\mathrm{crit}(f) = \{q_i\} \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ then S can be defined over a number field. As in the Riemann surface case this can be achieved in three steps.

Step 1. A non locally constant family of punctured Riemann surfaces of hyperbolic type $f : S \smallsetminus B \to \mathbb{P}^1$ is determined by its monodromy homomorphism $\mathrm{Mon} : \pi_1(\mathbb{P}^1 \smallsetminus \{q_i\}) \to \mathrm{Mod}_{g,r}$ up to finitely many choices.

Step 2. There are only finitely many monodromy homomorphisms

$$\chi : \quad \pi_1(\mathbb{P}^1 \smallsetminus \{q_i\}) \quad \to \mathrm{Mod}_{g,r}$$

arising from a Lefschetz pencil on a minimal complex surface with pre-assigned critical values $\{q_i\}$.

Step 3. Apply Criterion 13.

Steps 1 and 2 correspond to Theorems 17 and 18 respectively. As for Step 3, one has only to observe that the condition $\mathrm{crit}(f) \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ obviously implies that the family of critical value sets $\mathrm{crit}(f^\sigma) = \{q_i^\sigma\}$ obtained by letting the group $\mathrm{Gal}(\mathbb{C})$ act on the pair (S, f) contains only finitely many different sets.

In the above discussion two cases have been excluded, namely families of Riemann surfaces of type $(0, r)$, $r \leq 3$ and locally constant families.

In the first instance S is a rational surface, thus S is either $\mathbb{P}^2$ or one of the countably many ruled surfaces F_n of Hirzebruch (see [2] or [3]) which are known to be defined over $\overline{\mathbb{Q}}$. The second case accounts for the ruled surfaces whose exceptional behavior is contemplated in Theorem 20.

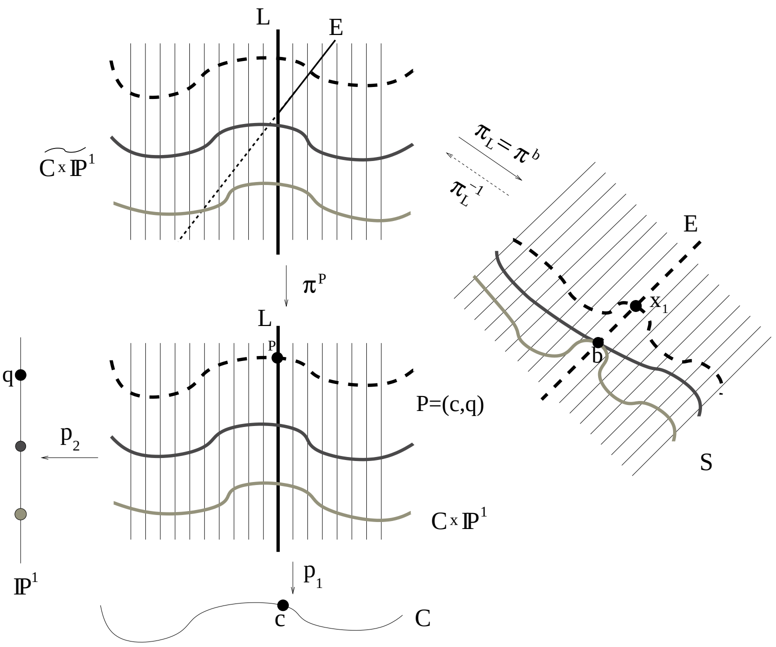

Figure 15: The Lefschetz pencil $f = p_2 \circ \pi^P \circ \pi_L^{-1}$ on the ruled surface S.

This behaviour is what Figure 15 is intended to illustrate. There, S is a ruled surface obtained by first blowing up one or several points $P = P_i = (c_i, q_i) \in C \times \mathbb{P}^1$ and then blowing down the line of the ruling on which P lies, which the blowing up map π^P has turned into an exceptional line. This process is known as an elementary transformation ([2]). In Figure 15 we use the same notation L for this line in $C \times \mathbb{P}^1$ and for its strict transform in $\widetilde{C \times \mathbb{P}^1}$. The composition $f = p_2 \circ \pi^P \circ \pi_L^{-1}$ defines a L.P. on S with $\mathrm{crit}(f) = \{q_i\}$. If the points P_i are chosen so that $q_i \in \overline{\mathbb{Q}}$, then this L.P. satisfies the requirements in Theorem 20.

