ON BEAUVILLE STRUCTURES ON THE GROUPS S_n AND A_n

YOLANDA FUERTES, GABINO GONZÁLEZ-DIEZ

Departamento de Matemáticas.
Universidad Autónoma de Madrid.
Cantoblanco Universidad. Madrid 28049.
e-mail: yolanda.fuertes@uam.es
e-mail: gabino.gonzalez@uam.es

Abstract. It is known that the symmetric group S_n, for $n \geq 5$, and the alternating group A_n, for large n, admit a Beauville structure. In this paper we prove that A_n admits a Beauville (resp. strongly real Beauville) structure if and only if $n \geq 6$ (resp $n \geq 7$). We also show that S_n admits a strongly real Beauville structure for $n \geq 5$.

1. Introduction and statement of results

A complex algebraic curve C will be termed *triangle curve* if it admits a finite group of automorphisms $G < \mathrm{Aut}(C)$ so that $C/G \cong \mathbb{P}^1$ and the natural projection $C \to C/G$ ramifies over three values, say $0, 1, \infty$. If the branching orders at these points are p, q and r we will say that C/G is an orbifold of type (p, q, r). Due to the celebrated theorem of Belyi, triangle curves are known to be defined over a number field.

Beauville surfaces arise as suitable quotients of a product of two triangle curves.

Definition [3] A *Beauville surface* is a compact complex surface S satisfying the following properties:

1) It is isogenous to a higher product, that is $S \cong C_1 \times C_2/G$, where $C_i (i = 1, 2)$ are curves of genera $g_i \geq 2$ and G is a finite group acting freely on $C_1 \times C_2$ by holomorphic transformations.

2) If $G^o < G$ denotes de subgroup consisting of the elements which preserve each of the factors, then G^o acts effectively on each curve C_i so that $C_i/G^o \cong \mathbb{P}^1$ and $C_i \to C_i/G^o$ ramifies over three points.

It is easy to see ([4]) that an automorphism of the product of two curves as above $f : C_1 \times C_2 \to C_1 \times C_2$ either preserves each factor or interchanges them. The latter case can only occur if $C_1 \cong C_2$. Clearly G^o has at most index 2 in G . A Beauville surface $C_1 \times C_2/G$ is said to be of *mixed* or *unmixed type* depending

Keywords: Beauville surfaces, Riemann surfaces, symmetric and alternating groups
2000 *Mathematics Subject Classification.* 30F10, 30F35, 14J29.
The first author is partially supported by Grants MTM2006-01859 and MTM2006-14688.
The second author is partially supported by Grant MTM2006-01859. .

on whether $[G : G^o] = 2$ or $G = G^o$. Accordingly the group G is said to *admit a mixed or unmixed Beauville structure.*

In this paper we focus on the symmetric and alternating groups S_n and A_n. We anticipate that these groups can never admit mixed Beauville structures (see section 3), therefore all Beauville structures occurring in this paper will be of unmixed type.

Beauville surfaces were introduced by F. Catanese in [4] generalizing a construction of A. Beauville in [1]. The first kind of questions that naturally arise regarding Beauville surfaces are questions such as which finite groups G can occur, which curves C_i, which genera g_i, etc. Most of what is known about these problems is due to Catanese ([4]) and Bauer-Catanese-Grunewald ([3], [2]). See also our article [6].

The following facts relative to the existence of Beauville structures on S_n and A_n are known.

(1) S_n admits an unmixed Beauville structure if and only if $n \geq 5$. ([3] and [6])
(2) A_n admits an unmixed Beauville structure for large n. ([2])

Here we prove

Theorem 1. *The alternating group A_n admits an unmixed Beauville structure if and only if $n \geq 6$.*

This is part of a much more ambitious conjecture according to which *all finite simple nonabelian groups except A_5 admit an unmixed Beauville structure* ([3]).

The relevance of Beauville surfaces lies mainly on the fact that they are the rigid ones among surfaces isogenous to a product. In fact, Catanese ([4], see also [3]) was able to prove the following powerful result:

If S is a Beauville surface and X is a complex surface with same topological number as S, and with isomorphic fundamental group, then X is diffeomorphic to S, and either X or the complex conjugate surface $\overline{X}$ is isomorphic to S.

