Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Generalized Fermat curves

Gabino González-Diez [a,*,1], Rubén A. Hidalgo [b,2], Maximiliano Leyton [c]

[a] Departamento de Matemáticas, UAM, Madrid, Spain
[b] Departamento de Matemáticas, Universidad Técnica Federico Santa María, Valparaíso, Chile
[c] Institut Fourier-UMR5582, Université Joseph Fourier, France

ARTICLE INFO

Article history:
Received 12 November 2007
Available online xxxx
Communicated by Corrado de Concini

Keywords:
Riemann surfaces
Fuchsian groups
Automorphisms

ABSTRACT

A closed Riemann surface S is a generalized Fermat curve of type (k,n) if it admits a group of automorphisms $H \cong Z_k^n$ such that the quotient $\mathcal{O} = S/H$ is an orbifold with signature $(0, n+1; k, \dots, k)$, that is, the Riemann sphere with $(n+1)$ conical points, all of same order k. The group H is called a generalized Fermat group of type (k,n) and the pair (S,H) is called a generalized Fermat pair of type (k,n). We study some of the properties of generalized Fermat curves and, in particular, we provide simple algebraic curve realization of a generalized Fermat pair (S,H) in a lower-dimensional projective space than the usual canonical curve of S so that the normalizer of H in $Aut(S)$ is still linear. We (partially) study the problem of the uniqueness of a generalized Fermat group on a fixed Riemann surface. It is noted that the moduli space of generalized Fermat curves of type (p,n), where p is a prime, is isomorphic to the moduli space of orbifolds of signature $(0, n+1; p, \dots, p)$. Some applications are: (i) an example of a pencil consisting of only non-hyperelliptic Riemann surfaces of genus $g_k = 1 + k^3 - 2k^2$, for every integer $k \geqslant 3$, admitting exactly three singular fibers, (ii) an injective holomorphic map $\psi : \mathbb{C} - \{0,1\} \to \mathcal{M}_g$, where $\mathcal{M}_g$ is the moduli space of genus $g \geqslant 2$ (for infinitely many values of g), and (iii) a description of all complex surfaces isogenous to a product with invariants $p_g = q = 0$ and covering group equal to $\mathbb{Z}_5^2$ or $\mathbb{Z}_2^4$.

* Corresponding author.
E-mail addresses: gabino.gonzalez@uam.es (G. González-Diez), ruben.hidalgo@usm.cl (R.A. Hidalgo), Maximiliano.Leyton@ujf-grenoble.fr (M. Leyton).

[1] Partially supported by project BFM2003-04964 of the MCYT.
[2] Partially supported by projects Fondecyt 1070271 and UTFSM 12.08.01.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.01.002

1. Preliminaries

If S denotes a closed Riemann surface, then $Aut(S)$ will denote its full group of conformal automorphisms. If $H < Aut(S)$, then we denote by $Aut_H(S)$ the normalizer of H inside $Aut(S)$, that is, the highest subgroup of $Aut(S)$ containing H as normal subgroup. If K is any group, we denote by K' its commutator subgroup.

An orbifold $\mathcal{O}$ of signature $(g, r; n_1, \ldots, n_r)$, where $g \geqslant 0$, $r \geqslant 0$, $n_j \geqslant 2$, are integers, is a closed Riemann surface of genus g together with a collection of r conical points of orders $n_1, \ldots, n_r$. Its orbifold fundamental group, $\pi_1^{orb}(\mathcal{O})$, is a group with generators $a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_r$ and relations $\prod_{j=1}^{g} [a_j, b_j] \prod_{s=1}^{r} c_j = 1 = c_1^{n_1} = \cdots = c_r^{n_r}$, where $[a, b] = aba^{-1}b^{-1}$. Generalities on orbifolds can be found in [26,29].

An orbifold of signature $(g, 0; -)$ is a closed Riemann surface of genus g. A conformal automorphism of an orbifold is a conformal automorphism of the corresponding Riemann surface which preserves the conical points of the same orders. The homology cover of an orbifold $\mathcal{O}$ is an orbifold $\widetilde{\mathcal{O}}$ providing the highest regular Abelian cover of it, that is, the regular covering induced by the commutator subgroup $\pi_1^{orb}(\mathcal{O})' \lhd \pi_1^{orb}(\mathcal{O})$. In many cases $\widetilde{\mathcal{O}}$ has no conical points, that is, it turns out to be a closed Riemann surface.

A closed Riemann surface S is called a *generalized Fermat curve of type* (k, n) if it is the homology cover of an orbifold of signature $(0, n+1; k, \ldots, k)$. It follows that there exists $H < Aut(S)$, so that $H \cong \mathbb{Z}_k^n$ and S/H is an orbifold with signature $(0, n+1; k, \ldots, k)$. In this case we say that H is a *generalized Fermat group of type* (k, n) and (S, H) a *generalized Fermat pair of type* (k, n). Reciprocally, each pair (S, H) so that S is a closed Riemann surface, $H < Aut(S)$ is isomorphic to $\mathbb{Z}_k^n$ and S/H is an orbifold with signature $(0, n+1; k, \ldots, k)$ is a generalized Fermat pair of type (k, n). Riemann–Hurwitz's formula asserts that the genus $g_{k,n}$ of a generalized Fermat curve of type (k, n) is

$$g_{k,n} = \frac{2 + k^{n-1}((n-1)(k-1) - 2)}{2}. \tag{1}$$

In particular, a generalized Fermat curve of type (k, n) is hyperbolic, that is, it has the hyperbolic plane $\mathbb{H}^2$ as universal cover Riemann surface, if and only if $(n-1)(k-1) > 2$. As an example, the classical Fermat curve $x^k + y^k + z^k = 0$ defines, in the complex projective plane $\mathbb{P}^2$, a generalized Fermat curve of type $(k, 2)$. Examples of generalized Fermat curves of type $(2, n)$ were studied in [9] from the point of view of Fuchsian and Schottky groups (see also Section 6).

We say that two generalized Fermat pairs (S_1, H_1) and (S_2, H_2) are topologically (holomorphically) equivalent if there is some orientation-preserving homeomorphism (holomorphic homeomorphism) $f : S_1 \to S_2$ so that $f H_1 f^{-1} = H_2$.

The non-hyperbolic generalized Fermat pairs are the following ones.

(i) $(k, n) = (2, 2)$: $S = \widehat{\mathbb{C}}$ and $H = \langle A(z) = -z, B(z) = 1/z \rangle$.

(ii) $(k, n) = (3, 2)$: $S = \mathbb{C}/\Lambda_{e^{2\pi i/3}}$, where $\Lambda_{e^{2\pi i/3}} = \langle A(z) = z + 1, B(z) = z + e^{2\pi i/3} \rangle$, and H is generated by the induced transformations of $J(z) = e^{2\pi i/3} z$ and $T(z) = z + (2 + e^{2\pi i/3})/3$. In this case, the 3 cyclic groups $\langle J \rangle$, $\langle TJ \rangle$ and $\langle T^2 J \rangle$ project to the only 3 cyclic subgroups in H with fixed points; 3 fixed points each one. This is provided by the degree 3 Fermat curve $x^3 + y^3 + z^3 = 0$.

(iii) $(k, n) = (2, 3)$: $S = \mathbb{C}/\Lambda_\tau$, where $\tau \in \mathbb{C}$ with $Im(\tau) > 0$, $\Lambda_\tau = \langle A(z) = z + 1, B(z) = z + \tau \rangle$, and H is generated by the induced transformations from $T_1(z) = -z$, $T_2(z) = -z + 1/2$ and $T_3(z) = -z + \tau/2$. In this case, the conformal involutions induced on the torus by T_1, T_2, T_3 and their product are the only ones acting with fixed points; 4 fixed points each. This is also described by the algebraic curve $\{x^2 + y^2 + z^2 = 0,\ \lambda x^2 + y^2 + w^2 = 0\}$, where $\lambda \in \mathbb{C} - \{0, 1\}$.

The hyperbolic generalized Fermat pair (S, H) of type (k, n) can be described in terms of Fuchsian groups as follows. Classical uniformization theorem asserts that the orbifold $\mathcal{O} = S/H$ is uniformized by a Fuchsian group $\Gamma < PSL(2, \mathbb{R})$, that is, $\mathbb{H}^2/\Gamma = \mathcal{O}$. The group Γ has presentation (we say that Γ is of type (k, n))

Please cite this article in press as: G. González-Diez et al., Generalized Fermat curves, J. Algebra (2009), doi:10.1016/j.jalgebra.2009.01.002

$$\Gamma = \langle x_1, \ldots, x_{n+1}\colon\ x_1^k = \cdots = x_{n+1}^k = x_1 x_2 \cdots x_{n+1} = 1 \rangle. \tag{2}$$

There is a torsion free normal subgroup $L \lhd \Gamma$ providing a uniformization of S and so that $H = \Gamma/L$. The commutator subgroup Γ' is torsion free [22] and uniformizes a closed Riemann surface S'. On S' we have the Abelian group $H' = \Gamma/\Gamma' \cong \mathbb{Z}_k^n$ so that $S'/H' = \mathcal{O}$, that is, S' is a generalized Fermat curve of type (k,n) with generalized Fermat group H' of type (k,n). As $L \lhd \Gamma$ satisfies that Γ/L is Abelian, we must have that $\Gamma' \lhd L$. As the index of L in Γ is equal to the index of Γ' in Γ, we have that $L = \Gamma'$. As Fuchsian groups of a fixed type (k,n) are topologically rigid and the commutator subgroup Γ' is a characteristic subgroup, all the above can be summarized in the following.

Theorem 1. *Let (S,H) be a generalized Fermat pair and Γ be a (orbifold) universal cover group of the orbifold S/H. Then (S,H) is holomorphically equivalent to $(U/\Gamma, \Gamma/\Gamma')$, where $U \in \{\widehat{\mathbb{C}}, \mathbb{C}, \mathbb{H}^2\}$ is the universal Riemann surface cover of S/H and Γ' is the commutator subgroup of Γ. In particular, any two generalized Fermat pairs of the same type are topologically equivalent.*

Denoting by a_j the congruence class of x_j mod Γ', we easily obtain the following consequences of Theorem 1, which we will need later.