But if the points $c_i \in C$ are trascendental there is no reason to think that S should be definable over a number field, even if C is.

We close this section by mentioning related work done independently by I. Ronkine [26]. Also with related aim (but different point of view) is the article by [25] by Paranjape.

11 Belyi's theorem via Griffiths uniformization

According to Theorem 1, a compact Riemann surface C is a Belyi surface if and only if there is a finite set $\Sigma \subset C$ such that $C \setminus \Sigma$ is isomorphic to a quotient of the form $\mathbb{H}/\Gamma$ where Γ is a torsion free finite index subgroup of $\mathbb{PSL}_2(\mathbb{Z})$.

This characterization may be regarded as a manifestation of how the arithmetic nature of an algebraic curve is reflected in that of its uniformizing group. This phenomenon can be also paralleled in the 2-dimensional case. Now the role of Fuchsian uniformization of algebraic curves is going to be played by Griffiths uniformization of algebraic surfaces.

11.1 Uniformization of certain Zariski open sets of an algebraic surface

A domain $\mathbb{B}$ in $\mathbb{C}^2$ is called a *Bergman domain* if it is the set of pairs (t, z) such that $t \in \mathbb{H}$, and $z \in D_t$ where D_t is a bounded Jordan domain whose boundary curve admits a parametric representation

$$z = W(t, e^{i\theta}),\ 0 \leq \theta \leq 2\pi$$

W being, for each fixed θ, a holomorphic function of t (see [5]). We can thus write $\mathbb{B} = \cup_{t\in\mathbb{H}} D_t$. It is clear that by choosing an isomorphism between the upper half plane and the unit disc $\mathbb{D}$ we can obtain an equivalent domain $\mathbb{B}_1$ in which $\mathbb{H}$ is replaced by $\mathbb{D}$. We shall say that $\mathbb{B}$ is bounded if $\mathbb{B}_1$ is.
By a *Bers transformation* of a Bergman domain $\mathbb{B} \subset \mathbb{H} \times \mathbb{C}$ we shall mean a holomorphic isomorphism $g(t, z) = (\widehat{g}(t), g_t(z))$ where $\widehat{g} \in \mathbb{PSL}_2(\mathbb{R})$ is a real Möbius transformation and $g_t : D_t \to D_{\widehat{g}(t)}$ is a biholomorphic map. If G is a group of Bers transformations acting freely on $\mathbb{B}$, there is an obvious short exact sequence

$$1 \hookrightarrow K \to G \xrightarrow{\rho} \Gamma \to 1$$

where the epimorphism ρ is defined by $\rho(g) = \widehat{g}$. We see that while Γ acts on $\mathbb{H}$, the group K acts freely on each simply connected region D_t as a Kleinian group K_t whose quotient space D_t/K_t is a Riemann surface. We will say that G is a *Bers-Griffiths extension* of Γ if for each $t \in \mathbb{H}$ the Riemann surface D_t/K_t is of finite hyperbolic type (p, r). In that case K itself will be said to be of type (p, r).

It is clear that in this situation G gives rise to a holomorphic fibration $f : \mathbb{B}/G \to \mathbb{H}/\Gamma$ whose fibre over a coset $[t] \in \mathbb{H}/\Gamma$, $t \in \mathbb{H}$, is the Riemann surface D_t/K_t. We shall be interested in the case in which Γ is a Fuchsian group of finite type.

Definition 22 We shall say that a 2-dimensional complex manifold U admits a Griffiths uniformization if its holomorphic universal cover $\widetilde{U} \to U$ is isomorphic to a bounded Bergman domain of $\mathbb{C}^2$ and its covering group G is a Bers-Griffiths extension of some Fuchsian group of finite type.

Griffiths uniformization theorem for algebraic surfaces states that every algebraic surface contains a Zariski open set which admits a Griffiths uniformization ([13], see also [5]).