In view of this fact a different kind of questions naturally arise. Namely, one should like to know when a Beauville surface S is isomorphic to $\overline{S}$ or, more sharply, when S is *real*, that is when there is a biholomorphic map $\sigma : S \to \overline{S}$ with $\sigma^2 = id$. Again, the existing answers to these questions are contained in the work by Catanese ([5]) and Bauer-Catanese-Grunewald ([3], [2]). The following facts relative to the existence of Beauville structures on S_n and A_n are known

(1) The alternating group A_n admits a Beauville structure whose corresponding surface S is not isomorphic to $\overline{S}$ when n satisfies the following conditions $n \geq 16,\ n \equiv 0 \ mod\ 4, n \equiv 1\ mod\ 3, n \not\equiv 3, 4\ mod\ 7$ ([3] and [2]).
(2) Let $p > 5$ be a prime number with $p \equiv\ 1\ mod\ 4,\ p \not\equiv 2, 4\ mod\ 5,\ p \not\equiv 5\ mod\ 13$ and $p \not\equiv 4\ mod\ 11$. Then there is a Beauville surface S with group A_{3p+1} which is biholomorphic to $\overline{S}$ but is not real ([3] and [2]).

and, in the opposite direction,

(3) Let $p > 5$ be a prime number with $p \equiv 1 \; mod \; 4$. Then, there is a real Beauville surface S with group A_{3p+1} ([2], Proposition 3.15).

In this article we prove

Theorem 2. *The alternating group A_n admits a strongly real unmixed Beauville structure if and only if $n \geq 7$.*

Again, this is part of a more ambitious conjecture according to which *all but finitely many finite simple groups have a strongly real Beauville structure* ([2]). The precise definition of a strongly real action is given in section 2.2. But, of course, if a group G admits a strongly real structure then the corresponding Beauville surface $S = (C_1 \times C_2)/G$ will automatically be a real surface.

Theorem 3. *The symmetric group S_n admits a strongly real unmixed Beauville structure for $n \geq 5$.*

The paper is organized as follows. The starting point of the article is the fact that, according to [2], the existence of Beauville and strongly real Beauville structures on a given group G is equivalent to the existence of a pair of triples of generators (a_i, b_i, c_i), (i=1,2), of G satisfying certain properties, one of which is the identity $a_i b_i c_i = 1$. That is why instead of generating triples we often speak of generating couples (a_i, c_i) with the understanding that $b_i = a_i^{-1} c_i^{-1}$. Section 2 is devoted to the quotation of these purely algebraic criteria for the existence of Beauville structures.
Then, with these criteria at one's disposal, we look for generators (a_i, c_i) of S_n such that they not only define a Beauville surface $S = (C_1 \times C_2)/S_n$ but in addition they satisfy the extra property that the surface $\widetilde{S} = (C_1 \times C_2)/A_n$ obtained when only the subgroup A_n is allowed to act on $C_1 \times C_2$ is still a Beauville surface. A peculiarity of this construction is that the Beauville surfaces $C_1 \times C_2/A_n$ so obtained will automatically be double covers of the Beauville surfaces $C_1 \times C_2/S_n$. All this is done in section 5.
Previously, in section 4, we deal with the low order groups not covered by the general approach; these turn out to be the groups A_n and S_n with $n \leq 10$.

Acknowledgment. *We would like to express our gratitude to our colleague Ernesto Girondo who checked some of our calculations by means of the GAP computer programme.*

2. Criteria for G to admit Beauville and real Beauville structures

There are purely algebraic criteria to detect when a finite group G admits a Beauville structure and when the corresponding surface S is real. In this section we quote these criteria, which can be found in [2] and [3].