Corollary 2. *Let (S,H) be a generalized Fermat pair of type (k,n) and let $P : S \to S/H$ be a branched regular covering with H as group of cover transformations.*

1.- *If $\mathrm{Aut}_H(S)$ denotes the normalizer of H inside $\mathrm{Aut}(S)$, then each orbifold automorphism of S/H lifts to an automorphism in $\mathrm{Aut}_H(S)$; that is, for each orbifold automorphism $\tau : S/H \to S/H$ there is a conformal automorphism $\widehat{\tau} : S \to S$ so that $P\widehat{\tau} = \tau P$.*
2.- *There exist elements of order k in H, say $a_1, \ldots, a_n$, so that:*
  (i) *$H = \langle a_1, \ldots, a_n \rangle$;*
  (ii) *each $a_1, \ldots, a_n$ and $a_{n+1} = a_1 a_2 \cdots a_n$ has exactly k^{n-1} fixed points;*
  (iii) *if $h \in H$ has fixed points, then $h \in \langle a_1 \rangle \cup \cdots \cup \langle a_n \rangle \cup \langle a_{n+1} \rangle$;*
  (iv) *if k is prime and $h \in H$ has no fixed points and it has order k, then no non-trivial power of h has fixed points; and*
  (v) *if $h \in H$ is an element of order k with fixed points and x, y are any two of these fixed points, then there is some $h^* \in H$ so that $h^*(x) = y$.*

  Such a set of generators $a_1, \ldots, a_n$ shall be called a standard set of generators for the generalized Fermat group H.

Remark 3. If (S,H) is a generalized Fermat pair of type (k,n), then for each $h \in H$ of order k with fixed points, we have that the Riemann surface structure of the orbifold $R = S/\langle h \rangle$ is a generalized Fermat curve of type $(k, n-1)$ with $K = H/\langle h \rangle$ as a generalized Fermat group of type $(k, n-1)$. This fact permits to construct towers of generalized Fermat curves starting from some non-hyperbolic one and adding an extra conical point at each step.

Having recalled all the necessary general basic facts and descriptions, we will proceed in the rest of this paper as follows. In Section 2 we note that generalized Fermat curves of genus at least 2 are non-hyperelliptic and we provide algebraic curve description of them (as fiber products of classical Fermat curves) in lower-dimensional projective spaces than the usual canonical curve (this may be of interest for computation on Riemann surfaces). In Section 3 we discuss the problem of uniqueness of generalized Fermat groups on each generalized Fermat curve. In Section 4 we describe the locus of generalized Fermat curves in the corresponding moduli space. In Section 5 we proceed to use the results obtained in the previous sections in order to produce three examples. The first one is the construction of a pencil of non-hyperelliptic Riemann surfaces of genus $g_k = 1 + k^3 - 2k^2$, for every integer $k \geqslant 3$, with exactly three singular fibers. The second one is the construction of an injective holomorphic map $\psi : \mathbb{C} - \{0,1\} \to \mathcal{M}_g$, where $\mathcal{M}_g$ is the moduli space of genus $g \geqslant 2$, for infinitely many values of g. The third one is the description of all complex surfaces isogenous to a product

Please cite this article in press as: G. González-Diez et al., Generalized Fermat curves, J. Algebra (2009), doi:10.1016/j.jalgebra.2009.01.002

JID:YJABR AID:12369 /FLA [m1G; v 1.77; Prn:15/01/2009; 8:13] P.4 (1-18)

$X = S_1 \times S_2/G$ with invariants $p_g = q = 0$ and group G equals either $G = \mathbb{Z}_5^2$ or $G = \mathbb{Z}_2^4$. Finally, in Section 6 we describe those generalized Fermat curves which can act as group of isometries of hyperbolic handlebodies.

2. Algebraic description

2.1. Non-hyperellipticity of general Fermat curves

Let us recall that a closed Riemann surface S of genus $g \geqslant 2$ is called *hyperelliptic* if it admits a (necessarily unique) conformal involution with exactly $2(g+1)$ fixed points, called the *hyperelliptic involution*. Equivalently, S is hyperelliptic if and only if there is a two-fold branched regular covering $f : S \to \widehat{\mathbb{C}}$. The algebraic equation of a hyperelliptic Riemann surface S is of the form $y^2 = \prod_{j=1}^{2(g+1)}(x - a_j)$ and, moreover, as the hyperelliptic involution belongs to the center of $Aut(S)$, a complete description of $Aut(S)$ can be done via such a curve description [6].

Unfortunately, generalized Fermat curves of genus at least 2 are non-hyperelliptic Riemann surfaces.

Theorem 4. *A hyperbolic generalized Fermat curve is non-hyperelliptic.*

Proof. Let (S, H) be a hyperbolic generalized Fermat pair of type (k, n). Let us assume S is hyperelliptic and let us denote by j its hyperelliptic involution. As j belongs to the center of $Aut(S)$, the action of H descends to a finite Abelian group of Möbius transformations $PSL(2, \mathbb{C})$ on the Riemann sphere. The fact that the finite Abelian groups inside $PSL(2, \mathbb{C})$ are either finite cyclic groups or the Klein group $\mathbb{Z}_2^2$ obligates to have either (i) $k > 2$ and $n = 1$, or (ii) $k = 2$ and $n = 1, 2$, a contradiction. □

2.2. Algebraic description

As a generalized Fermat curve S of genus $g \geqslant 2$ is non-hyperelliptic, it is well known that, if $w_1, \ldots, w_g$ is any basis of $H^{1,0}(S)$, the g-dimensional vector space of holomorphic 1-forms, then $\theta : S \to \mathbb{P}^{g-1}$ defined by $\theta(x) = [w_1(x) : \cdots : w_g(x)]$, is a holomorphic embedding and $\theta(S)$ is a non-singular projective algebraic curve of degree $2(g-1)$, called a canonical curve of S (see, for instance, [10]).

Assume (S, H) is generalized Fermat pair of type (k, n) and genus $g = g_{k,n} \geqslant 2$. Theorem 5 below provides a holomorphic embedding $\rho : S \to \mathbb{P}^n$ so that $\rho(S)$ is a non-singular projective algebraic curve of degree $k^{n-1} \leqslant 2(g-1)$ and the action of H is defined by a projective linear action $H_0 < PGL(n+1, \mathbb{C})$. Note that this description produces a projective curve of lower degree than a canonical curve and also the embedding is in lower dimension than the canonical embedding (this lower degree and dimension representation seems to be useful for practical computations). Not only the action of H will be linear, but also the action of $Aut_H(S)$.

Let us start with the following fiber product of $(n-1)$ classical Fermat curves, say the algebraic curve $C(\lambda_1, \ldots, \lambda_{n-2}) \subset \mathbb{P}^n$ given by the following $(n-1)$ homogeneous polynomials of degree k

$$\left\{\begin{array}{rcl} x_1^k + x_2^k + x_3^k & = & 0, \\ \lambda_1 x_1^k + x_2^k + x_4^k & = & 0, \\ \lambda_2 x_1^k + x_2^k + x_5^k & = & 0, \\ \vdots & \vdots & \vdots \\ \lambda_{n-2} x_1^k + x_2^k + x_{n+1}^k & = & 0, \end{array}\right. \tag{3}$$

where $\lambda_j \in \mathbb{C} - \{0, 1\}$, $\lambda_i \neq \lambda_j$, for $i \neq j$. The conditions on the parameters λ_j ensure that $C(\lambda_1, \ldots, \lambda_{n-2})$ is a non-singular projective algebraic curve, that is, a closed Riemann surface. On

Please cite this article in press as: G. González-Diez et al., Generalized Fermat curves, J. Algebra (2009), doi:10.1016/j.jalgebra.2009.01.002

$C(\lambda_1, \ldots, \lambda_{n-2})$ we have the Abelian group $H_0 \cong \mathbb{Z}_k^n$ of conformal automorphisms generated by the transformations

$$a_j([x_1 : \cdots : x_{n+1}]) = [x_1 : \cdots : x_{j-1} : \omega_k x_j : x_{j+1} : \cdots : x_{n+1}], \quad j = 1, \ldots, n,$$

where $\omega_k = e^{2\pi i/k}$. If we consider the degree k^n holomorphic map

$$\pi : C(\lambda_1, \ldots, \lambda_{n-2}) \to \widehat{\mathbb{C}}$$

given by

$$\pi([x_1 : \cdots : x_{n+1}]) = -\left(\frac{x_2}{x_1}\right)^k,$$

then $\pi \circ a_j = \pi$, for every a_j, $j = 1, \ldots, n$. It follows that $C(\lambda_1, \ldots, \lambda_{n-2})$ is a generalized Fermat curve with generalized Fermat group H_0 with standard generators $a_1, \ldots, a_n$, $a_{n+1} = a_1 a_2 \cdots a_n$. The fixed points of a_j on $C(\lambda_1, \ldots, \lambda_{n-2})$ are given by the intersection $Fix(a_j) = F_j \cap C(\lambda_1, \ldots, \lambda_{n-2})$ where $F_j = \{x_j = 0\}$. In this way, the branch values of π are given by the points

$$\pi(Fix(a_1)) = \infty, \qquad \pi(Fix(a_2)) = 0, \qquad \pi(Fix(a_3)) = 1,$$

$$\pi(Fix(a_4)) = \lambda_1, \quad \ldots, \quad \pi(Fix(a_{n+1})) = \lambda_{n-2}.$$

As every generalized Fermat pair (S, H) is uniquely determined by the orbifold S/H (by Theorem 1), we get the following algebraic description of generalized Fermat pairs.

Theorem 5. *Let (S, H) be a generalized Fermat pair of type (k, n) and, up to a Möbius transformation, let*

$$\{\infty, 0, 1, \lambda_1, \lambda_2, \ldots, \lambda_{n-2}\}$$

be the conical points of S/H, then (S, H) and $(C(\lambda_1, \ldots, \lambda_{n-2}), H_0)$ are holomorphically equivalent and the action of H is defined by the projective linear action $H_0 < PGL(n+1, \mathbb{C})$. We say that (S, H) is modeled by the algebraic curve $C(\lambda_1, \ldots, \lambda_{n-2})$.

The structure of equations in (3) and Theorem 5 provides the following.

Corollary 6. *A generalized Fermat curve of type (k, n) is the fiber product of $(n-1)$ classical Fermat curves of type $(k, 2)$.*

Remark 7. Note that there are $(n+1)!$ choices in the normalization of the conical points as described in Theorem 5. This normalization provides a canonical action of the symmetric group $\mathfrak{S}_{n+1}$ on the locus of ordered tuples $(\lambda_1, \ldots, \lambda_{n-2})$. We will come back to this discussion in Section 4.