11.2 Characterization of complex surfaces defined over $\overline{\mathbb{Q}}$ via Griffiths uniformization

Theorem 23 ([10]) *A minimal non ruled surface $S \subset \mathbb{P}^n(\mathbb{C})$ can be defined over a number field if and only if it contains a Zariski open set U admitting a Griffiths uniformization such that the uniformizing group G is a Bers-Griffiths extension of a genus zero finite index torsion free subgroup of $\mathbb{P}\mathrm{SL}_2(\mathbb{Z})$.*

The proof goes as follows. Let $f : S \dashrightarrow \mathbb{P}^1$ be a Lefschetz pencil with base locus $B = \{b_1, ..., b_r\}$, critical points $\{x_1, ..., x_d\}$ and critical values $\{q_1, ..., q_d\}$ in $\mathbb{P}^1(\overline{\mathbb{Q}})$. Then, by restriction to the regular values, one obtains a a family of *r-punctured* nonsingular Riemann surfaces $f : U \to \mathbb{P}^1 \smallsetminus \{q_i\}$, where $U \subset S$ is the Zariski open set $U = (S \smallsetminus B) \smallsetminus f^{-1}\{q_i\}$. Moreover the genus p of the fibres must be strictly positive, for otherwise the surface would be ruled.

Let $\Gamma \subset \mathbb{P}\mathrm{SL}_2(\mathbb{R})$ be the Fuchsian group uniformizing $\mathbb{P}^1 \smallsetminus \{q_i\}$. Then the results in [13] imply that U admits a Griffiths uniformization such that the covering transformations group G is a Bers-Griffiths extension of Γ by a group K of type (p, r). We would like to show that $\Gamma \subset \mathbb{P}\mathrm{SL}_2(\mathbb{Z})$. But, the values q_i being algebraic numbers, this is a straightforward consequence of classical Belyi theory.

Conversely, let us assume that our surface S contains a Zariski open set $U \subset S$ admitting a Griffiths uniformization $U \simeq \widetilde{U}/G$, where G is a Bers-Griffiths extension of a finite index torsion free subgroup $\Gamma \subset \mathbb{P}\mathrm{SL}_2(\mathbb{Z})$ of genus zero by a group of type (p, r) with $2p - 2 + r > 0$. Then, as noted above, the action of G induces a holomorphic fibration of Riemann surfaces of finite hyperbolic type $f : U \to \mathbb{P}^1 \smallsetminus \{q_i\}$ where $\mathbb{P}^1 \smallsetminus \{q_i\}$ is isomorphic to $\mathbb{H}/\Gamma$, hence, by the classical Belyi theorem, we can assume $\{q_i\} \subset \mathbb{P}^1(\overline{\mathbb{Q}})$. At this point one would like to bring the proof to an end by extending f to a L.P. on S whose singular fibres occur at the points q_i. However, this does not seem to be entirely straightforward. So, instead, using first compactification techniques of Imayoshi ([16], see also [15] and [17]) and then resolution of singularities, one extends f to a morphism

$$Y \xrightarrow{\pi} \widehat{U} \xrightarrow{f} \mathbb{P}^1$$

where $\widehat{U}$ is a compact normal surface bimeromorphic to S and $\pi : Y \to \widehat{U}$ is a resolution of singularities of $\widehat{U}$ (see e.g. [2] III.6.1). We see that Y is a surface bimeromorphic to $\widehat{U}$, and hence to the algebraic surface S. Therefore Y is a projective surface birationally equivalent to S. In fact, being non ruled, S is the unique minimal model of Y. Moreover, by the GAGA principle, the map $f_1 = f \circ \pi : Y \to \mathbb{P}^1$ is a regular map. In particular the action of the Galois group $\mathrm{Gal}(\mathbb{C})$ on $f_1 : Y \to \mathbb{P}^1$ is well defined. Thus, we are again in position to apply Criterion 13 which together with a suitable version of Arakelov's Theorem allows one to conclude the desired result.