2.1. Criterion for G to admit Beauville structure.

Definition 4. *Let G be a finite group and a, b, c three generators of order p, q, r respectively. We shall say that (a, b, c) is a hyperbolic triple of generators if the following conditions hold*

(i) $abc = 1$
(ii) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$

Now, set

$$\Sigma(a,c) := \bigcup_{g\in G} \bigcup_{i=1}^{\infty} \{ga^i g^{-1}, gb^i g^{-1}, gc^i g^{-1}\}$$

Criterion A([2]). *G admits a unmixed Beauville structure if and only if it has two hyperbolic triples of generators (a_i, b_i, c_i) of order (p_i, q_i, r_i), $i = 1, 2$, satisfying the following compatibility condition*

$$\Sigma(a_1, c_1) \bigcap \Sigma(a_2, c_2) = 1$$

The curves C_i on which G acts to produce the required Beauville surface

$$S = S(a_1, c_1; a_2, c_2) = C_1 \times C_2/G$$

arise as follows:

The triangle group

$$\Gamma_{(p_i,q_i,r_i)} = \langle x, y, z : x^{p_i} = y^{q_i} = z^{r_i} = xyz = 1\rangle$$

acts in the upper half-plane $\mathbb{H}$ as a discrete group of isometries (i.e. as a *Fuchsian group*) and $C_i = \mathbb{H}/K_i$ where K_i is the kernel of the epimorphism $\theta_i : \Gamma_{(p_i,q_i,r_i)} \to G$ which sends $x \to a_i$, $y \to b_i$ and $z \to c_i$. As for the action of an element $g = \theta_1(\gamma_1) = \theta_2(\gamma_2) \in G$ on $C_1 \times C_2$ this is induced by the action of (γ_1, γ_2) on $\mathbb{H} \times \mathbb{H}$.

We will say that the pair (a_i, c_i) (resp. the quadruple $(a_1, c_1; a_2, c_2)$) defines a *triangle* (resp. a *Beauville*) *structure* on G.

2.2. Criterion for G to admit real Beauville structures.

It has been noted in [2] that if, with the notation as above, $S = S(a_1, c_1; a_2, c_2)$ then $\overline{S} = S(a_1^{-1}, c_1^{-1}; a_2^{-1}, c_2^{-1})$. Starting with this observation, the authors have been able to give a purely group theoretical criterion to decide when S is isomorphic to $\overline{S}$ or, even, when S is real. The full characterization can be found in [2]. For our purposes it will be enough to have out our disposal the following sufficient condition for S to be real.

Criterion B([2]) Setting $G = \langle a_i, c_i\rangle$ and $S = S(a_1, c_1; a_2, c_2) = C_1 \times C_2/G$, as before, we have

- S is isomorphic to $\overline{S}$ whenever there is $\psi \in \mathrm{Aut}(G)$ and $\delta_i \in G$ $(i = 1, 2)$ such that $\delta_i\psi(a_i)\delta_i^{-1} = a_i^{-1}$ and $\delta_i\psi(c_i)\delta_i^{-1} = c_i^{-1}$ $(i = 1, 2)$.

The finite groups G enjoying this property are said to *admit a strongly real unmixed structure.*

3. The mixed case

In this short section we deal with the mixed case. We find that S_n and A_n do not admit a mixed Beauville structure. This is only a simple observation but we record it here for the sake of completeness.

Lemma 5. *Let $C \times C/G$ be a Beauville surface of mixed type and G^0 the subgroup of G consisting of the elements which preserve each of the factors, then the order of any element $f \in G \setminus G^0$ is divisible by 4.*

Proof. Clearly, it is enough to show that $f^2 \neq id$, for any $f \in G \setminus G^0$ since any odd power of f will lie again in $G \setminus G^0$. So, let us write f in the form $f(x,y) = (\lambda_1(y), \lambda_2(x))$, then $f^2(x,y) = (\lambda_1(\lambda_2(x)), \lambda_2(\lambda_1(y))) = (x,y)$ would imply $\lambda_2 = \lambda_1^{-1}$ which, in turn, implies that all points of the form $(\lambda_1(y), y)$ are fixed points, a contradiction. □

Corollary 6. *S_n and A_n do not admit a mixed Beauville structure.*

Proof. For $n \geq 5$ the group $G = A_n$ cannot admit a mixed Beauville structure simply because it does not contain subgroups G^0 of index 2. As for $n \leq 4$ let us just say that no group of order ≤ 120 admits a mixed Beauville structure ([2], [6]).

