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Automorphisms group of generalized Fermat curves of type $(k, 3)$

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ABSTRACT

The determination of the full group of automorphisms of a closed Riemann surface is in general a very complicated task. For hyperelliptic curves, the uniqueness of the hyperelliptic involution permits one to compute these groups in a very simple manner. Similarly, as classical Fermat curves of degree k admit a unique subgroup of automorphisms isomorphic to $\mathbb{Z}_k^2$, the determination of the group of automorphisms is not difficult.

In this paper we consider a family of non-hyperelliptic Riemann surfaces, obtained as the fibre product of two classical Fermat curves of the same degree k, which exhibit behaviors of both elliptic and hyperelliptic curves. These curves, called generalized Fermat curves of type $(k, 3)$, are the highest regular abelian branched covers of orbifolds of genus zero with four cone points, all of the same order k. More precisely, a generalized Fermat curve of type $(k, 3)$ is a closed Riemann surface S admitting a group $H < \mathrm{Aut}(S)$, called a generalized Fermat group of type $(k, 3)$, so that $H \cong \mathbb{Z}_k^3$ and S/H is an orbifold with signature $(0, 4; k, k, k, k)$. In this paper we prove the uniqueness of generalized Fermat groups of type $(k, 3)$. In particular, this allows the explicit computation of the full group of automorphisms of S.

1. Introduction and main result

If S is a Riemann surface, then we denote by $\mathrm{Aut}(S)$ its group of conformal automorphisms and by $\widehat{\mathrm{Aut}}(S)$ its group of conformal/anticonformal automorphisms. If $K < \mathrm{Aut}(S)$, then we denote by $\mathrm{Aut}_K(S)$ the normalizer of K in $\mathrm{Aut}(S)$. If S is a closed Riemann surface of genus $g \geq 2$, then H.A. Schwarz [22] proved that $\mathrm{Aut}(S)$ is finite and A. Hurwitz [13] that $|\mathrm{Aut}(S)| \leq 84(g-1)$, so $|\widehat{\mathrm{Aut}}(S)| \leq 168(g-1)$. W. Baily [1] showed that in genus $g \geq 2$ closed Riemann surfaces with non-trivial group of conformal automorphisms are very special. Breuer [3] provided a classification of all finite groups which can act as the group of conformal automorphisms up to genus 48. Natural problems are (i) to describe those groups which occur as the full group of conformal automorphisms of a genus g Riemann surface and (ii) to determine the group of automorphisms of a given Riemann surface. Closed Riemann surfaces can be represented as algebraic curves in projective space in many ways, the most important one being the canonical curve which, in the case of non-hyperelliptic Riemann surfaces, is determined by the choice of a basis of holomorphic 1-forms [8]. Although sometimes the use of a concrete curve description of the Riemann surface permits the construction of explicit examples of automorphisms, determining whether these automorphisms constitute the full automorphism group is usually intractable. In the literature there are few examples

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of Riemann surfaces whose defining algebraic equations and full automorphism groups are completely determined (see, for instance, [11,16–18,28,30]).

If S is a hyperelliptic Riemann surface and H is the cyclic group generated by the hyperelliptic involution, then H is unique in $\mathrm{Aut}(S)$ [4]. Also, if S is a classical Fermat curve of degree k (which is not hyperelliptic for $k \geq 4$), that is, $S = \{x_1^k + x_2^k + x_3^k = 0\} \subset \mathbb{P}^2$, then the group $H < \mathrm{Aut}(S)$ obtained by multiplying the coordinates by k-th roots of unity is the unique subgroup $K < \mathrm{Aut}(S)$ with the properties that $K \cong \mathbb{Z}_k^2$ and S/K is an orbifold with signature $(0, 3; k, k, k)$. In these two cases, the uniqueness of the corresponding group H permits one to obtain explicitly the groups $\mathrm{Aut}(S)$ and $\widehat{\mathrm{Aut}}(S)$ [24,29].

A closed Riemann surface S is called a generalized Fermat curve of type (k, n), where $k \geq 2$ and $n \geq 1$ are integers, if it admits a group $\mathbb{Z}_k^n \cong H < \mathrm{Aut}(S)$, called a generalized Fermat group of type (k, n), so that S/H is an orbifold with signature $(0, n + 1; k, \dots, k)$ (that is, S/H is the Riemann sphere $\widehat{\mathbb{C}}$ together with $n + 1$ conical points, each one of order k); the pair (S, H) is called a generalized Fermat pair of type (k, n). By the Riemann–Hurwitz formula, the genus of S is $g_{k,n} = (2 + k^{n-1}((n - 1)(k - 1) - 2))/2$ and, in particular, $g_{k,n} > 1$ except for the pairs $(k, n) \in \{(k, 1), (2, 2), (2, 3)\}$. A generalized Fermat curve of type $(k, 1)$ (respectively, of type $(2, 2)$) is just the Riemann sphere, with H being the cycle group generated by a Möbius transformation of order k (respectively, $H \cong \mathbb{Z}_2^2$). A generalized Fermat curve of type $(k, 2)$ is a classical Fermat curve and every genus one Riemann surface is a generalized Fermat curve of type $(2, 3)$.

If $g_{k,n} > 1$, then it was noticed in [10] that if Γ is a Fuchsian group so that $\mathbb{H}^2/\Gamma = S/H$, then $S = \mathbb{H}^2/\Gamma'$, where Γ' is the commutator subgroup of Γ. In particular, if M is the subgroup of Möbius transformations that keep invariant the conical points of S/H, then there is a natural epimorphism $\rho : \mathrm{Aut}_H(S) \to M$, whose kernel is H. Another consequence is that if (S_1, H_1) and (S_2, H_2) are both generalized Fermat pairs of the same type, then there is an orientation-preserving homeomorphism $F : S_1 \to S_2$ so that $F_1 H_1 F_1^{-1} = H_2$.

In [10] it was proven that every generalized Fermat curve of type (k, n), where $n \geq 2$, can be described as the fibre product of $(n - 1)$ classical Fermat curves so that the corresponding generalized Fermat group (and its normalizer in the group of conformal automorphisms) is linear. That description is as follows. Assume (S, H) is a generalized Fermat pair of type (k, n) and let $\pi : S \to \widehat{\mathbb{C}}$ be a branched regular covering with H as group of covering transformations. Up to composition with a Möbius transformation, we may assume that the branched values are given by the points $\{\infty, 0, 1, \lambda_1, \dots, \lambda_{n-2}\}$. Then (S, H) is conformally equivalent to $(S^k_{\lambda_1,\dots,\lambda_{n-2}}, H_0)$, where

$$S^k_{\lambda_1,\dots,\lambda_{n-2}} = \left\{ \begin{array}{rcl} x_1^k + x_2^k + x_3^k & = & 0 \\ \lambda_1 x_1^k + x_2^k + x_4^k & = & 0 \\ \lambda_2 x_1^k + x_2^k + x_5^k & = & 0 \\ \vdots & \vdots & \vdots \\ \lambda_{n-2} x_1^k + x_2^k + x_{n+1}^k & = & 0 \end{array} \right\} \subset \mathbb{P}^n \tag{1}$$

and $H_0 = \langle a_1, \dots, a_n \rangle$, where

$$a_j([x_1 : \cdots : x_{n+1}]) = [x_1 : \cdots : x_{j-1} : \omega_k x_j : x_{j+1} : \cdots : x_{n+1}], \quad j = 1, \dots, n, \tag{2}$$

with $\omega_k = e^{2\pi i/k}$, and the covering map is given by

$$\pi([x_1 : \cdots : x_{n+1}]) = -\left(\frac{x_2}{x_1}\right)^k \tag{3}$$

whose branch values certainly are

$$\infty, 0, 1, \lambda_1, \dots, \lambda_{n-2}. \tag{4}$$

Observe that the above smooth projective algebraic curve is of degree k^{n-1}. As the degree of the canonical curve is $2(g_k - 1) = k^{n-1}((k - 1)(n - 1) - 2)$, the above algebraic representation has lower degree than the canonical curve representation. If $g_{k,n} > 1$, then equality only holds for $(k, n) \in \{(2, 4), (4, 2)\}$. The case $(k, n) = (2, 4)$ corresponds to the so-called Humbert curves [6,7], in which case the generalized Fermat group of type $(2, 4)$ is unique and therefore the computation of $\mathrm{Aut}(S)$ can be easily done [6]. The case $(k, n) = (4, 2)$ corresponds to the classical Fermat curve of genus 3, given by $C = \{x_1^4 + x_2^4 + x_3^4 = 0\} \subset \mathbb{P}^2$, for which $\mathrm{Aut}(S) \cong \mathbb{Z}_4^2 \rtimes \mathfrak{S}_3$ [24,29].

If S is a generalized Fermat curve of type (p, n), where p is a prime and $g_{p,n} > 1$, then it was proved in [10] that any two generalized Fermat groups of same type (p, n) must be conjugate in $\mathrm{Aut}(S)$. Moreover, if the prime p is sufficiently large in comparison with n, then, as a consequence of results in [12], the generalized Fermat group of type (p, n) must be unique; in particular, $\mathrm{Aut}(S) = \mathrm{Aut}_H(S)$ can be explicitly computed. This leads us to conjecture the following.

Conjecture 1. *A generalized Fermat curve of type (k, n), where k is any integer greater than or equal to 2, has a unique generalized Fermat group of type (k, n), up to conjugation.*

In this paper we provide a positive answer to the above conjecture in the case of generalized Fermat curves of type $(k, 3)$, that is, closed Riemann surfaces described by curves of the form

$$S_\lambda^k = \left\{ \begin{matrix} x_1^k + x_2^k + x_3^k = 0 \\ \lambda x_1^k + x_2^k + x_4^k = 0 \end{matrix} \right\} \subset \mathbb{P}^3 \tag{5}$$

Theorem 2. *A generalized Fermat curve S of type $(k, 3)$, where $k \geq 3$ is an integer, admits a unique generalized Fermat group H of type $(k, 3)$. In particular, H is a normal subgroup of* $\mathrm{Aut}(S)$*, and* $\mathrm{Aut}(S)$ *and* $\widehat{\mathrm{Aut}}(S)$ *can be explicitly computed.*

In order to prove Theorem 2, we first prove that H is a normal subgroup and then we use this fact to prove the uniqueness. The proof make uses of Singerman's results [26,25].

Some of the consequences of Theorem 2 (see Section 2) are the following.

(1) Fuchsian groups of signature $(0, 4; k, k, k, k)$ are uniquely determined by their commutator subgroups (Corollary 3).
(2) Explicit descriptions of the groups $\mathrm{Aut}(S)$ and $\widehat{\mathrm{Aut}}(S)$ of each generalized Fermat curve S of type $(k, 3)$ are obtained.