Corollary 8. *If $(C(\lambda_1, \ldots, \lambda_{n-2}), H_0)$ is a generalized hyperbolic Fermat pair of type (k, n), then $Aut_{H_0}(C(\lambda_1, \ldots, \lambda_{n-2}))/H_0$ is isomorphic to the subgroup of Möbius transformations that preserves the finite set*

$$\{\infty, 0, 1, \lambda_1, \lambda_2, \ldots, \lambda_{n-2}\}.$$

Please cite this article in press as: G. González-Diez et al., Generalized Fermat curves, J. Algebra (2009), doi:10.1016/j.jalgebra.2009.01.002

Corollary 9. *If $(C(\lambda_1,\dots,\lambda_{n-2}),H_0)$ is a generalized hyperbolic Fermat pair of type (k,n), then $Aut_{H_0}(C(\lambda_1,\dots,\lambda_{n-2}))<PGL(n+1,\mathbb{C})$.*

Proof. Let (S,H) be a generalized Fermat curve. As consequence of Theorem 5 we may assume $(S,H)=(C(\lambda_1,\dots,\lambda_{n-2}),H_0)$. Let T be a Möbius transformation that permutes the conical points $\mu_1=\infty$, $\mu_2=0$, $\mu_3=1$, $\mu_4=\lambda_1,\dots,\mu_{n+1}=\lambda_{n-2}$. Corollary 8 asserts the existence of a conformal automorphism $\widehat{T}$ of $C=C(\lambda_1,\dots,\lambda_{n-2})$ so that $\pi\widehat{T}=T\pi$, that is, if $\widehat{T}([x_1:\dots:x_{n+1}])=[y_1:\dots:y_{n+1}]$ and $z=-(x_2/x_1)^k$, then $T(z)=-(y_2/y_1)^k$. As the only cyclic subgroups of H or order k acting with fixed points are $\langle a_j\rangle$, for $j=1,\dots,n+1$, there is a permutation $\sigma\in\mathfrak{S}_{n+1}$ (the symmetric group on $(n+1)$ letters) so that $\widehat{T}\langle a_j\rangle\widehat{T}^{-1}=\langle a_{\sigma(j)}\rangle$. Now, since the zeros and poles of the meromorphic function $x_j/x_1:C\to\widehat{\mathbb{C}}$ are $Fix(a_j)$ and $Fix(a_1)$ respectively, we see that the zeros and poles of the pullback function $\widehat{T}^*(x_j/x_1)=(x_j/x_1)\circ\widehat{T}:C\to\widehat{\mathbb{C}}$ are $Fix(a_{\sigma^{-1}(j)})$ and $Fix(a_{\sigma^{-1}(1)})$ respectively. It follows that there exist $c_2,\dots,c_{n+1}\in C-\{0\}$ such that $\widehat{T}^*(x_j/x_1)=c_j(x_{\sigma^{-1}(j)}/x_{\sigma^{-1}(1)})$ on C. This means that in the open set $\{x_1\neq 0\}$ the expression of the automorphism $\widehat{T}$ in terms of affine coordinates is

$$\widehat{T}(x_2/x_1,\dots,x_{n+1}/x_1)=(c_2x_{\sigma^{-1}(2)}/x_{\sigma^{-1}(1)},\dots,c_{n+1}x_{\sigma^{-1}(n+1)}/x_{\sigma^{-1}(1)}).$$

Therefore, in projective coordinates $\widehat{T}$ must be of the form

$$\widehat{T}\big([x_1:x_2:\dots:x_{n+1}]\big)=[x_{\sigma^{-1}(1)}:c_2x_{\sigma^{-1}(2)}:\dots:c_{n+1}x_{\sigma^{-1}(n+1)}].$$

The constants c_j can be easily computed from the algebraic equations (3). Now, as $T(z)=-c_2^k(x_{\sigma^{-1}(2)}/x_{\sigma^{-1}(1)})^k$, if we set $\mu_1=\infty$, $\mu_2=0$, $\mu_3=1$ and $\mu_{j+3}=\lambda_j$, for $j=1,\dots,n-2$, then

$$T(\mu_j)=\mu_{\sigma(j)},$$

that is, the transformation T induces the same permutation of the index set $\{1,\dots,n+1\}$ as $\widehat{T}$. This permits to compute easily $Aut_H(S)$ (this can be implemented into a computer program) and also to see that it is a subgroup of $PGL(n+1;\mathbb{C})$ whose matrix coordinates belong to the field $\mathbb{Q}(e^{2\pi i/k},c_2,\dots,c_{n-2})$. □

Remark 10. The procedure described in the above proof permits to define for each Möbius transformation T that permutes the conical points $\{\infty,0,1,\lambda_1,\dots,\lambda_{n-2}\}$ an element $\widehat{T}\in PGL(n+1,\mathbb{C})$. Such element is not uniquely defined by T, but it is unique up to composition with an element of H_0. Also, this representation satisfies that $\widehat{T_1T_2}=\widehat{T}_1\widehat{T}_2$ (up to composition with some element of H_0).

In the following examples we clarify the procedure given in the proof of Corollary 9.

Example 11. Let us clarify the above with the following example. Consider $n=4$ and $\lambda_2=\lambda_1^2=1+\lambda_1$, then we obtain a generalized Fermat curve S of genus $g_k=(2-5k^3+3k^4)/2$. If $k=2$, then S corresponds to the classical Humbert curve of genus 5 with 160 conformal automorphisms. In this case, the finite Möbius group that permutes the conical values ∞, 0, 1, λ_1 and λ_2 is the dihedral group D_5 generated by $u(z)=\lambda_2/(\lambda_2-z)$ (permuting cyclically ∞, 0, 1, λ_1 and λ_2) and $v(z)=\lambda_2-z$ (fixing ∞ and permuting 0 with λ_2 and 1 with λ_1). Liftings of u and v are given by

$$\widehat{u}\big([x_1:x_2:x_3:x_4:x_5]\big)=\big[x_5:\lambda_2^{1/k}x_1:x_2:(-\lambda_1)^{1/k}x_3:(-\lambda_2)^{1/k}x_4\big],$$
$$\widehat{v}\big([x_1:x_2:x_3:x_4:x_5]\big)=\big[x_1:x_5:(-1)^{1/k}x_4:(-1)^{1/k}x_3:x_2\big].$$

Note that $\widehat{u}^5=\widehat{v}^2=1$. Moreover, if we choose branches of $(-\lambda_2)^{1/k}$, $(-1)^{1/k}$, $\lambda_2^{1/k}$ and $\lambda_1^{1/k}$ so that they all satisfy $(-\lambda_2)^{1/k}(-1)^{1/k}=\lambda_2^{1/k}$ and $(\lambda_1^{1/k})^2=\lambda_2^{1/k}$, then $(\widehat{u}\,\widehat{v})^2=1$, that is, $D_5\cong\langle\widehat{u},\widehat{v}\rangle<$

Please cite this article in press as: G. González-Diez et al., Generalized Fermat curves, J. Algebra (2009), doi:10.1016/j.jalgebra.2009.01.002

$Aut_H(S)$. It follows that $Aut_H(S) = H \rtimes D_5$, in particular, $|Aut_H(S)| = 10k^4$. By Hurwitz's upper bound, if $k = 2$, then $Aut(S) = Aut_H(S)$.

Example 12. If $S = \{x_1^k + x_2^k + x_3^k = 0\} \subset \mathbb{P}^2$ is a classical Fermat curve of degree $k \geqslant 4$, that is, $n = 2$, then $Aut(S) \cong \mathbb{Z}_k^2 \rtimes \mathfrak{S}_3$ [27,30]. An easy way to see this is as follows. Let $H \cong \mathbb{Z}_k^2$ be a generalized Fermat group of type $(k, 2)$ inside $Aut(S)$. The quotient orbifold S/H has signature $(0, 3; k, k, k)$. Clearly, S/H admits the symmetric group $\mathfrak{S}_3$ as group of orbifold automorphisms. A lift of $\mathfrak{S}_3$ is generated by the permutations of the coordinates: $\sigma([x_1 : x_2 : x_3]) = [x_2 : x_1 : x_3]$ and $\tau([x_1 : x_2 : x_3]) = [x_3 : x_1 : x_2]$. In this case $Aut_H(S) = H \rtimes \mathfrak{S}_3$ and $S/Aut_H(S) = (S/H)/\mathfrak{S}_3$ has signature $(0, 3; 2, 3, 2k)$, which is maximal if $k \geqslant 4$ [28], in particular, $Aut(S) = Aut_H(S)$.

3. On the uniqueness of generalized Fermat groups

3.1. Uniqueness for fixed type

In this section we are concerned with the uniqueness of the generalized Fermat group, within a fixed type, on a fixed generalized Fermat curve. We are not able to provide a general answer to this problem, but in the case of types (p, n), where p is a prime, we have the following partial result.

Theorem 13. *Let $p \geqslant 2$ be a prime and $n \geqslant 2$ an integer so that $(n-1)(p-1) > 2$. If H_1 and H_2 are generalized Fermat groups of type (p, n) for the same Riemann surface S, then they are conjugate in $Aut(S)$.*

Proof. Let H be a generalized Fermat group of type (p, n), where $p \geqslant 2$ is a prime so that $(n-1)(p-1) > 2$. We have a chain of p-subgroups

$$H = K_0 \lhd K_1 \lhd K_2 \lhd \cdots \lhd K_k$$

where K_k is a p-Sylow subgroup and $K_{j+1}/K_j \cong \mathbb{Z}_p$. If $k = 0$, then we are done. Otherwise, Lemma 17 below asserts that K_0 is unique in K_1, hence K_0 is also normal subgroup in K_2, then (again by Lemma 17) unique in K_2. Proceeding in this way we end with K_0 being unique in K_k. □

Corollary 14. *If $n \geqslant 2$ is an integer and $p \geqslant 2$ is a prime so that $(n-1)(p-1) > 2$, then any Fuchsian group of type (p, n) is uniquely determined by its commutator subgroup up to conjugation by some isometry of hyperbolic plane. In particular, any two hyperbolic orbifolds of signature $(0, n+1; p, \ldots, p)$ are conformally equivalent if and only if their homology cover Riemann surfaces are conformally equivalent.*

In [15] (and for many classes of torsion free non-elementary Kleinian groups in [16–19,23]) was noted that any Fuchsian group of type (∞, n), $n \geqslant 2$, is uniquely determined by its commutator subgroup. A natural question is if there is some integer $q(n)$ so that if $k \geqslant q(n)$, then generalized Fermat groups of type (k, n) should be unique in $Aut(S)$. Related to this problem, the following was proved in [21].

Theorem 15. *(See [21].) If $n \geqslant 2$, then there exists a prime $q(n)$ so that for each prime $p \geqslant q(n)$ the generalized Fermat group of type (p, n) is unique in $Aut(S)$.*

Unfortunately, we do not have an explicit formula for $q(n)$ and n large. It was observed in [9] that a generalized Fermat group of type $(2, n)$, for $n = 4, 5$, of a closed Riemann surface S is unique in $Aut(S)$, in particular, a normal subgroup of $Aut(S)$. All the above permits to state the following fact.