As for the group $G = S_n$, its only subgroup of index 2 is $G^0 = A_n$ but then, there would be plenty of elements of order 2 in $G \setminus G^0$ to contradict lemma 5. □

Remark 7. *The number 4 appearing in lemma 5 as the minimum possible order of any element in $G \setminus G^0$ is sharp. The only known examples of groups admitting mixed Beauville structures have been constructed in* [2] *as a semidirect product of $\mathbb{Z}/4\mathbb{Z}$ and a certain class of special groups.*

4. The low order alternating and symmetric groups

In this section we produce Beauville structures on the few groups A_n and S_n which do not fit in the general pattern; these are the alternating and symmetric groups on $n \leq 10$ letters. By section 2 it will be enough to find a pair of suitable triples of generators of the group in question. The main tool to prove that the chosen triples of permutations are indeed generators will be Jordan's symmetric group theorem which we now state (see e.g. [8]).

Theorem (Jordan, 1873)
Let G be a primitive permutation group of degree n containing a prime cycle for some prime $q \leq n-3$. Then, G is either the alternating group A_n or the symmetric group S_n.

In the next example we employ this theorem to produce an unmixed Beauville structure on A_6. It has been already noted that A_n does not admit a Beauville structure for $n \leq 5$ ([2], [6]).

Example 8. *(A Beauville structure on A_6) Let us consider the following pair of triples of elements of A_6*

$$a_1 = (4,5,6), \quad b_1 = (1,5,4,3,2), \quad c_1 = (1,2,3,4,6)$$

$$a_2 = (1,2)(3,4,5,6), \quad b_2 = (1,5,6,4)(2,3) \quad c_2 = (1,3,5,6)(2,4).$$

Clearly the groups $H_i = \langle a_i, c_i \rangle$, $i = 1,2$ act primitively on $\{1,2,3,4,5,6\}$. By considering the 3-cycles $\alpha_1 = c_1 \in H_1$ and $\alpha_2 = b_2^{-1} a_2^{-1} b_2^{-1} c_2 b_2 c_2 = (2,4,3) \in H_2$ we conclude that $H_i = A_6$. Moreover, a quick look at the cycle decomposition of *these elements shows that Criterion A is satisfied. Thus, A_6 admits an unmixed Beauville structure.*

On the other hand computer inspection using the GAP programme has revealed that no strongly real structure exists on A_6. It should be mentioned that the same conclusion has been reached in [2] also by computer means.

Proposition 9. *The groups A_n, $7 \leq n \leq 10$ and S_m, $5 \leq m \leq 10$ admit a strongly real unmixed Beauville structure.*

Proof. We claim that the following pairs of couples of permutations define strongly real structures in each case

	$a_1; c_1$	$a_2; c_2$
A_7	(1,6,7,3,2); (1,2,3,4,5)	(1,6,4)(3,7,5); (1,2,3,4,5,6,7)
A_8	(6,3,5)(4,7,8); (1,2,3,4,5)	(1,7,5,3)(2,8,6,4);(1,2,3,4,5,6,7)
A_9	(1,6)(2,7)(3,8)(4,9); (1,2,3,4,5)	(1,8)(2,9)(5,4,6); (1,2,3,4,5,6,7)
A_{10}	(1,8,3,2)(6,5,9,10); (1,2,3,4,5,6,7)	(1,9,7,5,3)(2,10,8,6,4); (1,2,3,4,5,6,7,8,9)
S_5	(1,2,5); (4,5)(1,2,3)	(2,3,4,5); (1,4,2,5,3)
S_6	(5,6); (1,2,3,4,5)	(3,4,5,6) ; (1,2,3,5)
S_7	(2,6)(4,7); (1,2,3,4)(5,6,7)	(6,7); (1,2,3,4,5,6,7)
S_8	(2,7,3)(5,8,6); (1,2,3,4,5,6)	(7,8); (1,2,3,4,5,6,7,8)
S_9	(1,8)(2,5)(6,7); (1,2,3,4,5)(8,6,9,7)	(8,9); (1,2,3,4,5,6,7,8,9)
S_{10}	(1,9)(5,10)(2,3)(7,8); (1,2,3,4,5,6,7,8)	(9,10); (1,2,3,4,5,6,7,8,9,10)