This article is organized as follows. In Section 2 we provide some consequences of Theorem 2. In Section 3 we recall some basic facts about orbifolds and Fuchsian groups. We will especially need Singerman's results in [26,25]. In Section 5 we describe the group $\mathrm{Aut}_H(S)$ for an arbitrary generalized Fermat pair of type $(k, 3)$. In Section 6 we prove that every generalized Fermat group of type $(k, 3)$, $k \geq 3$, is necessarily a normal subgroup. In Section 6.2 we prove Theorem 2. In Section 7 we provide the proof of a key proposition needed in the proof of Theorem 2 whose proof we postponed for the reader's convenience.

2. Consequences of Theorem 2

In this section we provide some consequences of Theorem 2.

2.1. A commutator rigidity

In [10] it was noted that if (S, H) is a generalized Fermat pair of type (k, n), $k \geq 3$, and Γ is a Fuchsian group acting on the hyperbolic plane $\mathbb{H}^2$ so that $\mathbb{H}^2/\Gamma = S/H$, then $S = \mathbb{H}^2/\Gamma'$ and $H = \Gamma/\Gamma'$, where Γ' denotes the commutator subgroup of Γ. In the case $n = 3$, as a direct consequence of Theorem 2, we obtain the following commutator rigidity.

Corollary 3. *For $j = 1, 2$, let Γ_j be Fuchsian groups acting on the hyperbolic plane $\mathbb{H}^2$ so that $\mathbb{H}^2/\Gamma_j$ is an orbifold of signature $(0, 4; k, k, k, k)$ (hence necessarily $k \geq 3$). If $\Gamma_1' = \Gamma_2'$, then $\Gamma_1 = \Gamma_2$.*

Remark 4. Corollary 3 may be seen as a kind of Kleinian version of Torelli's theorem for orbifolds of type $(0, 4; k, k, k, k)$, where $k \geq 3$.

2.2. Equivalence of generalized Fermat curves

As already noted, from the results in [10], every generalized Fermat curve S of type $(k, 3)$, where $k \geq 3$, is conformally equivalent to one of the form S_λ^k, as in (5), for some $\lambda \in \mathbb{C} - \{0, 1\}$.

Consider the group $\mathbb{G} \cong \mathfrak{S}_3$ defined by

$$\mathbb{G} = \langle u(\lambda) = \lambda/(\lambda - 1), v(\lambda) = 1/\lambda \rangle = \mathrm{Aut}(\mathbb{C} - \{0, 1\}).$$

We observe that $\mathbb{G}$ is precisely the covering group of the classical elliptic modular function

$$j(\lambda) = \frac{(1 - \lambda + \lambda^2)^3}{\lambda^2(\lambda - 1)^2}.$$

Every orbifold of genus zero with exactly four conical points (all of them with the same order) is conformally equivalent to one whose conical points are ∞, 0, 1 and some $\lambda \in \mathbb{C} - \{0, 1\}$. Now, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are two such orbifolds, say with conical points $\{0, 1, \infty, \lambda_1\}$ and $\{0, 1, \infty, \lambda_2\}$ respectively, then $\mathcal{O}_1$ and $\mathcal{O}_2$ are isomorphic if and only if there is a Möbius transformation T sending the set $\{\infty, 0, 1, \lambda_1\}$ onto the set $\{\infty, 0, 1, \lambda_2\}$. It is not difficult to see that this is equivalent to the existence of some $t \in \mathbb{G}$ such that $\lambda_2 = t(\lambda_1)$, that is, to have that $j(\lambda_1) = j(\lambda_2)$. When $k = 2$ this amounts to saying that j classifies elliptic curves, a classical fact from which the function takes its name.

By the uniqueness of generalized Fermat groups of type $(k, 3)$ established in Theorem 2, we may show (see Section 3 in [10]) that the equivalence of generalized Fermat curves of type $(k, 3)$ is equivalent to the equivalence of orbifolds of genus zero with four points of equal multiplicity. More precisely, the following fact holds.

Corollary 5. *Two generalized Fermat curves of type $(k, 3)$, $k \geq 3$, say $S_{\lambda_1}^k$ and $S_{\lambda_2}^k$, are conformally equivalent if and only if there is some $t \in \mathbb{G}$ so that $\lambda_2 = t(\lambda_1)$, or equivalently, if and only if $j(\lambda_1) = j(\lambda_2)$. In particular, the moduli space of generalized Fermat curves of type $(k, 3)$, $k \geq 3$, is isomorphic to the moduli space of orbifolds of signature $(0, 4; k, k, k, k)$, which is also known to be isomorphic to the moduli space of genus one Riemann surfaces, this being the orbifold whose underlying Riemann surface is the complex plane $\mathbb{C}$ with two conical points, one of order 2 and the other of order 3.*

2.3. Group of conformal and anticonformal automorphisms

In Section 5 we compute the groups $\mathrm{Aut}_H(S_\lambda^k)$. Once this is done, Theorem 2 allows us to compute the full group of automorphisms of S_λ^k.

Corollary 6. *Let S_λ^k be a generalized Fermat curve of type $(k, 3)$, where $k \geq 3$, and $\lambda \in \mathbb{C} - \{0, 1\}$. Set*

$$(-1)^{1/k} = \begin{cases} e^{\pi i/k}, & \text{if } k \text{ is even} \\ -1, & \text{if } k \text{ is odd} \end{cases}$$

$$2^{1/k} \in \mathbb{R}, (-2)^{1/k} = (-1)^{1/k} 2^{1/k},$$

$$\begin{cases} \widehat{\alpha}([x_1 : x_2 : x_3 : x_4]) = [x_2 : \lambda^{1/k} x_1 : x_4 : \lambda^{1/k} x_3] \\ \\ \widehat{\beta}([x_1 : x_2 : x_3 : x_4]) = [(-1)^{1/k} x_3 : x_4 : (\lambda - 1)^{1/k} x_1 : (-1)^{1/k} (\lambda - 1)^{1/k} x_2] \\ \\ \widehat{\gamma}([x_1 : x_2 : x_3 : x_4]) = [x_4 : 2^{1/k} x_1 : x_2 : (-2)^{1/k} x_3], \quad \lambda = 2 \\ \widehat{\gamma}([x_1 : x_2 : x_3 : x_4]) = [x_3 : x_4 : 2^{1/k} x_2 : (-2)^{1/k} x_1], \quad \lambda = -1 \\ \widehat{\gamma}([x_1 : x_2 : x_3 : x_4]) = [2^{1/k} x_3 : x_1 : (-2)^{1/k} x_4 : x_2], \quad \lambda = 1/2 \\ \\ \widehat{\delta}([x_1 : x_2 : x_3 : x_4]) = [((1 + i\sqrt{3})/2)^{1/k} x_1 : (-1)^{1/k} x_4 : x_2 : x_3], \quad \lambda = (1 + i\sqrt{3})/2 \\ \widehat{\delta}([x_1 : x_2 : x_3 : x_4]) = [((1 - i\sqrt{3})/2)^{1/k} x_1 : x_4 : x_2 : (-1)^{1/k} x_3], \quad \lambda = (1 - i\sqrt{3})/2 \end{cases} \tag{6}$$

and $L = \langle \widehat{\alpha}, \widehat{\beta} \rangle$. Note that $\widehat{\gamma}^4 = 1$, $\widehat{\delta}^3 = 1$, and that (i) for k even, $L \cong D_{2k}$, where D_n denotes the dihedral group of order $2n$, and that (ii) for k odd, $L \cong \mathbb{Z}_2^2$.

Then

(i) *If $\lambda \notin \mathbb{G}(\{2, (1 + i\sqrt{3})/2\})$, then $\mathrm{Aut}(S_\lambda^k)/H \cong \mathbb{Z}_2^2$, $\mathrm{Aut}(S_\lambda^k) = H \rtimes L$ and the signature of $S_\lambda^k/\mathrm{Aut}(S_\lambda^k)$ is $(0, 4; 2, 2, 2, k)$;*

(ii) *if $\lambda \in \mathbb{G}(2) = \{-1, 1/2, 2\}$, then $\mathrm{Aut}(S_\lambda^k)/H \cong D_4$, $\mathrm{Aut}(S_\lambda^k) = H \rtimes \langle \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma} \rangle$ and the signature of $S_\lambda^k/\mathrm{Aut}(S_\lambda^k)$ is $(0, 3; 2, 4, 2k)$;*

(iii) *if $\lambda \in \mathbb{G}((1 + i\sqrt{3})/2) = \{(1 + i\sqrt{3})/2, (1 - i\sqrt{3})/2\}$, then $\mathrm{Aut}(S_\lambda^k)/H \cong \mathcal{A}_4$, $\mathrm{Aut}(S_\lambda^k) = H \rtimes \langle \widehat{\alpha}, \widehat{\beta}, \widehat{\delta} \rangle$ and the signature of $S_\lambda^k/\mathrm{Aut}(S_\lambda^k)$ is $(0, 3; 2, 3, 3k)$.*

The proof of Corollary 6 will be provided in Section 5.

A symmetry of a closed Riemann surface S is an anticonformal involution $\tau : S \to S$. As a consequence of Harnack's theorem [5,15,21], the locus of fixed points of a symmetry on a closed Riemann surface of genus g is either empty or it consists of $N \leq (g + 1)$ pairwise disjoint simple loops (the mirrors or ovals of the symmetry).

Corollary 7. *Let S_λ^k be a generalized Fermat curve of type $(k, 3)$, where $k \geq 3$, admitting a symmetry.*

(1) *If $\tau : S_\lambda^k \to S_\lambda^k$ is a symmetry and $t \in \mathbb{G}$, then the generalized Fermat curve $S_{t(\lambda)}^k$ admits a symmetry analytically equivalent to τ.*

(2) *Up to the action of $\mathbb{G}$, the values of λ belong to*

$$(\mathbb{R} - (-1, 1]) \cup (\mathbb{S}_+^1 - \{1\})$$

where $\mathbb{S}_+^1 = \{z \in \mathbb{C} : |z| = 1, \mathrm{Im}(z) \geq 0\}$.

(3) *If $\lambda \in \mathbb{R} - (-1, 1]$, then S_λ^k admits the following two symmetries*

$$\tau([x_1 : x_2 : x_3 : x_4]) = [\overline{x_1} : \overline{x_2} : \overline{x_3} : \overline{x_4}]$$
$$\rho([x_1 : x_2 : x_3 : x_4]) = [\overline{x_2} : \lambda^{1/k} \overline{x_1} : \overline{x_4} : \lambda^{1/k} \overline{x_3}]$$

In this case, $S_\lambda^k/\langle H, \tau \rangle$ is a closed disc with 4 conical points in its border, all of them of order k.

(a) *If $1 < \lambda$, then $S_\lambda^k/\langle H, \rho \rangle$ is a closed disc with two conical points in its interior, both of order k.*

(b) *If $\lambda \leq -1$, then $S_\lambda^k/\langle H, \rho \rangle$ is the real projective plane with two conical points, both of order k.*

(4) *If $\lambda \in \mathbb{S}_+^1$, then S_λ^k admits the symmetry*

$$\tau([x_1 : x_2 : x_3 : x_4]) = [\overline{x_2} : \overline{x_1} : \overline{x_3} : \lambda^{1/k} \overline{x_4}].$$

In this case, $S_\lambda^k/\langle H, \tau \rangle$ is a closed disc with two conical points in its border and one in its interior, all of them of order k.