Corollary 16. *Let (S, H) be a generalized hyperbolic Fermat pair of type (p, n), where p is a prime, and let $q(n)$ as in Theorem 15. If either*

Please cite this article in press as: G. González-Diez et al., Generalized Fermat curves, J. Algebra (2009), doi:10.1016/j.jalgebra.2009.01.002

(i) $p \geqslant q(n)$, *or*
(ii) $(p,n) = (2,n)$, *where* $n = 4, 5$,

then $Aut(S)/H$ *is isomorphic to the subgroup of Möbius transformations preserving the finite set of conical points of* $S/H \cong \widehat{\mathbb{C}}$.

Lemma 17. *Let* $p, n \geqslant 2$ *be so that* p *is a prime and* $(n-1)(p-1) > 2$. *Let* $L < Aut(S)$ *be any* p*-subgroup of conformal automorphisms of* S. *If* L *contains as normal subgroup a generalized Fermat group* H *of type* (p,n), *then* H *is the unique generalized Fermat group of type* (p,n) *contained in* L.

Proof. The case $p = 2$ was proved in [9]. From now on, we assume $p \geqslant 3$ a prime. We use induction on n.

If $n = 2$ and $p \geqslant 5$, then S is (conformally equivalent to) the classical Fermat curves $\{x^p + y^p + z^p = 0\} \subset \mathbb{P}^2$, whose full group of conformal automorphisms is $H \rtimes \mathfrak{S}_3$ [27,30] (see also Example 12), in particular, $H = L$.

Let $n = 3$, $p \geqslant 3$ and $H \lhd L$ as in the hypothesis. In this case, L/H is a finite p-group of orbifold automorphisms of S/H. As (i) L/H is, in particular, a group of conformal automorphisms of $\widehat{\mathbb{C}}$, (ii) the finite subgroups of $PSL(2,\mathbb{C})$ are cyclic groups, dihedral groups, alternating groups $\mathcal{A}_4$, $\mathcal{A}_5$ and the symmetric group $\mathfrak{S}_4$ [24] and (iii) $p \geqslant 3$, it follows that H/L is either trivial (then $H = L$ and we are done) or a cyclic group of order p. Let us assume $L/H = \langle \tau \rangle \cong \mathbb{Z}_p$. As τ must permute the 4 conical points, it follows that this case is only possible if $p = 3$. In this case, up to a Möbius transformation, we may assume the conical points are ∞, 0, 1 and $1 + w_3$ (where $w_3 = e^{2\pi i/3}$), and $\tau(z) = w_3 z + 1$. It follows that S is given by

$$\begin{cases} x_1^3 + x_2^3 + x_3^3 = 0, \\ (1 + w_3)x_1^3 + x_2^3 + x_4^3 = 0. \end{cases}$$

In this case (see also Remark 20), L is generated by H and (assuming τ fixes the conical point of S/H determined by the fixed points of $a_1([x_1 : x_2 : x_3 : x_4]) = [w_3 x_1 : x_2 : x_3 : x_4]$)

$$\alpha\big([x_1 : x_2 : x_3 : x_4]\big) = \big[(1 + w_3)^{1/3} x_1 : x_4 : x_2 : -x_3\big].$$

Note that $\alpha^3 \in H - \{I\}$. Now, direct computations permits to see that in L the generalized Fermat group is unique. In this case, $\mathcal{A}_4 = \langle \tau(z) = w_3 z + 1, \eta(z) = (z-1)/z \rangle$ is the group of orbifold automorphisms of S/H. As the 4 conical points of S/H are fixed points of elements of order 3 in $\mathcal{A}_3$ (the 4 vertices of a spherical tetrahedron in $\widehat{\mathbb{C}}$ invariant under $\mathcal{A}_4$) it follows that $(S/H)/\mathcal{A}_4$ has signature $(0,3;2,3,9)$ (the conical point of order 9 is the projection of one of the fixed point of τ, that is ∞, the conical point of order 3 is the projection of the other fixed point of τ, that is 0, and the conical point of order 2 is the projection of a fixed point of the involution $\tau^2\eta$). As the signature $(0,3;2,3,9)$ is a maximal one [28], it follows that $Aut(S) = Aut_H(S) = \langle H, \alpha, \beta \rangle$, where

$$\beta\big([x_1 : x_2 : x_3 : x_4]\big) = \big[x_2 : (1 + w_3)^{1/3} x_1 : x_4 : (1 + w_3)^{1/3} x_3\big] \quad \big(\beta^2 = I\big).$$

Let us assume our proposition to be true for types (p,m), where $m \leqslant n-1$ and $n \geqslant 4$.

Let us assume also that we have a generalized Fermat group $\widetilde{H} < L$, of type (p,n) such that $H \neq \widetilde{H}$. Set $G = \langle H, \widetilde{H} \rangle = H\widetilde{H}$ and $R = H \cap \widetilde{H} < Z(G)$ (where $Z(G)$ denotes the center of G). Clearly, $H \lhd G$. Let us consider the group G/H of conformal automorphisms of the orbifold S/H. As $G/H \cong \mathbb{Z}_p^l$, for some $l \geqslant 1$, and S/H has underlying Riemann surface the Riemann sphere, necessarily $l = 1$, that is, $G/H \cong \mathbb{Z}_p$. It follows that G has order p^{n+1} and, as $|G| = |H||\widetilde{H}|/|R|$, that R has order p^{n-1}, that is, $R \cong \mathbb{Z}_p^{n-1}$. This also asserts that G/R is a group of order p^2, then an Abelian group. We may write $H = \langle R, x \rangle$ and $\widetilde{H} = \langle R, y \rangle$, for some $x \in H - R$ and $y \in \widetilde{H} - R$ (both of them then have order p). As G/R is Abelian, it follows that $xyx^{-1}y^{-1} \in R$ and, in particular, that $\widetilde{H} \lhd G$.

Please cite this article in press as: G. González-Diez et al., Generalized Fermat curves, J. Algebra (2009), doi:10.1016/j.jalgebra.2009.01.002

If R contains some standard generator, say $t \in R$, then we consider the generalized Fermat curve $S/\langle t\rangle$ (the underlying Riemann surface structure) and the generalized Fermat groups $H/\langle t\rangle$ and $\widetilde{H}/\langle t\rangle$, both of type $(p, n-1)$. By the inductive hypothesis, $H/\langle t\rangle = \widetilde{H}/\langle t\rangle$; then $H = \widetilde{H}$, a contradiction. This fact together with Parts 2-(iii) and 2-(iv) of Corollary 2 implies that R acts freely on S. Set $M = S/R$. On M we have two commuting conformal automorphisms, say $\widetilde{x}$ and $\widetilde{y}$, both of order p, which are induced by x and y, respectively. The quotients of M by any of them is the Riemann sphere with exactly $(n+1)$ conical points of order p; that is, each of these automorphisms has exactly $(n+1)$ fixed points.

By Proposition 1.8, in [12], the surface M is one of the following ones:

$$F_p : w^p = u^p - 1,$$
$$D_p : w^p = (u^p - 1)(u^p - \lambda^p)^{p-1}, \quad \text{some } \lambda \in \mathbb{C},\ \lambda^p \neq 1,$$

and

$$\widetilde{x}(u, w) = (\eta^a u, \eta^b w),$$
$$\widetilde{y}(u, w) = (\eta^c u, \eta^d w)$$

where $1 \leqslant a, b, c, d \leqslant p$ and $\eta = e^{2\pi i/p}$. In there it is also noted that both automorphisms must have either p (if $M = F_p$) or $2p$ (if $M = D_p$) fixed points, that is, either $n+1 = p$ or $n+1 = 2p$ (it follows that S/G has signature either $(0, 3; p, p, p)$ or $(0, 4; p, p, p, p)$).

In the F_p case, the conical points in S/H are exactly p and they are invariant under some elliptic transformation u of order p (induced by y). We may assume, up to a Möbius transformation, that these conical points are ∞, 0, 1, $\lambda_1, \ldots, \lambda_{p-3}$ and that $u(\infty) = 0$, $u(0) = 1$, $u(1) = \lambda_1$, $u(\lambda_j) = \lambda_{j+1}$ (for $j = 1, \ldots, p-4$) and $u(\lambda_{p-3}) = \infty$. In this way, we can assume $S = C(\lambda_1, \ldots, \lambda_{p-3})$ and $H = H_0 \cong \mathbb{Z}_p^{p-1}$ is generated by standard generators $a_1, \ldots, a_{p-1}$ (as described in Section 2; the other standard generator is $a_p = a_1 \cdots a_{p-1}$). A lifting of u, under the natural projection $\pi[x_1 : \cdots : x_p] = -(x_2/x_1)^p$, has the form $\widehat{u}[x_1 : \cdots : x_p] = [x_p : c_2 x_1 : \cdots : c_p x_{p-1}]$, where $c_2, \ldots, c_p \in \mathbb{C} - \{0\}$ (see Corollary 9). All other liftings are of the form $h\widehat{u}$, where $h \in H$. In this way, $y = h\widehat{u}$ for suitable $h \in H$. As $h \in H$ and y commute with each $r \in R$, this commutativity property also holds for each of the liftings $\widehat{u}$. Now, the only elements of H commuting with $\widehat{u}$ are the powers of $\beta = a_1 a_2^2 a_3^3 \cdots a_{p-1}^{p-1}$. In fact, a non-trivial element of H is of the form $\alpha([x_1 : \cdots : x_p]) = [w_1 x_1 : w_2 x_2 : \cdots : w_p x_p]$, where $w_j^p = 1$. We may assume $w_p = 1$. As

$$\alpha\widehat{u}([x_1 : \cdots : x_p]) = [w_1 x_p : w_2 c_2 x_1 : w_3 c_3 x_3 : \cdots : w_{p-1} c_{p-1} x_{p-2} : c_p x_{p-1}],$$
$$\widehat{u}\alpha([x_1 : \cdots : x_p]) = [x_p : w_1 c_2 x_1 : w_2 c_3 x_3 : \cdots : w_{p-2} c_{p-1} x_{p-2} : w_{p-1} c_p x_{p-1}]$$

it follows that $w_2 = w_1^2$, $w_3 = w_1^3$, …, $w_{p-1} = w_1^{p-1}$, in particular, the above assertion. Since R has order p^{n-1} there are plenty of elements in R which cannot commute with $\widehat{u}$, a contradiction.