We only provide proof of the case of the group A_7, the remaining cases can be proved along the same lines or, else, using GAP. We first observe that the groups generated by the triples $(a_i, b_i = a_i^{-1}c_i^{-1}, c_i)$, $i = 1, 2$ satisfy the hypothesis of Jordan's symmetric group theorem for the p-cycles $\alpha_1 = (a_1c_1^3)^2 = (3, 7, 6)$ and $\alpha_2 = b_2^{-1}c_2 = (1, 3, 2)$. On the other hand it is clear that both triples are compatible in the sense of Criterion A. Therefore, the quadruple $(a_1, c_1; a_2, c_2)$ provides an unmixed Beauville structure on A_7. Finally, denoting by $\psi = \psi_\tau$ the automorphism of A_7 obtained by conjugation by an element $\tau \in S_n$, one easily checks that Criterion B is accomplished if we set $\tau = (1, 2)(6, 7)(3, 5) \in S_7$, $\delta_1 = c_1$ and $\delta_2 = c_2\alpha$ with $\alpha = (3, 5, 7, 4, 6) \in A_7$.

□

5. Real unmixed Beauville structures on A_n and S_n

While finding generators of all kinds on a trial an error basis turns out to be quite painless for the group S_n, achieving the same goal for A_n appears to be more difficult. For this reason, the way we construct our Beauville surfaces with group A_n is by finding Beauville structures (a_i, c_i) on S_n whose associated Beauville surfaces $C_1 \times C_2/S_n$ satisfy the property that the surface $C_1 \times C_2/A_n$ obtained when only the subgroup A_n is allowed to act on $C_1 \times C_2$ is still a Beauville surface. In turn these will explicitly give us the corresponding Beauville structures $(\widetilde{a}_i = b_i^2,\ \widetilde{c}_i = c_i)$ on A_n. We observe that the Beauville surfaces $(C_1 \times C_2)/A_n$ this way obtained will be double covers of the Beauville surfaces $(C_1 \times C_2)/S_n$.

We start by studying the relationship between the actions of these two groups.

5.1. Relating the actions of A_n and S_n.

Lemma 10. *Let (a,b,c) be a triangle structure on S_n of hyperbolic type $(2,2q,r)$, with r odd, and let us denote by C/S_n the associated orbifold. Consider the orbifold C/A_n obtained as quotient of the action of the subgroup A_n on the same curve C. Then C/A_n is the orbifold of type (q,r,r) associated to the triangle structure on A_n provided by the triple*

$$(\widetilde{a}=b^2,\ \widetilde{b}=b^{-1}cb,\ \widetilde{c}=c)$$

Proof. The situation is described in the following commutative diagram

$$\begin{array}{ccc} C & & \\ \downarrow & \searrow & \\ C/A_n & \longrightarrow & C/S_n \end{array}$$

The orbifold C/S_n is obtained as the quotient of the orbifold C/A_n by the group of order two $\langle j\rangle = S_n/A_n$. Since C/S_n is a Riemann surface of genus zero with three distinguished points, only two possibilities are left for C/A_n.
(i) C/A_n is an orbifold of genus zero with 4 branching values P,Q,R,S of orders $m,d,l,s\geq 2$ and j interchanges two points, say R and S, while fixing P and Q. In this case we would have $l=s$ and the type of C/S_n would be $(2m,2d,l)$ thus, different from $(2,2q,r)$. Contradiction.
(ii) C/A_n is an orbifold of genus zero with 3 branching values Q,R,S of orders q,r,l and j interchanges two points, say R and S, while fixing Q and another point P. In this case we would have $r=l$ and so the type of C/A_n would be (q,r,r) while that of C/S_n would have to be $(2,2q,r)$ as stated.