Proof. Let us assume that S_λ^k admits a symmetry τ.

As the generalized Fermat group H is unique in $\mathrm{Aut}(S_\lambda^k)$, by Theorem 2, the symmetry τ descends to an anticonformal involution, say η, of the Riemann sphere keeping invariant the set of conical points $\{\infty, 0, 1, \lambda\}$. Now, if $t \in \mathbb{G}$, then there is a Möbius transformation T sending the set $\{\infty, 0, 1, \lambda\}$ onto the set $\{\infty, 0, 1, t(\lambda)\}$. It follows (see Corollary 5) that there is an isomorphism $\widehat{T} : S_\lambda^k \to S_{t(\lambda)}^k$ so that $\pi \circ T = \widehat{T} \circ \pi$ [10], where $\pi([x_1 : x_2 : x_3 : x_4]) = -(x_2/x_1)^k$. This proves (1).

In order to obtain (2), observe that η is either a reflection (it has a circle of fixed points; so it is conjugate to $z \mapsto \overline{z}$) or an imaginary reflection (it has no fixed points; so it is conjugate to $z \mapsto -1/\overline{z}$). If η is a reflection and C is its circle of fixed points, then either (i) the four conical points belong to C or (ii) two of them belong to C and the other two are permuted by η or (iii) none of them belong to C and they are permuted in pairs by η. In case (i), it is clear that $C = \mathbb{R} \cup \{\infty\}$, so $\lambda \in \mathbb{R} - \{0, 1\}$. As $t(\lambda) = 1/\lambda$ belongs to $\mathbb{G}$, we may assume $\lambda \in \mathbb{R} - (-1, 1]$. In case (ii), up to some element of $\mathbb{G}$, we may assume that $1, \lambda \in C$ and that 0 and ∞ are permuted by η. In this case, the only possibility is to have $\eta(z) = 1/\overline{z}$, so C is the unit circle $\mathbb{S}^1$, and it follows that $\lambda \in \mathbb{S}^1$. Again, using $t(\lambda) = 1/\lambda$, we may assume that $\lambda \in \mathbb{S}^1_+$. In case (iii), again, up to some element of $\mathbb{G}$, we may assume that $\eta(\infty) = 0$ and that $\eta(1) = \lambda$. As η permutes 0 with ∞, it follows that $\eta(z) = R/\overline{z}$, for some $R > 0$. Now, as $\eta(1) = \lambda$, necessarily $R = \lambda$. As before, using $t(\lambda) = 1/\lambda$, we may also assume $\lambda > 1$. Now, let us assume η to be an imaginary reflection. Up to the action of $\mathbb{G}$, we may assume that $\eta(\infty) = 0$ and that $\eta(1) = \lambda$. In this case, as η permutes 0 with ∞, one has that $\eta(z) = R/\overline{z}$, where $R \notin [0, +\infty)$. The equality $\eta(\lambda) = 1$ asserts that $R = \overline{\lambda}$ and the equality $\eta(1) = \lambda$ asserts that $R = \lambda$, from which we see that $\lambda \in \mathbb{R}$. Proceeding as in the previous cases, we may assume $\lambda \leq -1$.

Parts (3) and (4) now follow easily. □

Corollary 8. *Let S^k_λ be a generalized Fermat curve of type $(k, 3)$, where $k \geq 3$. If S^k_λ admits an anticonformal automorphism, then it admits an anticonformal involution. In particular, $\widehat{\mathrm{Aut}}(S^k_\lambda)$ is completely determined from Corollaries 6 and 7.*

Proof. Let S^k_λ be a generalized Fermat curve of type $(k, 3)$, where $k \geq 3$, admitting some anticonformal automorphism, say η. One has that $\widehat{\mathrm{Aut}}(S^k_\lambda) = \langle \mathrm{Aut}(S^k_\lambda), \eta\rangle$. By the uniqueness of the generalized Fermat group $H < \mathrm{Aut}(S)$, η induces an anticonformal automorphism θ of the orbifold S^k_λ/H. There are two possibilities, either θ has order 2 or 4. In the case that θ has order 2, using the action of $\mathbb{G}$ on the parameter space of λ, one obtains that $\lambda \in (\mathbb{R} - (-1, 1]) \cup (\mathbb{S}^1_+ - \{1\})$. It follows from Corollary 7 that S^k_λ admits an anticonformal involution. If θ has order 4, then, up to the action of $\mathbb{G}$, we may assume that

$$\theta(\infty) = 0, \qquad \theta(0) = 1, \qquad \theta(1) = \lambda, \qquad \theta(\lambda) = \infty$$

and it follows that

$$\theta(z) = \frac{\overline{\lambda}}{\overline{\lambda} - \overline{z}}, \quad \lambda + \overline{\lambda} = |\lambda|^2,\ \lambda \in \mathbb{C} - \{0, 1\}$$

in which case η may be taken as

$$\eta([x_1 : x_2 : x_3 : x_4]) = [\overline{x_4} : \overline{\lambda}^{1/k}\overline{x_1} : \overline{x_2} : (-\overline{\lambda})^{1/k}\overline{x_3}].$$

Then, we note that $\rho(z) = (1 - \lambda)\overline{z} + \lambda$ is a reflection so that $\rho(\infty) = \infty$, $\rho(1) = 1$ and $\rho(0) = \lambda$. We now can see that

$$\tau([x_1 : x_2 : x_3 : x_4]) = [(1 - \overline{\lambda})^{1/k}\overline{x_1} : (-1)^{1/k}\overline{x_4} : \overline{x_3} : (-1)^{1/k}\overline{x_2}]$$

defines an anticonformal involution on S^k_λ, since each of the coefficients has modulus one. □

3. Orbifolds and Singerman's conditions

3.1. Orbifolds

An orbifold $\mathcal{O}$ with signature $\sigma = (g, r; n_1, \ldots, n_r)$, where $g \geq 0$, $r \geq 0$, $n_j \geq 2$ are integers, is a closed Riemann surface of genus g together a collection of r conical points of orders $n_1, \ldots, n_r$; these orders are called the periods of the signature. Its orbifold fundamental group is

$$\pi_1^{orb}(\mathcal{O}) = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_r : \prod_{j=1}^{g}[a_j, b_j]\prod_{s=1}^{r} c_j = 1 = c_1^{n_1} = \cdots = c_r^{n_r}\right\rangle \tag{7}$$

where $[a, b] = aba^{-1}b^{-1}$. If S is a closed Riemann surface and $H < \mathrm{Aut}(S)$ a finite group, then S/H is an example of an orbifold. Generalities on orbifolds can be found in [23,27].

Two orbifolds are conformally equivalent if there is a conformal homeomorphism between the respective underlying Riemann surfaces preserving conical points and conical orders. The set $\mathcal{M}_{(g,r;n_1,\ldots,n_r)}$ of conformally equivalent classes of orbifolds of signature $(g, r; n_1, \ldots, n_r)$ is called the moduli space of orbifolds of such signature. Details on Teichmüller theory and moduli spaces can be found in [20] and the references therein.

An orbifold with signature $(g, r; n_1, \ldots, n_r)$ is called hyperbolic if

$$2g - 2 + \sum_{j=1}^{r}\left(1 - \frac{1}{n_j}\right) > 0. \tag{8}$$

If $\mathcal{O}$ is a hyperbolic orbifold with hyperbolic signature $(g, r; n_1, \ldots, n_r)$, then there is a Fuchsian group Γ acting on the hyperbolic plane $\mathbb{H}^2$ with presentation

$$\Gamma = \left\langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \delta_1, \ldots, \delta_r : \prod_{j=1}^{g} [\alpha_j, \beta_j] \prod_{s=1}^{r} \delta_j = 1 = \delta_1^{n_1} = \cdots = \delta_r^{n_r} \right\rangle \tag{9}$$

such that $\mathbb{H}^2/\Gamma$ is an orbifold conformally equivalent to $\mathcal{O}$. We also say that Γ as above has signature $(g, r; n_1, \ldots, n_r)$. It is well known [2,19] that each finite order element of a Fuchsian group Γ, with presentation as in (9), is conjugate in Γ to some power of one of the generators δ_j (this is part of the Poincaré polygon theorem).

We say that a signature σ_1 is of index D in a signature σ_2 if there are Fuchsian groups Γ_j with signature σ_j, for $j = 1, 2$, so that Γ_1 is contained in Γ_2 with index D. If moreover, $\Gamma_1 \lhd \Gamma_2$, then we say that the signature σ_1 is normal in the signature σ_2.

By the hyperbolic area of a Fuchsian group (respectively, an orbifold) of signature $(g, r; n_1, \ldots, n_r)$ we refer to the hyperbolic area of any of its fundamental polygon domains. This is given by

$$A(g, r; n_1, \ldots, n_r) = 2\pi \left(2g - 2 + \sum_{j=1}^{r} \left(1 - \frac{1}{n_j} \right) \right). \tag{10}$$

If a signature σ_1 is of index D in a signature σ_2, then

$$A(\sigma_1) = DA(\sigma_2) \tag{11}$$

3.2. Singerman's conditions

Let us recall the following result due to D. Singerman [25] which provides necessary and sufficient conditions for possible inclusions among Fuchsian groups of finite type in terms of their signatures.

Proposition 9 (*[25]*). *Let Γ be a Fuchsian group with signature $(g, r; n_1, \ldots, n_r)$. Then Γ contains a subgroup Γ_1 of index D with signature*

$$(\gamma, \rho_1 + \rho_2 + \cdots + \rho_r; m_{11}, \ldots, m_{1\rho_1}, \ldots, m_{r1}, \ldots, m_{r\rho_r})$$

if and only if the following conditions hold

(a) *there exists a finite permutation group J transitive on D points, and an epimorphism $\theta : \Gamma \rightarrow J$ satisfying that each permutation $\theta(\delta_j)$ has precisely ρ_j cycles of lengths less than n_j, the lengths of these cycles being*

$$n_j/m_{j1}, \ldots, n_j/m_{j\rho_j};$$

(b) $A(\Gamma_1)/A(\Gamma) = D$.

Remark 10. (i) In Proposition 9 the subgroup Γ_1 is (up to conjugation in Γ) equal to $\theta^{-1}(J_1)$, where $J_1 = \{h \in J : h(1) = 1\}$. It follows from the transitivity of J that if, for some $x \in \Gamma$, $\theta(x)$ fixes a point, then there is a conjugate $y \in \Gamma$ of x so that $\theta(y)$ fixes 1, that is, $y \in J_1$. This fact will be used frequently in the applications of Singerman's result. (ii) In the same proposition it may occur that some of the values $\rho_j = 0$.

4. Key propositions

In this section we state the key propositions we will need to prove Theorem 2. The proof of one of them will be postponed to the last section.