In the D_p case, the arguments are similar but in that case we should use $\widehat{u}[x_1 : \cdots : x_{2p}] = [x_p : c_2 x_1 : \cdots : c_p x_{p-1} : c_{p+1} x_{2p} : c_{p+2} x_{p+1} : \cdots : c_{2p} x_{2p-1}]$, where $c_j \in \mathbb{C} - \{0\}$. □

Remark 18. If p is a prime and either (i) $p > n+1$ or (ii) $n+1$ is not congruent to 0, 1 or 2 module p, then a generalized Fermat group of type (p, n) is a p-Sylow subgroup of $Aut(S)$. In fact, if H is a generalized Fermat group of type (p, n), then H is either a p-Sylow subgroup of $Aut(S)$ or it is a normal subgroup of index p of some subgroup $K < Aut(S)$. The last case will provide an elliptic Möbius transformation of order p that preserves $n+1$ points on the Riemann sphere S/H, a contradiction to the conditions (i) or (ii).

Please cite this article in press as: G. González-Diez et al., Generalized Fermat curves, J. Algebra (2009), doi:10.1016/j.jalgebra.2009.01.002

Proposition 19. *Let S be a generalized Fermat curve of type (p,n). If $p \geqslant 5$ is a prime, then a generalized Fermat group of type (p,n) cannot belong to two different p-Sylow subgroups of $Aut(S)$. In particular, the number of different generalized Fermat groups in S is equal to the number of p-Sylow subgroups in $Aut(S)$.*

Proof. Lemma 17 asserts that inside every p-Sylow subgroup of $Aut(S)$ there is a unique generalized Fermat group of type (p,n). Assume $p \geqslant 5$ is prime. If H is a generalized Fermat group contained in two different p-Sylow subgroups, say G_1 and G_2, then S/H will admit two different automorphisms of order p, both of them permuting the conical points; this would imply that there is a finite group of Möbius transformations generated by two different transformations of order $p \geqslant 5$, a contradiction (all finite subgroups of Möbius transformations are either cyclic, $\mathbb{Z}_2^2$, dihedral groups, alternating groups $\mathcal{A}_4$ or $\mathcal{A}_5$ or the symmetric group $\mathfrak{S}_4$ [24]). □

Remark 20. Proposition 19 is false for $p=3$. In fact, let us consider the generalized Fermat curve of type $(3,3)$

$$S = \begin{cases} x_1^3 + x_2^3 + x_3^3 = 0, \\ (1+w_3)x_1^3 + x_2^3 + x_4^3 = 0. \end{cases}$$

We already noted in the proof of Lemma 17 that $Aut(S) = \langle H_0, \alpha, \beta \rangle$, where

$$\alpha\big([x_1 : x_2 : x_3 : x_4]\big) = \big[(1+w_3)^{1/3}x_1 : x_4 : x_2 : -x_3\big] \quad \big(\alpha^9 = I,\ \alpha^3 \in H_0 - \{I\}\big),$$
$$\beta\big([x_1 : x_2 : x_3 : x_4]\big) = \big[x_2 : (1+w_3)^{1/3}x_1 : x_4 : (1+w_3)^{1/3}x_3\big] \quad \big(\beta^2 = I\big).$$

The generalized Fermat group $H_0 = \langle a_1, a_2 \rangle$ is contained in the following two different 3-Sylow subgroups of $Aut(S)$: $G_1 = \langle H_0, \alpha \rangle$ and $G_2 = \langle H_0, \delta \rangle$, where

$$\delta\big([x_1 : x_2 : x_3 : x_4]\big) = \big[x_4 : w_3^{1/3}x_1 : -x_3 : (1+w_3)^{1/3}x_2\big] \quad \big(\delta^3 = 1\big),$$
$$\beta = \delta\alpha^2 \text{ (up to an element of } H_0\text{)}.$$

3.2. On the uniqueness of the type

In this section we provide a partial discussion on the problem about the uniqueness of the type on a fixed Riemann surface S admitting a generalized Fermat group. Let us assume we have a closed Riemann surface S which is a generalized Fermat curve of type (k,n). As already noted at the beginning, the genus of S is of the form

$$g = 1 + \frac{\phi(k,n)}{2} \tag{4}$$

where $\phi(k,n) = k^{n-1}((n-1)k - n - 1)$. The hyperbolicity on S ensures that the possible pairs (k,n) belong to the set

$$D = \big\{(k,n):\ k,n \in \{2,3,4,\ldots\},\ (k-1)(n-1) > 2\big\}. \tag{5}$$

If we set $n(2)=4$, $n(3)=3$ and $n(k)=2$, for $k \geqslant 4$, then we easily see that

(P1) for each fixed $k \geqslant 2$ and $n \geqslant n(k)$, we have that $\phi(k,n)$ is strictly increasing in n;
(P2) for each $n \geqslant 2$ and $k \geqslant 2$ so that $n \geqslant n(k)$, we have that $\phi(k,n)$ is strictly increasing in k.

Please cite this article in press as: G. González-Diez et al., Generalized Fermat curves, J. Algebra (2009), doi:10.1016/j.jalgebra.2009.01.002

The function ϕ is injective for most fixed values of $g \geqslant 2$. The first two values of g for which ϕ fails to be injective are $g = 10$ and $g = 55$, in which case, the possible values of (k, n) are $(3, 3)$, $(6, 2)$ and $(3, 4)$, $(12, 2)$, respectively. However as the next proposition shows the uniqueness of type is maintained in these two cases.

Proposition 21. *No Riemann surface of genus* 10 *(respectively,* 55*) is simultaneously a generalized Fermat curve of types* $(3, 3)$ *and* $(6, 2)$ *(respectively,* $(3, 4)$ *and* $(12, 2)$*).*

Proof. Assume we have a closed Riemann surface S, of genus $g = 10$, so that $H_1, H_2 < Aut(S)$ are generalized Fermat groups of types $(3, 3)$ and $(6, 2)$, respectively. As we have that S/H_2 is an orbifold of signature $(0, 3; 6, 6, 6)$, it follows that S is the classical Fermat curve $S = \{x_1^6 + x_2^6 + x_3^6 = 0\} \subset \mathbb{P}^2$. It is well known that $Aut(S) = \mathbb{Z}_6^2 \rtimes \mathfrak{S}_3$ [27,30] (see also Example 12) and that $S/Aut(S)$ has signature $(0, 3; 2, 3, 12)$. But, a group $\mathbb{Z}_6^2 \rtimes \mathfrak{S}_3$ admits no subgroup isomorphic to $\mathbb{Z}_3^3$, a contradiction.

Next, assume we have a closed Riemann surface S, of genus $g = 55$, so that $H_1, H_2 < Aut(S)$ are generalized Fermat groups of types $(3, 4)$ and $(12, 2)$, respectively. As we have that S/H_2 is an orbifold of signature $(0, 3; 12, 12, 12)$, it follows that S is the classical Fermat curve $S = \{x_1^{12} + x_2^{12} + x_3^{12} = 0\} \subset \mathbb{P}^2$. It is well known that $Aut(S) = \mathbb{Z}_{12}^2 \rtimes \mathfrak{S}_3$ [27,30] (see also Example 12) and that $S/Aut(S)$ has signature $(0, 3; 2, 3, 24)$. But, a group $\mathbb{Z}_{12}^2 \rtimes \mathfrak{S}_3$ admits no subgroup isomorphic to $\mathbb{Z}_3^4$, a contradiction. □

Proposition 21 makes us wonder for the existence of a closed Riemann surface admitting generalized Fermat groups of different type. A partial answer to this question is provided by the following.

Proposition 22. *If S is a hyperbolic generalized Fermat curve of even genus of types (k, n) and $(\widehat{k}, \widehat{n})$, so that k and $\widehat{k}$ are even, then $k = \widehat{k}$ and $n = \widehat{n} = 2$.*

This is consequence of Lemma 23 below and property (P2) which says that $\phi(k, 2)$ is strictly increasing in k.

Lemma 23. *If S is a hyperbolic generalized Fermat curve of genus g and type (k, n), where k and g are even, then $n = 2$ and k is not divisible by* 4.

Proof. As in $g = 2$ we have no generalized Fermat curves, we must assume $g \geqslant 4$. Let us write

$$g - 1 = p_1^{a_1} \cdots p_r^{a_r},$$

where $p_j \geqslant 3$ are prime integers and $a_j \geqslant 1$. In this way, we have that a type (k, n) for this genus g must satisfy from (4) the equality

$$k^{n-1}\big((n - 1)k - (n + 1)\big) = 2p_1^{a_1} \cdots p_r^{a_r}.$$

As we are assuming k even, the above obligates to have

$$k^{n-1} = 2p_1^{b_1} \cdots p_r^{b_r},$$
$$(n - 1)k - (n + 1) = p_1^{a_1 - b_1} \cdots p_r^{a_r - b_r},$$

for certain values $0 \leqslant b_j \leqslant a_j$. As all primes p_j are different from 2, the first equality obligates to have $n = 2$ and that k is not divisible by 4. □

Please cite this article in press as: G. González-Diez et al., Generalized Fermat curves, J. Algebra (2009), doi:10.1016/j.jalgebra.2009.01.002

4. Moduli of generalized Fermat pairs

4.1. Some generalities

We now recall several known facts concerning parametrization of Riemann surfaces endowed with a group action such as ours [13], see also [14]. Let S_0 be closed Riemann surface of genus g and $H_0 < Aut(S_0)$. There is an irreducible complex analytic space $\widetilde{\mathcal{M}}(H_0)$ whose points are in bijection with the isomorphy classes of pairs (S, H) topologically conjugate to (S_0, H_0). This space is the normalization of the locus in the moduli space of genus g, say $\mathcal{M}_g$, consisting of the classes of Riemann surfaces admitting a group of conformal automorphisms topologically conjugate to H_0. Let $\mathcal{T}_g$ the Teichmüller space of genus g and Mod_g the corresponding mapping class group of genus g. Then, $\mathcal{M}_g = \mathcal{T}_g/Mod_g$. Let us consider H_0 as a subgroup of Mod_g and let $\mathcal{T}_g(H_0) \subset \mathcal{T}_g$ be its locus of fixed points. It is well known that $\mathcal{T}_g(H_0)/N(H_0) = \widetilde{\mathcal{M}}(H_0)$, where $N(H_0)$ denotes the normalizer of H_0 inside Mod_g.