Remark 24 It should be noted that Theorem 23 does not hold for arbitrary minimal ruled surfaces $p : S \to C$, even if the base curve C is defined over $\overline{\mathbb{Q}}$. This can be seen as follows. Any such surface contains a Zariski open set $V \simeq (C \setminus \Sigma) \times \mathbb{P}^1$, with Σ a finite subset of C, therefore, it surely contains a smaller one of the form $U \simeq (C \setminus \Sigma) \times (\mathbb{P}^1 \setminus \{0, 1, \infty\})$. The latter admits a Griffiths uniformization with uniformizing group $G \simeq \Gamma(2) \times K$, where K is the Fuchsian group uniformizing $(C \setminus \Sigma)$, so that $C \setminus \Sigma = \mathbb{D}/K$, and the action of G is the obvious product action on $\mathbb{H} \times \mathbb{D}$. But, on the other hand, the moduli space of minimal ruled surfaces over a given curve C of genus g is known to depend on $3g - 3$ complex parameters (see [3]), thus, for mere cardinality reasons, not all of them can be defined over $\overline{\mathbb{Q}}$.

11.3 Dessins on 4-manifolds?

Having established for complex surfaces a result analogous to Belyi's theorem for complex curves, one wonders how far the analogy goes. In particular, since we spent a good deal of the first part of this paper discussing the notion of dessins d'enfants and the action of $\mathrm{Gal}(\overline{\mathbb{Q}})$ on them, it is natural to ask if this theory can also be extended to complex dimension two. Let us make more precise what we mean by that.

1.- First of all, given a compact oriented 4-manifold M, one would like to single out a distinguished class of suitable topological decompositions of M (that would play the role of dessins) which in a natural way correspond to pairs (M, f) where f is a differentiable map of a certain type from M to $\mathbb{S}^2$.

This first requirement is satisfactorily solved by the so called topological Lefschetz pencils.

Definition 25 A *topological Lefschetz pencil* (T.L.P.) $f : M \dashrightarrow \mathbb{S}^2$ on a four-manifold comprises an oriented four-manifold M and a surjective map f defined on the complement of a finite set $B = \{b_i, i = 1, \ldots, r\}$ to the sphere with finitely many critical points $\{x_j, j = 1, \ldots, d\}$, all in distinct fibres, such that

i) Near each x_j there are local complex coordinates with respect to which the map f takes the form $f(z_1, z_2) = z_1^2 + z_2^2$.

ii) Near each b_i there are complex coordinates with respect to which the map f takes the form $f(z_1, z_2) = z_1/z_2$.

From the topological point of view L.P and T.L.P. are of course indistinguishable.

Using the function $|f|^2$ as a Morse function on M (or rather the four-manifold $\widetilde{M}$ obtained by blowing up the points $\{b_i\}$), Kas ([19]) gave a handlebody decomposition $\mathcal{D}$ of M on which the elementary pieces are of the form $\Sigma_g \times D$, where D is the topological disc and Σ_g is the topological surface homeomorphic to the generic fibre. We can call this type of handlebody decomposition of a four-manifold M a *Kas handlebody decomposition*.

It is known ([19], [6], [12]) that there is an essentially bijective correspondence between Kas handlebody decompositions $\mathcal{D}$, T.L.P. (M, f) and monodromy representations Mon $= (D_{\alpha_1}, \ldots, D_{\alpha_d})$ as defined in Section 9.1. Thus, it looks natural to reserve the term *four-dimensional dessin* to any of the following equivalent objects

$$\mathcal{D} \equiv (M, f) \equiv (D_{\alpha_1}, \ldots, D_{\alpha_d}) .$$

2.- Once we have decided what a possible candidate for the concept of four-dimensional dessins could be, one needs to be able to associate to each dessin $\mathcal{D}$ a holomorphic structure on M for which f becomes a holomorphic L.P. This will make of M a projective variety on which the action of Gal($\overline{\mathbb{Q}}$) would make sense.

This is the point where difficulties emerge for not all T.L.P. can be endowed with a complex structure. One way to see this is that, because of fundamental work by Donaldson [6] and Gompf [11], one knows that T.L.P. are in correspondence with symplectic four-manifolds. But, on the other hand, as it was first observed by Thurston ([29]), not all symplectic four-manifolds are Kähler manifolds.

Thus, it seems that in order to make sense of four-dimensional dessins one needs first to understand which monodromies $\mathcal{D} \equiv (D_{\alpha_1}, \ldots, D_{\alpha_d})$ arise as monodromies of (complex) L.P.

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