Proposition 11. *Let K be a Fuchsian group of signature $(0, 3; t, s, r)$, where $2 \le r \le s \le t$, containing as a subgroup of index 2 a Fuchsian group U of signature $(0, 4; 2, 2, 2, k)$, where $k \ge 3$. Then $r = 2$, $s = 4$ and $t = 2k$.*

Proof. As U is a normal subgroup of index two, it follows that the orbifold $\mathcal{O} = \mathbb{H}^2/U$ admits a conformal involution $\rho : \mathcal{O} \rightarrow \mathcal{O}$ (permuting the conical points and keeping invariant their orders) so that $\mathcal{O}/\langle\rho\rangle = \mathbb{H}^2/K$. Then necessarily ρ must fix the conical point of order k and one of the conical points of order 2, permuting the other two conical points. It follows that $\mathcal{O}/\langle\rho\rangle$ must be an orbifold of signature $(0, 3; 2k, 4, 2)$. □

Proposition 12. *Let (S, H) be a generalized Fermat pair of type $(k, 3)$, $k \ge 3$. Let K be a Fuchsian group acting on the hyperbolic plane $\mathbb{H}^2$ so that $\mathbb{H}^2/K = S/\mathrm{Aut}(S)$. If $\Gamma < K$ is so that $\mathbb{H}^2/\Gamma = S/H$, then $\Gamma' \lhd K$.*

Proof. This is consequence of the fact that $\mathbb{H}^2/\Gamma' = S$ [10]. □

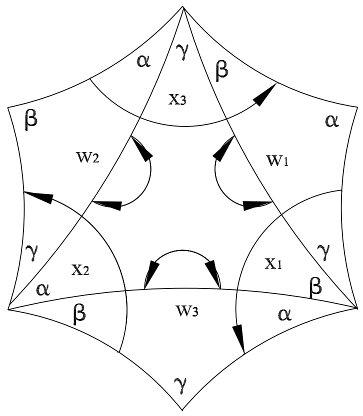

Fig. 1. Fundamental domains for U and Γ.

Proposition 13. *Any Fuchsian group U, of signature $(0, 4; 2, 2, 2, k)$, where $k \geq 3$, contains a unique normal subgroup Γ of index 4 and signature $(0, 4; k, k, k, k)$. Moreover, $U/\Gamma \cong \mathbb{Z}_2^2$ and, if*

$$U = \langle w_1, w_2, w_3, w_4 : w_1^2 = w_2^2 = w_3^2 = w_4^k = w_4w_3w_2w_1 = 1\rangle,$$

then

$$\Gamma = \langle x_1, x_2, x_3 : x_1^k = x_2^k = x_3^k = (x_3x_2x_1)^k = 1\rangle$$

where

$$\begin{cases} x_1 = w_3w_2w_1, \\ x_2 = w_2w_1w_3, \\ x_3 = w_1w_3w_2. \end{cases} \tag{12}$$

Proof. Let us consider a presentation of U as follows

$$U = \langle w_1, w_2, w_3, w_4 : w_1^2 = w_2^2 = w_3^2 = w_4^k = w_4w_3w_2w_1 = 1\rangle.$$

It is clear that the group Γ as defined in the proposition is a normal subgroup so that $U/\Gamma \cong \mathbb{Z}_2^2$ and of signature $(0, 4; k, k, k, k)$ (see Fig. 1). We claim that Γ is unique with such a property. In fact, if Γ^* is a normal subgroup of U of index 4, then there is a homomorphism $\eta : U \to \mathfrak{S}_4$ so that $\eta(U)$ is a transitive subgroup and $\Gamma^* = \ker(\eta)$. As the signature of Γ^* is required to be $(0, 4; k, k, k, k)$, according to Proposition 9, we must have that $\eta(w_4) = (1)(2)(3)(4)$ and, for each $j = 1, 2, 3$, that $\eta(w_j)$ consists of two 2-cycles. Up to an automorphism of $\mathfrak{S}_4$, we may assume that $\eta(w_1) = (12)(34)$ and $\eta(w_2) = (13)(24)$. As $\eta(w_1w_2w_3) = \eta(w_1w_2w_3w_4) = 1$, then $\eta(w_3) = (14)(23)$. As for each automorphism τ of $\mathfrak{S}_4$ one has that $\ker(\tau\eta) = \ker(\eta)$, then we obtain that $\Gamma^* = \Gamma$. □

Proposition 14. *Let K be a Fuchsian group with signature $(0, 3; t, s, r)$, where $2 \leq r \leq s \leq t$, containing with finite index D a Fuchsian group $U < K$ of signature $(0, 4; 2, 2, 2, k)$ for some $k \geq 3$. Let Γ be the unique Fuchsian normal subgroup of U of index 4 and of signature $(0, 4; k, k, k, k)$. If U is a normal subgroup of K, then Γ' is also a normal subgroup of K.*

Proof. Let us assume U is a normal subgroup of K. As Γ is the unique normal subgroup of U of index 4 and signature $(0, 4; k, k, k, k)$, it follows that Γ is also a normal subgroup of K. As Γ' is a characteristic subgroup of Γ, it also follows that Γ' is a normal subgroup of K. □

The following will be proved in Section 7.

Proposition 15. *Let K be a Fuchsian group with signature $(0, 3; t, s, r)$, where $2 \leq r \leq s \leq t$.*

(1) *The group K does not contain a Fuchsian group of index D and signature $(0, 4; 2, 2, 2, k)$ in the following cases.*

r	s	t	k	D
2	3	12	6	4
2	4	8	8	3
2	5	5	5	3
3	3	4	4	3

(2) *Assume K contains a Fuchsian group $U < K$ of signature $(0, 4; 2, 2, 2, k)$, for some $k \geq 3$, as a finite index D subgroup. Let Γ be the unique Fuchsian normal subgroup of U of index 4 and of signature $(0, 4; k, k, k, k)$. Then, Γ' is not normal in K in the following cases. In particular, U cannot be normal either.*

r	s	t	k	D
2	3	7	3	7
2	3	7	7	15
2	3	8	3	4
2	3	8	4	6
2	3	8	8	9
2	3	10	10	6
2	4	5	5	6
2	4	5	4	5
2	4	6	4	3
2	4	6	6	4

5. $\mathrm{Aut}_H(S)$ for a generalized Fermat pair (S, H) of type $(k, 3)$

As already noted in the Introduction, a generalized Fermat curve of type $(k, 3)$, where $k \geq 3$, can be described as an algebraic curve of the form [10]

$$S_\lambda^k = \begin{Bmatrix} x_1^k + x_2^k + x_3^k = 0 \\ \lambda x_1^k + x_2^k + x_4^k = 0 \end{Bmatrix} \subset \mathbb{P}^3 \tag{13}$$

where $\lambda \in \mathbb{C} - \{0, 1\}$ and the generators of the generalized Fermat group $\mathbb{Z}_k^3 \cong H = \langle a_1, a_2, a_3 \rangle < \mathrm{Aut}(S_\lambda^k)$, as

$$\begin{cases} a_1([x_1 : x_2 : x_3 : x_4]) = [w_k x_1 : x_2 : x_3 : x_4] \\ a_2([x_1 : x_2 : x_3 : x_4]) = [x_1 : w_k x_2 : x_3 : x_4] \\ a_3([x_1 : x_2 : x_3 : x_4]) = [x_1 : x_2 : w_k x_3 : x_4]. \end{cases} \tag{14}$$

The orbifold S_λ^k/H is precisely $\mathcal{O}_\lambda$, the Riemann sphere with conical points ∞, 0, 1 and λ, the four of them of order k. The corresponding degree k^3 holomorphic branched covering map with H as covering group is

$$\pi : S_\lambda^k \to \widehat{\mathbb{C}} : [x_1 : x_2 : x_3 : x_4] \mapsto -\left(\frac{x_2}{x_1}\right)^k,$$

whose branch values are the points

$$\pi(\mathrm{Fix}(a_1)) = \infty, \pi(\mathrm{Fix}(a_2)) = 0, \pi(\mathrm{Fix}(a_3)) = 1, \pi(\mathrm{Fix}(a_4)) = \lambda, \tag{15}$$

where $a_4 = a_1 a_2 a_3$. It is not difficult to see that if $b \in H - \{I\}$ has fixed points, then $b \in \langle a_j \rangle$, for some $j = 1, 2, 3, 4$. We say that a_1, a_2, a_3 and a_4 are the standard generators of H. This permits us to speak of the 4 standard generators of any generalized Fermat group of type $(k, 3)$ [10].

5.1. Generic automorphisms

We observe in passing the fact that any Riemann surface S_λ^k possesses a larger group of automorphisms, say G, so that $H \lhd G$, $G/H \cong \mathbb{Z}_2^2$ and S_λ^k/G is of signature $(0, 4; 2, 2, 2, k)$. In fact, the orbifold S_λ^k/H always admits $\mathbb{Z}_2^2$ as group of orbifold automorphisms, generated by the transformations

$$\alpha(z) = \lambda/z \quad \text{and} \quad \beta(z) = (z - \lambda)/(z - 1). \tag{16}$$

If we choose values for $\lambda^{1/k}$, $(-1)^{1/k}$, $(\lambda - 1)^{1/k}$, then the transformations

$$\begin{Bmatrix} \widehat{\alpha}([x_1 : x_2 : x_3 : x_4]) = [x_2 : \lambda^{1/k} x_1 : x_4 : \lambda^{1/k} x_3] \\ \widehat{\beta}([x_1 : x_2 : x_3 : x_4]) = [(-1)^{1/k} x_3 : x_4 : (\lambda - 1)^{1/k} x_1 : (-1)^{1/k} (\lambda - 1)^{1/k} x_2] \end{Bmatrix} \in \mathrm{Aut}(S_\lambda^k) \tag{17}$$

satisfy the properties that $\alpha\pi = \pi\widehat{\alpha}$ and $\beta\pi = \pi\widehat{\beta}$, therefore $G = \langle H, \widehat{\alpha}, \widehat{\beta} \rangle < \mathrm{Aut}_H(S_\lambda^k)$. Note that $\widehat{\alpha}^2 = \widehat{\beta}^2 = 1$, but $\widehat{\alpha}\widehat{\beta}$ has order 2 only if k is odd and $(-1)^{1/k} = -1$, in which case, $\mathbb{Z}_2^2 = \langle \widehat{\alpha}, \widehat{\beta} \rangle < \mathrm{Aut}_H(S_\lambda^k)$.

The above only reflects the fact that the pair of signatures $(0, 4; k, k, k, k)$ and $(0, 4; 2, 2, 2, k)$ is one of the few that occur in Singerman's list of pairs of signatures representing strict inclusions between Fuchsian groups having the same Teichmüller space [26].