4.2. Generalized Fermat curves

Let us now assume (S_0, H_0) is a generalized Fermat pair of type (k, n) and genus $g = g_{k,n}$, where (k, n) is so that $(n-1)(k-1) > 2$. As any two generalized Fermat pairs are topologically conjugate, by Theorem 1, $\widetilde{\mathcal{M}}(H_0)$ contains the isomorphy classes of all generalized Fermat pairs of type (k, n). We denote $\widetilde{\mathcal{M}}(H_0)$ simply by $\mathcal{M}(k, n)$ and by $\mathcal{F}(k, n) \subset \mathcal{M}_g$ the locus consisting of those classes $[S] \in \mathcal{M}_g$ where S is a generalized Fermat curve of type (k, n). The open connected set

$$\mathcal{P}_n = \left\{(\lambda_1, \ldots, \lambda_{n-2}) \in \mathbb{C}^{n-2}\colon\ \lambda_j \neq 0, 1,\ \lambda_j \neq \lambda_i\right\} \subset \mathbb{C}^{n-2}$$

is a model for the moduli space of ordered $(n+1)$ pointed sphere. Next we proceed to recall the construction of a model of moduli space of unordered $(n+1)$ pointed sphere. We say that $(\lambda_1, \ldots, \lambda_{n-2}), (\mu_1, \ldots, \mu_{n-1}) \in \mathcal{P}_n$ are equivalent if there is a Möbius transformation $A \in PSL(2, \mathbb{C})$ so that A sends the set

$$\{\infty, 0, 1, \lambda_1, \lambda_2, \ldots, \lambda_{n-2}\}$$

onto

$$\{\infty, 0, 1, \mu_1, \mu_2, \ldots, \mu_{n-2}\}.$$

The above equivalence relation is in fact given by action of the symmetric group $\mathfrak{S}_{n+1}$ on $n+1$ letters as follows. For each $\sigma \in \mathfrak{S}_{n+1}$ and each $(\lambda_1, \ldots, \lambda_{n-2}) \in \mathcal{P}_n$ we form the $(n+1)$-tuple $(x_1 = \infty, x_2 = 0, x_3 = 1, x_4 = \lambda_1, \ldots, x_{n+1} = \lambda_{n-2})$. Then, we consider the $(n+1)$-tuple $(x_{\sigma(1)}, \ldots, x_{\sigma(n+1)})$ and let $T_\sigma \in PSL(2, \mathbb{C})$ be the unique Möbius transformation so that $T_\sigma(x_{\sigma(1)}) = \infty$, $T_\sigma(x_{\sigma(2)}) = 0$ and $T_\sigma(x_{\sigma(3)}) = 1$. We set $\mu_j = T_\sigma(x_{\sigma(j+3)})$, for $j = 1, \ldots, n-2$. In this way, we obtain that $(\mu_1, \ldots, \mu_{n-1}) \in \mathcal{P}_n$ and an action of $\mathfrak{S}_{n+1}$ on $\mathcal{P}_n$. The quotient space $\mathcal{Q}_n$ obtained by this action is a model of the moduli space $\mathcal{M}_{0,n+1}$ of the unordered $(n+1)$ pointed sphere.

The space $\mathcal{F}(k, n)$ is related to the moduli space $\mathcal{Q}_n$ introduced above, as follows. Let us denote by $[S]$ (resp. $[S, H]$, $[S/H]$) the isomorphy class of the Riemann surface S (resp. the pair (S, H), and the orbifold S/H), then the rule that sends $[S, H]$ to $[S]$ (resp. to $[S/H]$) defines a surjective holomorphic mapping $\pi_1 : \mathcal{M}(k, n) \to \mathcal{F}(k, n) \subset \mathcal{M}_g$ (resp. $\pi_2 : \mathcal{M}(k, n) \to \mathcal{Q}_n$). Moreover, π_1 is known to be a finite surjective mapping which fails to be injective if and only if there are surfaces S admitting two generalized Fermat groups H_1 and H_2, both of type (k, n), which are not conjugate within the full group $Aut(S)$; so that, $[S, H_1]$ and $[S, H_2]$ are different points of $\mathcal{M}(k, n)$ which map to the same point $[S] \in \mathcal{F}(k, n) \subset \mathcal{M}_g$. As for π_2, Theorem 1 tells us that it induces an isomorphism between $\mathcal{Q}_n$ and $\mathcal{M}(k, n)$. We can now state the following results.

Theorem 24. *The moduli space $\mathcal{Q}_n$ defines a generically one-to-one cover of $\mathcal{F}(k, n)$.*

Please cite this article in press as: G. González-Diez et al., Generalized Fermat curves, J. Algebra (2009), doi:10.1016/j.jalgebra.2009.01.002

Proof. As explained above, injectivity of $\pi_1 : \mathcal{Q}_n \to \mathcal{M}_g$ only fails at points corresponding to Fermat curves S such that $Aut(S)$ contains a subgroup G generated by two non-conjugate Fermat groups H_1, H_2 of same type (k, n). Since G strictly contains H_1, we see that S/G is an orbifold of genus zero with $r < n+1$ branched points. This implies that the locus of unwanted points is a finite union of spaces of dimension $r-3$ with $r < n+1$, hence its dimension is strictly lower than that of $\mathcal{Q}_n$. □

The space $\mathcal{Q}_n$ is the normalization of $\mathcal{F}(k,n)$. In the particular case that $k = p$ is prime, the argument given to prove Theorem 24 combined with Theorem 13 clearly implies the following more precise statement.

Theorem 25. *Let $p, n > 0$ be integers with $(n-1)(p-1) > 2$ and p is a prime. Then, $\mathcal{Q}_n$ is isomorphic to $\mathcal{F}(p,n)$, that is, $\mathcal{F}(p,n)$ is a normal variety.*

Corollary 26. *Let $p, n > 0$ be integers with $(n-1)(p-1) > 2$ and p a prime. Then, the moduli space of $n+1$ unordered points in the sphere ($\cong \mathcal{Q}_n$) embeds holomorphically inside the moduli space $\mathcal{M}_g$, where $g = 1 + \phi(p,n)/2$, as the locus of generalized Fermat curves of type (p,n).*

5. Applications of generalized Fermat curves

In this section we provide three higher-dimensional applications of generalized Fermat curves. The first one is the construction of a pencil of non-hyperelliptic Riemann surfaces with exactly three singular fibers. The second one is related to the previous one and consists of the definition of an injective holomorphic map from $\mathbb{C} - \{0,1\}$ into some moduli space $\mathcal{M}_g$. The third one is the description of all complex surfaces isogenous to a product $X = S_1 \times S_2/G$ with invariants $p_g = q = 0$ and group G equals either $G = \mathbb{Z}_5^2$ or $G = \mathbb{Z}_2^4$.

The link between the first two applications is made as follows. If $\pi : X \to \mathbb{P}^1$ is a pencil of curves of genus g with singular fibers at $t \in \{0, 1, \infty\}$, then, by the universal property of moduli space, there is a holomorphic map $\psi : \mathbb{C} - \{0,1\} \to \mathcal{M}_g$ defined by sending $t \in \mathbb{C} - \{0,1\}$ to the point in $\mathcal{M}_g$ representing the isomorphism class of the curve $\pi^{-1}(t)$. Conversely, $\mathcal{M}_g$ comes equipped with a fiber space $\pi : \mathcal{C}_g \to \mathcal{M}_g$, *the universal curve*, whose fiber over a point $x \in \mathcal{M}_g$ is generically the Riemann surface C_x that this point represents, and so our initial pencil is obtained as pull-back of $\mathcal{C}_g$ by ψ. However, this correspondence is not a perfect one because if x represents a curve with automorphisms the true fiber is $C_x/Aut(C_x)$. Therefore, if a map $\psi : \mathbb{C} - \{0,1\} \to \mathcal{M}_g$, such as the one we construct below, parametrizes Riemann surfaces with automorphisms it is not, in principle, clear that it must be induced by a pencil of curves.

5.1. A pencil of generalized Fermat curves with three singular fibers

It was observed by A. Beauville [4] that every family of curves over $\mathbb{P}^1$, with variable moduli, admits at least 3 singular fibers. In that paper an example for which the minimal number of singular fibers occurs is provided by certain family of hyperelliptic Riemann surfaces. Another example of this kind was constructed by González-Aguilera and Rodríguez [11]; in this case all curves in the family possess an automorphism group which contains the alternating group in 5 letters $\mathcal{A}_5$. Next we provide another example of this sort in which the members of the family are generalized Fermat curves, hence never hyperelliptic.

Let us consider $n = 3$ and $k \geqslant 3$. In this case $\mathcal{P}_3 = \mathbb{C} - \{0,1\}$. Inside $\mathbb{P}^1 \times \mathbb{P}^3$ let us consider the complex surface

$$X = \left\{ \begin{matrix} x_1^k + x_2^k + x_3^k = 0, \\ a x_1^k + b x_2^k + b x_4^k = 0 \end{matrix} : [a:b] \in \mathbb{P}^1,\ [x_1 : \cdots : x_4] \in \mathbb{P}^3 \right\}.$$

Equipped with the projection into the first factor, this is a pencil whose fibers we denote by $V([a:b])$.

Please cite this article in press as: G. González-Diez et al., Generalized Fermat curves, J. Algebra (2009), doi:10.1016/j.jalgebra.2009.01.002

By taking $b = 1$ and $a \in \mathbb{C} - \{0, 1\}$, we see that this is a moduli family of non-hyperelliptic curves with variable moduli. We already known that the only singular fibers are provided by $V([0:1])$, $V[1:1]$ and $V([1:0])$.

The singular fiber $V([1:0]) = \{[0:1:w:z] \in \mathbb{P}^3 : w^k = -1\} \cup \{[0:0:0:1]\}$ is given by k different $\mathbb{P}^1 \subset \mathbb{P}^3$ all of them sharing a unique point $[0:0:0:1]$. The singular fiber $V([0:1]) = \{[x_1 : x_2 : x_3 : x_4] \in \mathbb{P}^3\colon x_1^k + x_2^k + x_3^k = 0, x_4 = wx_2, w^k = -1\}$ is given by k different classical Fermat curves of order k all of them sharing exactly k points, these being $[1:0:w:0]$, where $w^k = -1$. Similarly, the singular fiber $V([1:1])$ is given by k different classical Fermat curves of order k all of them sharing exactly k points, these being $[1:w:0:0]$, where $w^k = -1$. In particular, $V([0:1]) \equiv V([1:1])$.

5.2. An injective embedding of the thrice punctured sphere in moduli space

Let $q(59)$ as in Theorem 15, enjoying the property that for every prime $p \geqslant q(59)$ we have that a closed Riemann surface S has at most one generalized Fermat group of type $(p, 59)$, which then must be a normal subgroup of $Aut(S)$. Let us choose once for all such a prime integer $p \geqslant q(59)$ and, throughout this section, let us denote by g_p the genus of the generalized Fermat curve of type $(p, 59)$.