5.2. Extra automorphisms

For generic values of λ, there are no more orbifold automorphisms of $\mathcal{O}_\lambda = S^k_\lambda/H$, that is, generically, $\mathrm{Aut}_H(S^k_\lambda) = G = \langle H, \widehat{\alpha}, \widehat{\beta}\rangle$ ($\cong H \rtimes \mathbb{Z}_2^2$, for k odd). For some particular values of $\lambda \in \mathbb{C} - \{0, 1\}$ it happens that $G \neq \mathrm{Aut}_H(S^k_\lambda)$. These cases occur when the orbifold $\mathcal{O}_\lambda$ admits extra automorphisms.

Note that the set of orbifolds $\mathcal{O}$ of genus zero with exactly four conical points (all of them of a same order k) can be identified to the space $\mathcal{C}_4$ of unordered quadruples of distinct points of $\widehat{\mathbb{C}}$, which itself can be viewed as the quotient space $\widehat{\mathcal{C}}_4/\mathfrak{S}_4$, where $\widehat{\mathcal{C}}_4$ is the space of ordered quadruples of distinct points of $\widehat{\mathbb{C}}$ and $\mathfrak{S}_4$ is the symmetric group in four letters. The group $\mathrm{PGL}(2, \mathbb{C})$ acts by Möbius transformations on both $\widehat{\mathcal{C}}_4$ and $\mathcal{C}_4$. The quotient $\mathcal{C}_4/\mathrm{PGL}(2, \mathbb{C})$ is the moduli space of such orbifolds and the stabilizer of a given orbifold $\mathcal{O}$ is isomorphic to its automorphism group $\mathrm{Aut}(\mathcal{O})$. For any such a orbifold $\mathcal{O}$, the group $\mathrm{Aut}(\mathcal{O})$ is usually identified to a subgroup of $\mathfrak{S}_4$ via its action on its conical points z_1, z_2, z_3, z_4. This subgroup contains always Klein's four group $V = \langle(1, 2)(3, 4), (1, 3)(2, 4)\rangle < \mathfrak{S}_4$, which corresponds to the group $\mathbb{Z}_2^2$ generated by the transformations α and β introduced above. Therefore, there is a natural action of $\mathfrak{S}_3 \cong \mathfrak{S}_4/V$ on the intermediate covering $\mathcal{C}_{(4)} := \mathcal{C}_4/V$ such that the orbifolds with extra automorphisms, that is, the orbifolds $\mathcal{O}$ such that $\mathrm{Aut}(\mathcal{O})/V$ is not trivial, correspond to the points fixed by this action.

On the other hand, there is a classical function on $\mathcal{C}_{(4)}$, called *the cross ratio*, defined by the formula

$$[z_1, z_2, z_3, z_4] = \frac{(z_4 - z_2)(z_3 - z_1)}{(z_4 - z_1)(z_3 - z_2))}$$

that is invariant under the action of $\mathrm{PGL}(2, \mathbb{C})$ and that in fact produces an isomorphism between the Galois covers $\mathcal{C}_{(4)}/\mathrm{PGL}(2, \mathbb{C}) \to \mathcal{C}_4/\mathrm{PGL}(2, \mathbb{C})$ with deck group $\mathfrak{S}_3$ and $j : \mathbb{C}\{0, 1\} \to \mathbb{C}$ with deck group $\mathbb{G} \cong \mathfrak{S}_3$ (see 2.2) defined by associating to the orbifold with conical points z_1, z_2, z_3 and z_4 the orbifold $\mathcal{O}_\lambda$ with conical points $0, 1, \infty$ and $\lambda = [z_1, z_2, z_3, z_4]$. It follows that the orbifolds $\mathcal{O}_\lambda$ with extra automorphisms correspond to the values of λ which are ramification points of the modular function j. These values are $\{1 \pm \sqrt{3}/2\}$ which constitute the $\mathbb{G}$-orbit that maps over 0 together with $\{-1, 1/2, 2\}$ which constitute the $\mathbb{G}$-orbit that maps over 1 (see Remark 4.61 of [9]). Thus, up to the action of the group $\mathbb{G}$, we may assume these values to be $\lambda \in \{2, (1+i\sqrt{3})/2\}$. For instance, for $\lambda = 2$ we have the extra Möbius transformation of order 4 given by $T(z) = 2/(2-z)$ that permutes the set of points $\{\infty, 0, 1, 2\}$. Similarly, for $\lambda = (1+i\sqrt{3})/2$ we have that the extra Möbius transformation of order 3 given by $S(z) = 1/(1-z)$ (for details, see Chapter II of Part I in Klein's book [14]).

If $\lambda_1 = 2$ (and for values of λ in its orbit under the action of $\mathbb{G}$), then $S^k_{\lambda_1}/H$ admits an extra automorphism of order 4 given by $\gamma(z) = 2/(2 - z)$. In this case $S^k_{\lambda_1}/H$ has a dihedral group D_4 as group of automorphisms and $(S^k_{\lambda_1}/H)/D_4$ has signature $(0, 3; 2, 4, 2k)$. As such a signature is a maximal one [26], then $S^k_{\lambda_1}/\mathrm{Aut}(S^k_{\lambda_1}) = (S^k_{\lambda_1}/H)/D_4$ and $\mathrm{Aut}(S^k_{\lambda_1})/H = D_4$. Since

$$\widehat{\gamma}([x_1 : x_2 : x_3 : x_4]) = [x_4 : 2^{1/k}x_1 : x_2 : (-2)^{1/k}x_3] \quad (\widehat{\gamma}^4 = 1)$$

is a lifting of γ to $S^k_{\lambda_1}$, then $\mathrm{Aut}(S^k_{\lambda_1}) = \langle G, \widehat{\gamma}\rangle$.

Similarly, if $\lambda_2 = (1 + i\sqrt{3})/2$ (and for values of λ in its orbit under the action of $\mathbb{G}$), then $S^k_{\lambda_2}/H$ admits an extra automorphism of order 3 given by $\delta(z) = (1 + i\sqrt{3} - 2z)/(1 + i\sqrt{3})$. In this case $S^k_{\lambda_2}/H$ has the alternating group $\mathcal{A}_4$ as group of automorphisms, $(S^k_{\lambda_2}/H)/\mathcal{A}_4$ has signature $(0, 3; 2, 3, 3k)$ and $\mathrm{Aut}(S^k_{\lambda_2})/H = \mathcal{A}_4$. As such a signature is a maximal one [26], then $S^k_{\lambda_2}/\mathrm{Aut}(S^k_{\lambda_2}) = (S^k_{\lambda_2}/H)/\mathcal{A}_4$ and $\mathrm{Aut}(S^k_{\lambda_2})/H = \mathcal{A}_4$. Since

$$\widehat{\delta}([x_1 : x_2 : x_3 : x_4]) = [((1 + i\sqrt{3})/2)^{1/k}x_1 : (-1)^{1/k}x_4 : x_2 : x_3] \quad (\widehat{\delta}^3 = 1)$$

is a lifting of δ to $S^k_{\lambda_2}$, then $\mathrm{Aut}(S^k_{\lambda_2}) = \langle G, \widehat{\delta}\rangle$.

All the above determines $\mathrm{Aut}_H(S^k_\lambda)$, for generic λ, and $\mathrm{Aut}(S^k_\lambda)$, for $\lambda \in \{-1, 1/2, 2, (1 \pm i\sqrt{3})/2\}$.

The above, together with Theorem 2, provides a proof of Corollary 6.

6. Proof of Theorem 2

6.1. Normality property

In this section, we prove that, if (S, H) is a generalized Fermat pair of type $(k, 3)$, with $k \geq 3$, then H is a normal subgroup of $\mathrm{Aut}(S)$, that is, $\mathrm{Aut}(S) = \mathrm{Aut}_H(S)$. In the previous section we have already obtained this for $\lambda \in \{-1, 1/2, 2, (1\pm i\sqrt{3})/2\}$.

Let (S, H) be a generalized Fermat pair of type $(k, 3)$, where $k \geq 3$ and choose $\lambda \in \mathbb{C} - \{0, 1\}$ so that $S = S^k_\lambda$, and, as in the previous section, let $G = \langle H, \widehat{\alpha}, \widehat{\beta}\rangle < \mathrm{Aut}(S)$. We denote by D the index of G in $\mathrm{Aut}(S)$.

As S/H has signature $(0, 4; k, k, k, k)$, it follows from Hurwitz's formula [8] that the genus of S is $g = 1 - 2k^2 + k^3 \geq 10$ (as $k \geq 3$). Since the orbifold S/G has signature $(0, 4; 2, 2, 2, k)$, it follows that the signature of the orbifold $S/\mathrm{Aut}(S)$ is either of the form (i) $(0, 4; r, s, t, u)$ $(2 \leq r, s, t, u)$ or (ii) $(0, 3; r, s, t)$ $(2 \leq r, s, t)$.

Case (i)

If $S/\mathrm{Aut}(S)$ has signature $(0, 4; r, s, t, u)$, then it follows from [26] that the only possibility is to have $r = s = t = 2$ and $u = k$, that is, $G = \mathrm{Aut}_H(S) = \mathrm{Aut}(S)$.

Case (ii)

Let $S/\mathrm{Aut}(S)$ be of signature $(0, 3; r, s, t)$, where $r^{-1} + s^{-1} + t^{-1} < 1$. As $G \neq \mathrm{Aut}(S)$, there is a natural branched covering of degree $D \geq 2$, say $P : S/G \to S/\mathrm{Aut}(S)$, induced by the inclusion $G < \mathrm{Aut}(S)$. Proposition 11, together with the computations done in Section 5, asserts that the index $D = 2$ corresponds to the value $\lambda = 2$ (up to the action of $\mathbb{G}$) and the explicit determination of $\mathrm{Aut}(S_2^k)$ shows that $\mathrm{Aut}(S) = \mathrm{Aut}_H(S)$ as required.

So, from now on we assume $D \geq 3$.

We consider co-compact Fuchsian groups U and K so that $S/G = \mathbb{H}^2/U$ and $S/\mathrm{Aut}(S) = \mathbb{H}^2/K$. Moreover, these groups have presentations as follows:

$$U = \langle w_1, w_2, w_3, w_4 : w_1^2 = w_2^2 = w_3^2 = w_4^k = w_1w_2w_3w_4 = 1\rangle$$
$$K = \langle x, y, z : x^r = y^s = z^t = xyz = 1\rangle$$

By Proposition 9, there is a homomorphism $\theta : K \to \mathfrak{S}_D$, so that $\theta(K)$ is transitive and $\theta^{-1}(J_1) = U$, where J_1 is the stabilizer of 1 in $\theta(U)$.

Lemma 16. *If $D = 3$ and $\{r, s, t\} \neq \{2, 4, 6\}$, then U is a normal subgroup of K, that is, G is normal in $\mathrm{Aut}(S)$.*

Proof. In this case, there is a homomorphism $\theta : K \to \mathfrak{S}_3$ as in Proposition 9 with $x = \delta_1$, $y = \delta_2$ and $z = \delta_3$. $J = \theta(K)$ is a transitive subgroup of $\mathfrak{S}_3$ and the only transitive subgroups of $\mathfrak{S}_3$ are $\langle(1, 2, 3)\rangle$ and $\mathfrak{S}_3$.

If $J = \langle(1, 2, 3)\rangle$, then $J_1 = \{(1)(2)(3)\}$, and U is a normal subgroup of K.

If $J = \mathfrak{S}_3$, as $\delta_1\delta_2\delta_3 = 1$, then at least two of the permutations $\theta(\delta_1)$, $\theta(\delta_2)$ and $\theta(\delta_3)$ must be permutations of order two and different from each other. We may assume, without loss of generality, that $\theta(\delta_1) = (1)(2, 3)$ and $\theta(\delta_2) = (2)(1, 3)$. So, in this case $\theta(\delta_3) = (1, 3, 2)$. As the signature of K is hyperbolic, at least two of the values r, s and t must be at least 3. Without loss of generality, we may assume $r \geq 3$. In this way, $\rho_1 = 2$ and we must have that

$$1 = \frac{r}{m_{11}}, \qquad 2 = \frac{r}{m_{12}}.$$

It follows that $m_{11} = r$, $m_{12} = r/2 \in \{2, k\}$. The only possibility is to have $k = r = 4$. Now, $\rho_2 \in \{1, 2\}$. But $\rho_2 = 2$ will assert that $m_{21} = s$, $m_{22} = s/2 \in \{2\}$, a contradiction. It follows that $\rho_2 = 1$, from which we must have that $s = 2$, $m_{21} = 2$ and $\rho_3 = 1$ (as $\rho_1 + \rho_2 + \rho_3 = 4$). As we must have that $\{m_{11}, m_{12}, m_{21}, m_{31}\} = \{2, 2, 2, 4\}$, one has that $m_{31} = 2$ and, in particular, that

$$3 = \frac{t}{m_{31}} = \frac{t}{2}$$

that is $t = 6$. □

Lemma 17. *At least of one the integers r, s, t, must be divisible by k.*

Proof. This is consequence of the facts: (i) each finite order element of a Fuchsian group Γ, with presentation as in (9), is conjugate in Γ to some power of one of the generators δ_j, and (ii) $G < \mathrm{Aut}(S)$. □

As a consequence of the previous lemma, from now on in this section can assume $t = Tk$. We also assume that $2 \leq r \leq s$ (if $r = 2$, then $s \geq 3$). By comparison of hyperbolic area, one has

$$\frac{1}{2} - \frac{1}{k} = D\left(1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t}\right) = D\left(1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{Tk}\right) \tag{18}$$

In particular,

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} \geq \frac{3}{4} + \frac{1}{2k} \tag{19}$$

Proposition 18. *If $D \geq 3$, then $T \in \{1, 2, 3\}$. Furthermore the value $T = 3$ corresponds to $D = 3$, signature $(0, 3; 2, 3, 3k)$ and $\lambda = \frac{1+i\sqrt{3}}{2}$ (up to the action of $\mathbb{G}$).*

Proof. As $D \geq 3$, it follows from (18) that $1/r + 1/s + 1/t \geq 5/6 + 1/(3k)$. As $r \geq 2$ and $s \geq 3$, we obtain that $5/6 + 1/t \geq 5/6 + 1/(3k)$, from which $t \leq 3k$, that is $T \in \{1, 2, 3\}$. Equality $t = 3k$ (that is, $T = 3$) asserts equality in the above, so $r = 2$ and $s = 3$ and, by (18), that $D = 3$. Therefore, in this case, the signature of our triangular group is $(0, 3; 2, 3, 3k)$ which, by Lemma 16, corresponds (see Section 5) to the case $\lambda = \frac{1+i\sqrt{3}}{2}$. □

By Proposition 18 the only cases we need to consider are $T \in \{1, 2\}$ which we proceed to analyze below.

6.1.1. Case $T = 1$

In this case, $t = k$ and, since we are assuming that $D \geq 3$, relation (18) yields

$$\frac{1}{2} - \frac{1}{k} \geq 3\left(1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{k}\right) \Rightarrow \left(\frac{1}{r} + \frac{1}{s}\right) \geq \frac{5}{6} - \frac{2}{3k} \geq \frac{11}{18}. \tag{20}$$

The above inequality, together with the fact that $1 > r^{-1} + s^{-1} + k^{-1}$, asserts that

$$(r, s, k) = \begin{cases} (2, 3, k), & k \geq 7 \\ (2, 4, k), & k = 5, 6, 7, 8 \\ (2, 5, k), & k = 3, 4, 5 \\ (2, 6, 4), (2, 7, 3), (2, 8, 3), (2, 9, 3), (3, 3, 4). \end{cases}$$

The case $(2, 9, 3)$ produces $D = 3$ and, by Lemma 16 and Section 5, it corresponds to $\lambda = (1 + i\sqrt{3})/2$, in which case, the explicit determination of the automorphism group shows that $\mathrm{Aut}_H(S) = \mathrm{Aut}(S)$.

We claim that the rest of the cases do not occur.

(i) The cases $(r, s, k) = (2, 6, 4)$ $(D = 3)$, $(r, s, k) = (2, 7, 3)$ $(D = 7)$, $(r, s, k) = (2, 8, 3)$ $(D = 4)$ and $(r, s, t) = (3, 3, 4)$ $(D = 3)$ are not possible by Proposition 15.

(ii) In the case $(r, s, k) = (2, 3, k)$, with $k \geq 7$, we have $D = \frac{3(k-2)}{k-6}$. We now claim that D is divisible by 3. Clearly any pre-image in S of the conical point $q \in S/\mathrm{Aut}(S)$ of order 3 is a point with ramification order 3, and, since no cone point of S/G has this order, it follows that the fibre of q in the intermediate orbifold S/G must also consist of, say, d points, all of them with ramification order 3 and each of them different from the fourth conical point of S/G. Therefore, $D = 3d$ where $d = \frac{k-2}{k-6}$. It follows that $k \in \{7, 8, 10\}$. Thus we are left with the following three cases

$$(r, s, k; D) \in \{(2, 3, 7; 15), (2, 3, 8; 9), (2, 3, 10; 6)\}$$

which are not possible by Proposition 15.

(iii) Let us consider the case $(r, s, k) = (2, 4, k)$, where $k \in \{5, 6, 7, 8\}$, in which case $D = \frac{2(k-2)}{k-4}$. As D is an integer, it follows that $k \in \{5, 6, 8\}$. Thus we are left with the cases

$$(r, s, k; D) = \{(2, 4, 5; 6), (2, 4, 6; 4), (2, 4, 8; 3)\}$$

which are again not possible by Proposition 15.

(iv) Let us consider the case $(r, s, k) = (2, 5, k)$, where $k \in \{3, 4, 5\}$. In this case $D = \frac{5(k-2)}{3k-10}$. Again, as D is an integer, it follows that $k \in \{4, 5\}$. Thus we are left with the cases

$$(r, s, k; D) = \{2, 5, 4; 5), (2, 5, 5; 3)\}$$

which are again not possible by Proposition 15.

6.1.2. Case $T = 2$

In this case, $t = 2k$ and relation (19) now yields

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{2k} \geq \frac{3}{4} + \frac{1}{2k} \Rightarrow \frac{1}{r} + \frac{1}{s} \geq \frac{3}{4}. \tag{21}$$

It follows that $(r, s) \in \{(2, 3), (2, 4)\}$. So we are left with only the following possible signatures:

(1) $(0, 3; 2, 4, 2k)$. In this case $D = 2$ and this signature occurs for $\lambda = 2$ (see Section 5). Then we know that $\mathrm{Aut}(S) = \mathrm{Aut}_H(S)$ as required.
(2) $(0, 3; 2, 3, 2k)$. In this case $D = 3 + 3/(k - 3)$ and, in particular, $k \in \{4, 6\}$. For $k = 4$ we obtain signature $(0, 3; 2, 3, 8)$ and degree $D = 6$, which is not possible by Proposition 15. The value $k = 6$ gives signature is $(0, 3; 2, 3, 12)$ and degree $D = 4$. This case is again impossible by Proposition 15.

6.2. Uniqueness property

In the previous section we have proved that every generalized Fermat group of type $(k, 3)$, with $k \geq 3$, is a normal subgroup of the full group of conformal automorphisms. We use this property to show that the generalized Fermat group is unique; which is the content of Theorem 2.

Lemma 19. *Let S be a generalized Fermat curve of type $(k, 3)$, $k \geq 3$, and let H_1 and H_2 be generalized Fermat groups for S, both of type $(k, 3)$. If both of them have in common an element $b \neq I$ acting with fixed points, then $H_1 = H_2$.*

Proof. Let $b \in H_1 \cap H_2 - \{I\}$ be a common element acting with fixed points. Let $p \in S$ be a fixed point of b. We know that there is a standard generator $b_j \in H_j$, for $j = 1, 2$, so that $b \in \langle b_j \rangle \cong \mathbb{Z}_k$. Moreover, the point p must be also a fixed point of b_j [10]. As the stabilizer in $\mathrm{Aut}(S)$ of p is a cyclic group, it must follow that $\langle b_1 \rangle = \langle b_2 \rangle$, in particular, H_1 and H_2 have a common standard generator. Let $a \in H_1 \cap H_2$ be a common standard generator. Let R be the underlying Riemann surface associated to the orbifold $S/\langle a \rangle$. Then R is a classical Fermat curve of degree k with Fermat group $K_j = H_j/\langle a \rangle \cong \mathbb{Z}_k^2 < \mathrm{Aut}(R)$, for $j = 1, 2$. Therefore $K_1 = K_2$ [24,29] so in particular, $H_1 = H_2$. □

Let (S, H) be a generalized Fermat pair of type $(k, 3)$. Assume there is another different generalized Fermat group $\widehat{H} < \mathrm{Aut}(S)$ of type $(k, 3)$. Let $\widehat{a} \in \widehat{H}$ be a standard generator of $\widehat{H}$. It follows from Lemma 19 that $\widehat{a}^m \notin H$ for $m = \{1, \ldots, k-1\}$. It follows, from the normality of H in $\mathrm{Aut}(S)$ (proved previously), that $\widehat{a}$ induces a conformal automorphism a of order k of the orbifold S/H. As the signature of S/H is $(0, 4; k, k, k, k)$ and $k \geq 3$, it follows from Section 5.2 that $k \in \{3, 4\}$; so up to $\mathbb{G}$-equivalence

$$S = \begin{cases} S^3_{(1+i\sqrt{3})/2} & \text{for } k = 3 \\ S^4_2 & \text{for } k = 4. \end{cases}$$

We already know the full group of conformal automorphisms of S in those cases and the uniqueness of H can be checked directly. We may assume (because $k \geq 3$ and the non-trivial elements of $L = \langle \widehat{\alpha}, \widehat{\beta} \rangle$ (Section 5.1) induce order 2 automorphisms on S/H) that

$$\widehat{a} = \begin{cases} \widehat{\delta} & \text{if } k = 3 \text{ (of order 3)} \\ \widehat{\gamma} & \text{if } k = 4 \text{ (of order 4)} \end{cases}$$

Case k = 4

In this case the conformal automorphism (of order 4)

$$\widehat{\gamma}([x_1 : x_2 : x_3 : x_4]) = [x_4 : 2^{1/4}x_1 : x_2 : (-2)^{1/4}x_3] \tag{22}$$

(where we have fixed choices for $2^{1/4}$ and $(-2)^{1/4}$) has exactly 8 fixed points on S. In fact, the fixed points of $\widehat{\gamma}$ in $\mathbb{P}^3$ are the points of the form

$$\mathrm{Fix}(\widehat{\gamma}) = \{[1 : 2^{1/4}/\rho : 2^{1/4}/\rho^2 : \rho] : \rho^4 = \sqrt{2}(-1)^{1/4}\} \subset \mathbb{P}^3, \tag{23}$$

and $\mathrm{Fix}(\widehat{\gamma}) \cap S$ corresponds to the case $(-1)^{1/4} = (-1 \pm i)/\sqrt{2}$. As the number of fixed points of a standard generator in this case is $4^2 = 16$, this case is not possible.