Now, let us regard $\mathcal{A}_5$ as a finite group of Möbius transformations and let $\pi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a holomorphic branched covering map induced by the action of $\mathcal{A}_5$. Let us assume the branch values of π are given by ∞, 0 and 1. For each $t \in \mathbb{C} - \{0, 1\}$, we consider the preimage $\pi^{-1}(t) = \{\mu_1, \ldots, \mu_{60}\}$. We may then consider a Möbius transformation to send 3 of these preimages onto ∞, 0 and 1; the others are denoted by $\lambda_1, \ldots, \lambda_{57}$. This gives us a generalized Fermat curve of type $(p, 59)$, say $C(\lambda_1, \ldots, \lambda_{57}) = C(t)$. The previous choices are not unique, but the conformal class $[C(t)]$ of the resulting generalized Fermat curve is uniquely determined by the value of t. This asserts that there is a well defined holomorphic map

$$\psi : \mathbb{C} - \{0, 1\} \to \mathcal{F}(p, 59) \subset \mathcal{M}_{g_p}.$$

Assume $\psi(t_1) = \psi(t_2)$. In this case, the uniqueness of the generalized Fermat group, asserts the existence of a Möbius transformation T so that $T(\pi^{-1}(t_1)) = \pi^{-1}(t_2)$. If $\pi^{-1}(t_1) = \{\mu_1, \ldots, \mu_{60}\}$ and $\pi^{-1}(t_2) = \{\mu_1', \ldots, \mu_{60}'\}$, then as both sets are invariant by (the same) $\mathcal{A}_5$, it follows that T normalizes $\mathcal{A}_5$. Set $G = \langle T, \mathcal{A}_5\rangle$. As T leaves invariant the set of fixed points of the order 3 elements in $\mathcal{A}_5$ (20 points) and each Möbius transformation is uniquely determined by its action at three different points, it follows that T has finite order and that G is finite. But, it is well known that there is no finite subgroup of Möbius transformations containing $\mathcal{A}_5$ as a proper subgroup [24], in particular, $T \in \mathcal{A}_5$. It follows that ψ is an injective map.

As the set $\{\lambda_1, \ldots, \lambda_{60}\}$ is invariant under the action of $\mathcal{A}_5$, $C(\lambda_1, \ldots, \lambda_{57})$ has the property that $Aut(C(\lambda_1, \ldots, \lambda_{57}))$ contains the group $L = \langle H \cong \mathbb{Z}_p^{59}, \mathcal{A}_5^*\rangle$, where $\mathcal{A}_5^*$ denotes the set formed by a lifting of each element of the group $\mathcal{A}_5$, which exists by Corollary 2. As there is no finite group of Möbius transformations containing strictly $\mathcal{A}_5$, it follows that in fact $L = Aut(C(\lambda_1, \ldots, \lambda_{57}))$. As a consequence, for every $t \in \mathbb{C} - \{0, 1\}$, the automorphism group is the same at each surface $C(t)$.

All the above is summarized in the following.

Proposition 27. *Let $q(59)$ be as in Theorem 15 and $p \geqslant q(59)$ a prime number. Then the above construction provides an injective holomorphic map $\psi : \mathbb{C} - \{0, 1\} \to \mathcal{F}(p, 59) \subset \mathcal{M}_{g_p}$ which sends $t \in \mathbb{C} - \{0, 1\}$ to the point in $\mathcal{M}_{g_p}$ representing the generalized Fermat curve $C(t)$. Moreover, for each $t \in \mathbb{C} - \{0, 1\}$ $Aut(C(t))$ is an extension of $\mathcal{A}_5$ by $\mathbb{Z}_p^{59}$.*

Remark 28. Let us go back to the group L which arises as the full automorphism group of the curves occurring in Proposition 27. In the language of Section 4.1 consider the quotient space $\widetilde{\mathcal{M}}(L) = \mathcal{T}_{g_p}(L)/N(L)$. By general results of Teichmüller theory [25] one knows that the stabilizer of a point $x \in \mathcal{T}_{g_p}(L)$ is precisely the group $Stab(x) = Aut(x) \cap N(L)$ where $Aut(x)$ is the full group of automorphisms of a suitable model S_x of the Riemann surface parametrized by the point x. This fact has two implications. Firstly, we see that L acts trivially on $\mathcal{T}_{g_p}(L)$, so we may as well write

Please cite this article in press as: G. González-Diez et al., Generalized Fermat curves, J. Algebra (2009), doi:10.1016/j.jalgebra.2009.01.002

$\widetilde{\mathcal{M}}(L) = \mathcal{T}_{g_p}(L)/(N(L)/L)$. Secondly, since $Aut(x) = L$ for each $x \in \mathcal{T}_{g_p}(L)$, we infer that $N(L)/L$ acts fixed point freely on $\mathcal{T}_{g_p}(L)$. Moreover, since S_x/L is an orbifold of genus zero with four branching values (in fact, with signature $(0,4;2,3,5,p)$), it follows, again by general results of Teichmüller theory, that $\mathcal{T}_{g_p}(L)$ is isomorphic to the Teichmüller space of a 4-puncture sphere, which we may identify with the hyperbolic plane $\mathbb{H}^2$. The conclusion is that $\widetilde{\mathcal{M}}(L)$ is a Riemann surface and that $N(L)/L$ is its fundamental group. Now by Proposition 27 this Riemann surface contains $\mathbb{C} - \{0,1\}$. Hence, we can only have $\widetilde{\mathcal{M}}(L) = \mathbb{C} - \{0,1\}$ and so $N(L)/L$ is isomorphic to $\Gamma(2) \cong \mathbb{Z} * \mathbb{Z}$, the principal congruence subgroup of level 2 of $PSL(2,\mathbb{Z})$.

5.3. Complex surfaces isogenous to a product

By a complex surface we shall mean a compact holomorphic manifold of complex dimension two, hence a real four manifold.

A complex surface X is said to be isogenous to a higher product if there are Riemann surfaces S_1 and S_2 of genus $\geqslant 2$ and a finite group G acting freely on $S_1 \times S_2$ by biholomorphic transformations such that X is isomorphic to the quotient $S_1 \times S_2/G$. If the action of G preserves each of the factors then X is said to be of unmixed type. In what follows we shall assume that this is always the case. Surfaces isogenous to a higher product have been extensively studied by Bauer, Catanese and Grunewald [3,7,8].

The first example of a complex surface isogenous to a higher product was given by Beauville as exercise number 4 in page 159 of [5]. In his example the algebraic curves S_1, S_2 are both the Fermat curve S: $X_1^5 + X_2^5 + X_3^5 = 0$ and G is the group $\mathbb{Z}_5^2$ acting on $S \times S$ by, say,

$$(a,b)\big([x_1 : x_2 : x_3], [y_1 : y_2 : y_3]\big) = \big(\big[\xi^a x_1 : \xi^b x_2 : x_3\big], \big[\xi^{a+3b} y_1 : \xi^{2a+4b} y_2 : y_3\big]\big)$$

where ξ is 5th root of unity.

The results in this paper allow us to extend Beauville's construction to generalized Fermat pairs as follows. Let us consider two generalized Fermat pairs (S_1, H_1) and (S_2, H_2), both of type (k,n), where $k, n \geqslant 2$, $(k-1)(n-1) > 2$.

Let $a_1, \ldots, a_n$ and $a_{n+1} = a_1 \cdots a_n$ be standard generators of H_1 and let $b_1, \ldots, b_n$ and $b_{n+1} = b_1 \cdots b_n$ be standard generators of H_2.

Remember that the only non-trivial elements of the generalized Fermat group H_2 acting with fixed points are contained in the cyclic groups generated by the $n+1$ standard generators $b_1, \ldots, b_{n+1}$.

Remark 29. If $n = 2$ and $k \geqslant 4$, a necessary and sufficient condition for the existence of generators of H_2 acting freely on S_2 is that $LCM(k,6) = 1$, where LCM stands for "least common multiple" (see [7,8]).

If $n = 2$ (in which case $k \geqslant 4$) and $LCM(k,6) = 1$, a set of generators acting freely is given by

$$c_1 = b_1 b_2^2, \qquad c_2 = b_1 b_2^{-1}.$$

If $n = 3$, $k \geqslant 4$ and $LCM(6,k) = 1$, then a set of generators of H_2, each one acting freely on S_2, is given by

$$c_1 = b_1 b_2^2, \qquad c_2 = b_1 b_2^{-1}, \qquad c_3 = b_1 b_3.$$

If $n = 3 = k$, then a set of generators of H_2, each one acting freely on S_2, is given by

$$c_1 = b_1 b_2^2, \qquad c_2 = b_1 b_2, \qquad c_3 = b_2 b_3.$$

If $n \geqslant 4$, then a set of generators of H_2, each one acting freely on S_2, is given by

$$c_1 = b_1b_2b_3, \qquad c_2 = b_1b_2, \qquad c_3 = b_2b_3, \qquad c_j = b_1b_j, \quad j = 4, \dots, n.$$

Note that, in any of the above case, $c_{n+1} = c_1c_2 \cdots c_n$ also acts freely on S_2.

Let us consider an action $\Phi : \mathbb{Z}_k^n \to Aut(S_1 \times S_2)$ as follows. If $e_1, \dots, e_n$ denotes the standard basis of $\mathbb{Z}_k^n$, then set

$$\Phi(e_j) : S_1 \times S_2 \to S_1 \times S_2,$$
$$\Phi\big([x_1 : \cdots : x_{n+1}], [y_1 : \cdots : y_{n+1}]\big) \mapsto \big(a_j\big([x_1 : \cdots : x_{n+1}]\big), c_j\big([y_1 : \cdots : y_{n+1}]\big)\big)$$

and extend it to the rest of $\mathbb{Z}_k^n$ in the natural way.