Case k = 3

In this case the conformal automorphism (of order 3)

$$\widehat{\delta}([x_1 : x_2 : x_3 : x_4]) = [((1 + i\sqrt{3})/2)^{1/3}x_1 : (-1)^{1/3}x_4 : x_2 : x_3] \tag{24}$$

(where we have fixed choices for $((1 + i\sqrt{3})/2)^{1/3}$ and $(-1)^{1/3}$) has no fixed points on S. In fact, the fixed points of $\widehat{\delta}$ in $\mathbb{P}^3$ are the points of the form

$$\mathrm{Fix}(\widehat{\delta}) = \{[0 : 1 : 1/\rho : 1/\rho^2], [1 : x_2 : x_2/\rho : x_2/\rho^2] : \rho^3 = (-1)^{1/3}\} \subset \mathbb{P}^3, \tag{25}$$

and so $\mathrm{Fix}(\widehat{\delta}) \cap S = \emptyset$. As a standard generator must have fixed points, this case is not possible.

7. Proof of Proposition 15

Let K be a Fuchsian group of signature $(0, 3; t, s, r)$, where $2 \leq r \leq s \leq t$, say

$$K = \langle x_1, x_2 : x_1^t = x_2^r = (x_2x_1)^s = 1 \rangle.$$

In this section, if $A < K$, then we will use the notation $\langle\langle A \rangle\rangle$ to denote the smallest normal subgroup of K containing A. If $A < C < K$, then we use the notation $\langle\langle A \rangle\rangle_C$ to denote the smallest normal subgroup of C containing A.

Assume that K contains a Fuchsian subgroup U of signature $(0, 4; 2, 2, 2, k)$, $k \geq 3$, of index D. By Proposition 13, U contains a unique index 4 normal subgroup Γ of signature $(0, 4; k, k, k, k)$. Moreover, if we write

$$U = \langle w_1, w_2, w_3, w_4 : w_1^2 = w_2^2 = w_3^2 = w_4^k = w_4w_3w_2w_1 = 1 \rangle,$$

then

$$\Gamma = \langle y_1, y_2, y_3 : y_1^k = y_2^k = y_3^k = (y_3y_2y_1)^k = 1 \rangle,$$

where

$$\begin{cases} y_1 = w_3w_2w_1, \\ y_2 = w_2w_1w_3, \\ y_3 = w_1w_3w_2. \end{cases} \tag{26}$$

A set of generators of U as above will be called a *nice set of generators* of U. Note that

$$\Gamma' = \langle\langle [y_i, y_j];\, i, j \in \{1, 2, 3\}\rangle\rangle_\Gamma$$

and

$$\langle\langle \Gamma' \rangle\rangle = \langle\langle [y_i, y_j];\, i, j \in \{1, 2, 3\}\rangle\rangle.$$

As $U < K$ is of index D and signature $(0, 4; 2, 2, 2, k)$, it follows from Proposition 9 the existence of an epimorphism

$$\theta : K \to J < \mathfrak{S}_D,$$

with J acting transitively, $U = \theta^{-1}(J_1)$, where $J_1 = \{h \in J : h(1) = 1\}$, and non-negative integer values ρ_1, ρ_2 and ρ_3 satisfying $\rho_1 + \rho_2 + \rho_3 = 4$, such that

(1) $\theta(x_1)$ has precisely ρ_1 cycles of lengths less that t; these lengths being divisors of t and each of the short cycles, say of length α, produces a conical point of $\mathbb{H}^2/U$ of order $t/\alpha \in \{2, k\}$;
(2) $\theta(x_2)$ has precisely ρ_2 cycles of lengths less that r; these lengths being divisors of r and each of the short cycles, say of length β, produces a conical point of $\mathbb{H}^2/U$ of order $r/\beta \in \{2, k\}$;
(3) $\theta(x_2x_1)$ has precisely ρ_3 cycles of lengths less that s; these lengths being divisors of s and each of the short cycles, say of length γ, produces a conical point of $\mathbb{H}^2/U$ of order $s/\gamma \in \{2, k\}$.

Now, if $T \in \mathrm{Aut}(\mathfrak{S}_D)$, then we may also consider the homomorphism

$$\psi = T \circ \theta : K \to T(J) < \mathfrak{S}_D.$$

The subgroup $\widehat{U} = \psi^{-1}(J_1)$ is conjugate to U in K. Moreover, if $\widehat{\Gamma}$ is the unique normal subgroup of index 4 of signature $(0, 4; k, k, k, k)$ of $\widehat{U}$, then Γ' is a normal subgroup of K if and only if $\widehat{\Gamma}'$ is a normal subgroup of K (since U and $\widehat{U}$ are conjugate in K). This observation permits us to search for all the possibilities for $\theta(x_1)$ and $\theta(x_2)$ satisfying conditions (1), (2) and (3) up to conjugation in $\mathfrak{S}_D$ and to assume that $\theta^{-1}(J_1)$ equals U (otherwise, we will be working with a conjugate copy of U in K). In this way, in order to prove Proposition 15, we only need to prove that Γ' cannot be a normal subgroup of K.

If our choices for $\theta(x_1)$ and $\theta(x_2)$ happen to be inconsistent with the identity $\theta(x_2x_1) = \theta(x_2)\theta(x_1)$, that would mean that K cannot contain such a subgroup U and we are done in that case.

If on the contrary these choices are consistent, then the group J_1 can be easily described. This being done we will determine a nice set of generators of U by making a choice of elements of U satisfying the desired relations, and then using GAP [31] to check that the index of the subgroup U_1 of U generated by them is equal to 1. Next we write generators y_1, y_2 and y_3 for Γ as above. Clearly, Γ' is normal subgroup of K if and only if $\Gamma' = \langle\langle \Gamma' \rangle\rangle$. As $[K : \Gamma'] = [K : U][U : \Gamma][\Gamma : \Gamma'] = 4Dk^3$, in order to get a contradiction we only need to see that $[K : \langle\langle \Gamma' \rangle\rangle] \neq 4Dk^3$. The computation of the index $[K : \langle\langle \Gamma' \rangle\rangle]$ is done with the help of GAP.

Remark 20. In using GAP, we should be aware of the following fact. We multiply cycles from right to left whereas GAP does it from left to right. In order to get the same results, we need to make the computations in GAP reversing the order of the cycles in each permutation.

Proof of part (1)

Case $k = 6$, $D = 4$, $(r, s, t) = (2, 3, 12)$

As x_2x_1 has order 3, it follows that $\theta(x_2x_1)$ must fix some point. We may assume $\theta(x_2x_1)$ fixes 1. It follows then that $x_2x_1 \in U$, which implies that $\mathbb{H}^2/U$ should have a conical point of order 3, a contradiction.

Case $k = 8$, $D = 3$, $(r, s, t) = (2, 4, 8)$

As in the signature of U there is not a period 4, it follows that no conjugate of x_2x_1 can belong to U, that is, $\theta(x_2x_1)$ should not fix a point. But, as x_2x_1 has order 4 and $D = 3$, it follows that this must be the case, a contradiction.

Case $k = 5$, $D = 3$, $(r, s, t) = (2, 5, 5)$

As in the signature of U there is a period 5, we may assume $x_1 \in U$. As $\theta(x_1)^5 = 1$, it follows that $\theta(x_1) = (1)(2)(3)$. It follows we must have at least three periods of order 5 in the signature of U, a contradiction.

Case $k = 4$, $D = 3$, $(r, s, t) = (3, 3, 4)$

As in the signature of U there is a period 4, we may assume $x_1 \in U$. As $\theta(x_1)^4 = 1$, it follows that $\rho_1 \in \{2, 3\}$. As $D = 3$, it is clear that $\rho_2, \rho_3 \in \{0, 3\}$. With such a possibilities $\rho_1 + \rho_2 + \rho_3 \neq 4$, a contradiction.

Proof of part (2)

We have to analyze the ten cases listed in the first table of Proposition 15. Since the method is similar in most of them, we only describe it in three representative cases.

Case $k = 5$, $D = 6$, $(r, s, t) = (2, 4, 5)$

In this case, since U has a period of order 5 in its signature, we may assume, up to conjugation in K, that $\theta(x_1) \in J_1$. Next, up to permutation of the indices $2, \ldots, 6$, we may assume (as U has only one period of order 5 in its signature) that $\theta(x_1) = (1)(2\ 3\ 4\ 5\ 6)$. As $\rho_1 = 1$, we have that $\rho_2 + \rho_3 = 3$. If $\rho_2 \in \{1, 3\}$, then $\theta(x_2)$ should contain exactly ρ_2 cycles of length 1 and all the others of length 2. But this is impossible for $D = 6$; it follows that $\rho_2 \in \{0, 2\}$.

Subcase 1

If $\rho_2 = 0$, then $\rho_3 = 3$ and we must have

$$\theta(x_2) = (a_1\ a_2)(a_3\ a_4)(a_5\ a_6),\ \theta(x_2x_1) = (b_1\ b_2)(b_3\ b_4)(b_5\ b_6).$$

We may assume $a_1 = 1$ and, up to conjugation by a power of $\theta(x_1)$, that $a_2 = 2$. In this way, the possibilities are

$$\theta(x_2) \in \{(1\ 2)(3\ 4)(5\ 6),\ (1\ 2)(3\ 5)(4\ 6),\ (1\ 2)(3\ 6)(4\ 5)\}$$

In the case $\theta(x_2) = (1\ 2)(3\ 4)(5\ 6)$ we have that $\theta(x_2x_1) = \theta(x_2)\theta(x_1) = (1\ 2\ 4\ 6)(3)(5)$; in the case $\theta(x_2) = (1\ 2)(3\ 5)(5\ 6)$ we have that $\theta(x_2x_1) = \theta(x_2)\theta(x_1) = (1\ 2\ 5\ 4\ 3\ 6)$; and in the case $\theta(x_2) = (1\ 2)(3\ 6)(4\ 5)$ we have that $\theta(x_2x_1) = \theta(x_2)\theta(x_1) = (1\ 2\ 6)(3\ 5)(4)$. Each of these cases produces a contradiction with the desired cycle decomposition of $\theta(x_2x_1)$.

Subcase 2

If $\rho_2 = 2$, then $\rho_3 = 1$ and we must have

$$\theta(x_2) = (a_1\ a_2)(a_3\ a_4),\ \theta(x_2x_1) = (b_1\ b_2)(b_3\ b_4\ b_5\ b_6).$$

If $\theta(x_2)(1) = 1$, then $\theta(x_2x_1)(1) = 1$, a contradiction to our assumption. So, we may assume $a_1 = 1$. Also, up to conjugation by a power of $\theta(x_1)$, we may also assume that $a_2 = 2$. The possibilities are

$$\theta(x_2) \in \{(1\ 2)(3\ 4),\ (1\ 2)(3\ 5),\ (1\ 2)(3\ 6),\ (1\ 2)(4\ 5),\ (1\ 2)(4\ 6),\ (1\ 2)(5\ 6)\}$$

Therefore, the only chances to have a cycle decomposition of $\theta(x_2x_1)$ as desired are given by

$$\theta(x_2) = (1\ 2)(3\ 5) \implies \theta(x_2x_1) = (1\ 2\ 5\ 6)(3\ 4)$$
$$\theta(x_2) = (1\ 2)(4\ 6) \implies \theta(x_2x_1) = (1\ 2\ 3\ 6)(4\ 5).$$

Both cases are the same up to the (geometric) automorphism of K defined by $\phi(x_1) = x_1^{-1}$, $\phi(x_2) = x_2$. So we only need to consider one of them, say $\theta(x_2) = (1\ 2)(3\ 5)$. In this case, $J = \theta(K) = \langle (2\ 3\ 4\ 5\ 6), (1\ 2)(3\ 5) \rangle$ and $J_1 = \langle (2\ 3\ 4\ 5\ 6), (3\ 4\ 5) \rangle$.

Let us consider the subgroup U_1 generated by the elements $z_1 = x_1$, $z_2 = x_2x_1^{-1}x_2x_1^{-2}x_2$, $z_3 = x_2x_1^3x_2x_1^{-3}x_2$ and $z_4 = x_2x_1x_2x_1^{-1}x_2$ (see Fig. 1 for a fundamental polygon for U_1 and the side pairings). We have that $U_1 < U$ and that $[K : U_1] = 6 = [K : U]$; so $U_1 = U$.

Notice that $w_1 = x_2x_1x_2x_1^{-1}x_2$, $w_2 = x_2x_1^3x_2x_1^{-3}x_2$, $w_3 = x_2x_1^3(x_1x_2)^2x_1^{-3}x_2$ and $w_4 = x_1^{-1}$ are nice generators of U. So, generators of Γ are given by $y_1 = x_1$, $y_2 = x_2x_1^{-2}x_2x_1^{-2}x_2x_1^2x_2x_1^{-1}x_2x_1x_2$ and $y_3 = x_2x_1x_2x_1^{-2}x_2x_1^{-2}x_2$. We obtain, by using GAP, that $[K : \langle\langle \Gamma' \rangle\rangle] = 10 \neq 4Dk^3$, in particular, Γ' cannot be normal subgroup of K.

Case $k = 8$, $D = 9$, $(r, s, t) = (2, 3, 8)$

As U has an element of order 8, then we may assume $\theta(x_1) \in J_1$, that is, $\rho_1 \geq 1$. Let us recall that ρ_1 denotes the number of cycles of $\theta(x_1)$ whose lengths are divisors of 8 and different from 8; therefore $\rho_1 \neq 2$. Let us now note that $\rho_2 \geq 1$; otherwise every cycle of $\theta(x_2)$ must have length 2, not possible using $D = 9$ letters. As $\rho_1 + \rho_2 \leq 4$, it follows that $\rho_1 \in \{1, 3\}$. Similarly, $\rho_3 \neq 1$; otherwise $\theta(x_2x_1)$ has exactly one cycle of length less than 3 and, as such a length is a divisor of 3, it has length 1. This clearly is again impossible to achieve with $D = 9$ letters.

Up to permutation of the indices 2, 3, 4, 5, 6, 7, 8, 9, we may assume that $\theta(x_1)$ is one of the following ones:

$$\theta(x_1)=\begin{cases}(2\ 3\ 4\ 5\ 6\ 7\ 8\ 9), & \rho_1=1\\ (2\ 3\ 4\ 5)(6\ 7\ 8\ 9), & \rho_1=3.\end{cases} \tag{27}$$

Subcase 1

Let us assume $\theta(x_1)=(2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)$. As in this case, $\rho_2+\rho_3=3$, $\rho_2\geq 1$ and $\rho_3\neq 1$, we have the following possibilities: $(\rho_2,\rho_3)\in\{(1,2),(3,0)\}$. The case $\rho_3=2$ ensures that $\theta(x_2x_1)$ has exactly two cycles of length 1 and all the others of length 3; this is not possible with $D=9$ letters. So, we have that $\rho_2=3$ and $\rho_3=0$; that is $\theta(x_2)=(a_1\ a_2)(a_3\ a_4)(a_5\ a_6)$ and $\theta(x_2x_1)=(b_1\ b_2\ b_3)(b_4\ b_5\ b_6)(b_7\ b_8\ b_9)$. The only possibilities for $\theta(x_2)$, up to conjugation by powers of $\theta(x_1)$, are

$$\theta(x_2)\in\{(1\ 3)(2\ 4)(6\ 9),(1\ 3)(2\ 4)(5\ 8)\}.$$

If we consider the groups (both of order 432)

$$L_1=\langle(2\ 3\ 4\ 5\ 6\ 7\ 8\ 9),(1\ 3)(2\ 4)(5\ 8)\rangle$$

and

$$L_2=\langle(2\ 3\ 4\ 5\ 6\ 7\ 8\ 9),(1\ 3)(2\ 4)(6\ 9)\rangle=\langle(2\ 9\ 8\ 7\ 6\ 5\ 4\ 3),(1\ 4)(2\ 7)(3\ 5)\rangle,$$

then there is an isomorphism $\phi:L_1\to L_2$ defined by

$$\phi((2\ 3\ 4\ 5\ 6\ 7\ 8\ 9))=(2\ 9\ 8\ 7\ 6\ 5\ 4\ 3)$$
$$\phi((1\ 3)(2\ 4)(5\ 8))=(1\ 4)(2\ 7)(3\ 5).$$

As a consequence, we only need to consider one of the two possibilities for $\theta(x_2)$, say $\theta(x_2)=(1\ 3)(2\ 4)(6\ 9)$. In this case,

$$\begin{cases}J=\langle(2\ 3\ 4\ 5\ 6\ 7\ 8\ 9),\ (1\ 3)(2\ 4)(5\ 8)\rangle\\ J_1=\langle(3\ 5)(4\ 8)(7\ 9),\ (2,\ 6)(3,\ 8)(4,;7),\ (2,\ 7)(3,\ 6)(5,\ 9),\ (2\ 5\ 6\ 9)(3\ 8\ 7\ 4)\rangle.\end{cases}$$

The group U is generated by the elements $w_1=x_2x_1^{-2}x_2x_1^2x_2x_1^3x_2$, $w_2=x_2x_1^{-3}x_2x_1^3x_2$, $w_3=x_2x_1^2x_2x_1^{-2}x_2$ and $w_4=x_1$. In this case, Γ is generated by $y_1=x_1^{-1}$, $y_2=x_2x_1^3x_2x_1^{-4}x_2x_1^{-2}x_2$ and $y_3=x_2x_1^{-2}x_2x_1^2x_2x_1^{-3}x_2x_1^3x_2x_1^3x_2$. Now it can be seen that $[K:\langle\langle\Gamma'\rangle\rangle]=2\neq 4Dk^3$, from which we infer that Γ' cannot be normal in K.

Subcase 2

Let us first assume $\theta(x_1)=(2\ 3\ 4\ 5)(6\ 7\ 8\ 9)$. In this case $\rho_2+\rho_3=1$ and $\rho_2\geq 1$. It follows that $\rho_3=0$ and $\rho_2=1$. This now ensures that $\theta(x_2x_1)$ is product of exactly three disjoint 3-cycles and that $\theta(x_2)$ has exactly one cycle of length 1 and all the others of length 2. Clearly, $\theta(x_2)$ cannot fix 1, for otherwise, $\theta(x_2x_1)$ will have a 1-cycle, a contradiction. Up to conjugation by $\theta(x_1)$, we may assume that

$$\theta(x_2)=(2)(1\ a_2)(a_3\ a_4)(a_5\ a_6)(a_7\ a_8)$$

and

$$\theta(x_2x_1)=(b_1\ b_2\ b_3)(b_4\ b_5\ b_6)(b_7\ b_8\ b_9).$$

Using GAP, one can check that the above cannot happen.

Case $k=7$, $D=15$, $(r,s,t)=(2,3,7)$

In this case (and only in this case) we use a different kind of argument (one may also proceed similarly as in the other cases, but the computations are lengthy in the symmetric group in 15 letters). Let $S=\mathbb{H}^2/\Gamma'$, let $H=\Gamma/\Gamma'<\mathrm{Aut}(S)$ and let us assume that Γ' is a normal subgroup of K. By means of GAP we see that in this situation $F=K/\Gamma'<\mathrm{Aut}(S)$ is of order $|\ F\ |=[K:\Gamma][\Gamma:\Gamma']=60*7^3=20\,580$. Note that the number of 7-Sylow subgroups is greater than 1, for otherwise, $H\lhd F$ and F/H will be a group of (orbifold) automorphisms of the orbifold $\mathbb{H}^2/\Gamma$ of order 60, which is not possible by the computations in Section 5: the only possible orbifold automorphism groups are $\mathbb{Z}_2^2$, D_4 and $\mathcal{A}_4$. On the other hand, it is known that there is no simple group of order 20 580, therefore we may construct a non-trivial chain

$$F_1\lhd F_2\lhd\cdots F_r\lhd F_{r+1}=F, \tag{28}$$

with F_{j+1}/F_j a simple group and F/F_r non-trivial. As the only order of a non-cyclic simple group that divides 20 580 is 60, the group F/F_r is either cyclic or $\mathcal{A}_5$. This quotient cannot be $\mathcal{A}_5$ as in that case F_r would be a normal 7-Sylow subgroup, a contradiction. It follows that F/F_r is a non-trivial abelian group. But, $F=K/\Gamma'$ and $F_r=K_r/\Gamma'$, for some $K_r\lhd K$, and $K/K_r\cong F/F_r$. It follows that $K'\lhd K_r$. Now, as $K/K'\cong\langle a,b:a^7=b^2=(ba)^3=1,ab=ba\rangle=\langle a,b:a^1=b^1=1\rangle$, it follows that $[K:K']=1$. We get that $K_r=K$, so $F_r=F$, a contradiction.

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