Proposition 30. *Let (S_1, H_1) and (S_2, H_2) be two generalized Fermat pairs of same type (k, n), where $(n-1)(k-1) > 2$. If $n \in \{2, 3\}$ and $k \geqslant 4$ we assume $LCM(6, k) = 1$. Let $\Phi : \mathbb{Z}_k^n \to Aut(S_1 \times S_2)$ be the action constructed above. Then, $\Phi(\mathbb{Z}_k^n)$ acts freely on $S_1 \times S_2$ giving rise to a complex surface $X = S_1 \times S_2/\mathbb{Z}_k^n$ with geometric genus p_g and irregularity $q = 0$ where*

$$p_g = \frac{k^{n-2}((n-1)(k-1)-2)^2}{4} - 1.$$

Proof. That the action is free is consequence of the previous observations and construction. The fact that the irregularity q vanishes is a consequence of the fact that each of the quotient Riemann surfaces S_i/H_i is isomorphic to $\mathbb{P}^1$ which does not admit non-zero holomorphic differentials (cf. [2], p. 303). The statement relative to the geometric genus p_g follows from the following identity (see [2,4,8])

$$1 + p_g - q = \chi(X) = \frac{1}{4}\chi_{top}(X) = \frac{(g_{k,n}-1)^2}{4|\mathbb{Z}_k^n|}$$

where $\chi_{top}(S)$ stands for the Euler–Poincaré characteristic of X, $\chi(X) =: \chi(\mathcal{O}_X)$ for the Euler characteristic of the structure sheaf of X and $g_{k,n}$ denotes the genus of a Fermat curve of type (k, n). □

Corollary 31. *The above construction gives complex surfaces $X = S_1 \times S_2/\mathbb{Z}_k^n$ with invariants $p_g = q = 0$ if and only if $(k, n) = (5, 2)$ or $(k, n) = (2, 4)$.*

Conversely, if $X = S_1 \times S_2/G$ is a complex surface isogenous to a higher product with invariants $p_g = q = 0$ and group $G = \mathbb{Z}_5^2$ (resp. $G = \mathbb{Z}_2^4$) then S_1 and S_2 are generalized Fermat curves of type $(5, 2)$ (resp. $(2, 4)$).

Proof. As we are looking for complex surfaces with $q = 0$, both quotients S_1/G and S_2/G are isomorphic to $\mathbb{P}^1$. Exploiting the Riemann–Hurwitz formula, it is shown in [3] that the orbifolds S_i/G have to have both signature $(0, 3; 5, 5, 5)$ (resp. $(0, 5; 2, 2, 2, 2, 2)$), hence S_1 and S_2 are generalized Fermat curves of the required type. □

Remark 32. Bauer and Catanese [3] have proved that when $G = \mathbb{Z}_5^2$ one obtains exactly two non-isomorphic complex surfaces whereas for $G = \mathbb{Z}_2^4$ the complex surfaces so obtained form an irreducible connected component of dimension 4 in their moduli space. The dimensions of these components are in accordance with our Theorem 25 which implies that these components are parametrized by the space $\mathcal{Q}_3 \times \mathcal{Q}_3$ (resp. $\mathcal{Q}_5 \times \mathcal{Q}_5$) via the rule

$$(S_1, S_2) \to X = S_1 \times S_2/G.$$

Furthermore, by the uniqueness of this representation of X (Proposition 3.13 in [8]), this map is injective modulo the action of $\mathbb{Z}_2$ which interchanges the factors of $\mathcal{Q}_n \times \mathcal{Q}_n$. In the first case the two different components (of dimension 0) arise from two different actions of the group $\mathbb{Z}_5^2$ whereas in the second case the above seems to indicate that all possible actions of the group $\mathbb{Z}_2^4$ are equivalent and that the irreducible component in question is $\mathcal{Q}_5 \times \mathcal{Q}_5/\mathbb{Z}_2$.

In [3] it is also shown that there are only two other Abelian groups G which can give rise to complex surfaces $X = S_1 \times S_2/G$ with invariants $p_g = q = 0$, namely $G = \mathbb{Z}_2^3$ and $G = \mathbb{Z}_3^2$.

6. Connection with handlebodies

A *Schottky uniformization* of a closed Riemann surface S is a triple $(\Omega, G, p : \Omega \to S)$, where G is a Schottky group with region of discontinuity Ω and $p : \Omega \to S$ is a regular holomorphic covering with G as covering group. The 3-manifold with boundary $M_G = (\mathbb{H}^3 \cup \Omega)/G$ is a handlebody body of genus g whose interior admits a complete hyperbolic metric with injectivity radius bounded from below by zero. Let t be a hyperbolic isometry of the interior of M_G. By lifting it to the universal cover $\mathbb{H}^3$, we obtain a Möbius transformation in the normalizer of the Schottky group G; so it defines naturally a continuous extension of t as a conformal automorphism of S. Reciprocally, a conformal automorphism of S extends continuously as an isometry of the interior of M_G if and only if it lifts to Ω (by the given Schottky uniformization) as a conformal automorphism (this because the region of discontinuity of a Schottky group is of class O_{AD}). As a consequence, a group $H < Aut(S)$ extends as a group of isometries of the interior of M_G if and only if H lifts under the Schottky uniformization as a group of automorphisms of Ω.

Theorem 33. *Let (S, H) be a generalized Fermat pair of type (k, n). Then H can be extended as group of hyperbolic isometries of some handlebody whose conformal boundary is S if and only if $k = 2$.*

Proof. Necessary and sufficient conditions for H Abelian to lift to some Schottky uniformization of S are given in [20]. A necessary condition is that every non-trivial element of H must have an even number of fixed points. Now, let H be a generalized Fermat group of type (k, n). The above together with part (2.-(ii)) of Corollary 2 asserts that k should be even. Assuming k even, choose any element a of the standard set of generators of H, and choose any two fixed points of it, say x and y. Part (2.-(v)) of Corollary 2 asserts the existence of some $h \in H$ so that $h(x) = y$, so the angles of rotation of a_1 at both x and y are the same. This violates the necessary conditions of [20] for H to lift to some Schottky uniformization if $k > 2$. The case $k = 2$ it is known to be of Schottky type. □

Uncited references

[1]

Acknowledgment

We would like to thank the referee for her/his valuable comments which improved the presentation of the paper and saved us from several mistakes.

References

[1] S.S. Abhyankar, Algebraic Space Curves, Les Presses de L'Université de Montreal, Montreal, Canada, 1971.
[2] Wolf P. Barth, Klaus Hulek, Chris A.M. Peters, Antonius Van de Ven, Compact Complex Surfaces, second edition, Ergeb. Math. Grenzgeb. (3), vol. 4, Springer-Verlag, Berlin, 2004.
[3] I. Bauer, F. Catanese, Some new surfaces with $p_g = q = 0$, in: The Fano Conference, Univ. Torino, Turin, 2004, pp. 123–142.
[4] A. Beauville, Le nombre minimum de fibres singuliéres d'une courbe stable sur $\mathbb{P}^1$, in: Lucien Szpiro (Ed.), Séminaire sur les pinceaux de courbes de genre au moins deux, Astérisque 86 (1981) 97–108.
[5] A. Beauville, Surfaces algébriques complexes, Astérisque 54 (1978).
[6] E. Bujalance, J.J. Etayo, E. Martinez, Automorphisms groups of hyperelliptic Riemann surfaces, Kodai Math. J. 10 (1987) 174–181.
[7] I. Bauer, F. Catanese, F. Grunewald, Beauville surfaces without real structures, I, in: F. Bogomolov, Y. Tschinkel (Eds.), Geometric Methods in Algebra and Number Theory, in: Progr. Math., vol. 235, Birkhäuser, 2005, pp. 1–42.
[8] F. Catanese, Fibred surfaces, varieties isogenous to a product and related moduli spaces, Amer. J. Math. 122 (2000) 1–44.
[9] A. Carocca, V. González, R.A. Hidalgo, R. Rodríguez, Generalized Humbert curves, Israel J. Math. 64 (1) (2008) 165–192.

[10] H. Farkas, I. Kra, Riemann Surfaces, second edition, Grad. Texts in Math., vol. 71, Springer-Verlag, New York, 1992.
[11] V. González-Aguilera, R.E. Rubi, A pencil in $\widehat{\mathcal{M}}_6$ with three boundary points, Geom. Dedicata 42 (1992) 255–265.
[12] G. González-Diez, On prime Galois coverings of the Riemann sphere, Ann. Mat. Pura Appl. (4) CLXVIII (1995) 1–15.
[13] G. González-Diez, W.J. Harvey, Moduli of Riemann surfaces with symmetry, in: London Math. Soc. Lecture Note Ser., vol. 173, 1992, pp. 75–93.
[14] G. González-Diez, R.A. Hidalgo, Conformal versus topological conjugacy of automorphisms on compact Riemann surfaces, Bull. London Math. Soc. 29 (1997) 280–284.
[15] R.A. Hidalgo, Homology coverings of Riemann surfaces, Tôhoku Math. J. 45 (1993) 499–503.
[16] R.A. Hidalgo, Kleinian groups with common commutator subgroup, Complex Var. Theory Appl. 28 (1995) 121–133.
[17] R.A. Hidalgo, A commutator rigidity for Kleinian groups, Revista Proyecciones 14 (1995) 75–81.
[18] R.A. Hidalgo, Noded function groups, in: Contemp. Math., vol. 240, 1999, pp. 209–222.
[19] R.A. Hidalgo, A note on the homology covering of analytically finite Klein surfaces, Complex Var. Theory Appl. 42 (2000) 183–192.
[20] R.A. Hidalgo, Schottky uniformizations of closed Riemann surfaces with abelian groups of conformal automorphisms, Glasgow Math. J. 36 (1994) 17–32.
[21] R.A. Hidalgo, M. Leyton, On uniqueness of automorphisms groups of Riemann surfaces, Rev. Mat. Iberoamericana 23 (3) (2007) 793–810.
[22] C. Maclachlan, Abelian groups of automorphisms of compact Riemann surfaces, Proc. London Math. Soc. (3) 15 (1965) 699–712.
[23] B. Maskit, The homology covering of a Riemann surface, Tôhoku Math. J. 38 (1986) 561–562.
[24] B. Maskit, Kleinian Groups, Grundlehren Math. Wiss., vol. 287, Springer-Verlag, 1988.
[25] S. Nag, The Complex Analytic Theory of Teichmüller Spaces, A Wiley–Interscience Publication, John Wiley & Sons, Inc., New York, 1988.
[26] P. Scott, The geometries of 3-manifolds, Bull. London Math. Soc. 15 (1983) 401–487.
[27] T. Shioda, Arithmetic and geometry of Fermat curves, in: M. Nagata, T.A. Peng (Eds.), Proceedings of the Alg. Geo. Seminar, Singapure, 1987, World Scientific, 1988.
[28] D. Singerman, Finitely maximal Fuchsian groups, J. London Math. Soc. (2) 6 (1972) 29–38.
[29] W.P. Thurston, The geometry and topology of 3-manifolds, Princeton Lecture Notes, 1978–1981.
[30] P. Tzermias, The group of automorphisms of the Fermat curve, J. Number Theory 53 (1995) 173–178.

Please cite this article in press as: G. González-Diez et al., Generalized Fermat curves, J. Algebra (2009), doi:10.1016/j.jalgebra.2009.01.002