In order to obtain the triple of generators of A_n that gives rise to this orbifold, let us recall (see section 2.1) that the orbifold C/S_n determined by the triple (a,b,c) is obtained as $C/S_n = \frac{\mathbb{H}/K}{\Gamma/K}$, where K is the kernel of the group epimorphism $\theta:\Gamma=\Gamma_{(2,2q,r)}\to S_n$ defined by $x\to a$, $y\to b$ and $z\to c$. Now if we set $\theta^{-1}(A_n)=\Lambda$, then Λ agrees with the unique index two subgroup of $\Gamma_{(2,2q,r)}$ generated by $\widetilde{x}=y^2$, $\widetilde{y}=y^{-1}zy$, $\widetilde{z}=z$ (see [7]). Clearly the restriction of θ to Λ is nothing but the epimorphism which sends $\widetilde{x}$ to $\widetilde{a}$, $\widetilde{y}$ to $\widetilde{b}$ and $\widetilde{z}$ to $\widetilde{c}$. This means that the orbifold corresponding to the triple $(\widetilde{a},\ \widetilde{b},\ \widetilde{c})$ is $\frac{\mathbb{H}/K}{\Lambda/K} =: C/A_n$. □

Corollary 11. *Let $(a_1,c_1;a_2,c_2)$ be an unmixed Beauville structure on S_n. Let $(p_i,2q_i,r_i)$, with r_i odd, be the type of the triple (a_i,b_i,c_i) (i=1,2), and let $S=S(a_1,c_1;a_2,c_2)=C_1\times C_2/S_n$ be the corresponding Beauville surface. Let $\widetilde{S}=C_1\times C_2/A_n$ be the double cover of S obtained as quotient of $C_1\times C_2$ by the action of the subgroup A_n. Then $\widetilde{S}$ is the Beauville surface $\widetilde{S}=S(\widetilde{a}_1,\widetilde{c}_1;\widetilde{a}_2,\widetilde{c}_2)$ where*

$$\widetilde{a}_1=b_1^2,\ \widetilde{c}_1=c_1;\ \widetilde{a}_2=b_2^2,\ \widetilde{c}_2=c_2$$

Moreover, if $(a_1,c_1;a_2,c_2)$ defines a strongly real unmixed structure on S_n so that S is a real surface, then $(\widetilde{a}_1,\widetilde{c}_1;\widetilde{a}_2,\widetilde{c}_2)$ defines a strongly real unmixed structure on A_n so that $\widetilde{S}$ is also real.

Proof. The first statement is a direct consequence of Lemma 10. As for the question of reality, suppose that there is $\psi \in \mathrm{Aut}(S_n)$ and $\delta_i \in S_n,\ (i=1,2)$, such that

$$\delta_i\psi(a_i)\delta_i^{-1} = a_i^{-1} \text{ and } \delta_i\psi(c_i)\delta_i^{-1} = c_i^{-1}, (i=1,2)$$

Then, setting $\widetilde{\psi} = \psi_{|A_n}$, $\widetilde{\delta}_i = c_i\delta_i$, and using the identity $\delta_i\psi(b_i)\delta_i^{-1} = a_ic_i$, one easily checks that

$$\widetilde{\delta}_i\widetilde{\psi}(\widetilde{a}_i)\widetilde{\delta}_i^{-1} = \widetilde{a}_i^{-1} \text{ and } \widetilde{\delta}_i\widetilde{\psi}(\widetilde{c}_i)\widetilde{\delta}_i^{-1} = \widetilde{c}_i^{-1}, (i=1,2)$$

as was to be seen. □

5.2. **Real unmixed Beauville structures on A_n and S_n, n even.**

Lemma 12. *For $n \geq 10$ even, the permutations*

$$a_1 = (1,n)(2,n-1)\left(\frac{n}{2}, n-2\right)\left(\frac{n}{2}-1, \frac{n}{2}-2\right)\left(\frac{n}{2}+1, \frac{n}{2}+2\right)$$

and $c_1 = (1,2,3,\ldots,n-3)$ generate S_n.

Proof. If we consider

$$d := a_1c_1 = \left(1, n-1, 2, 3, \ldots, \frac{n}{2}-3, \frac{n}{2}-1, n-2, \frac{n}{2}, \frac{n}{2}+2, \frac{n}{2}+3, \ldots, n-3, n\right)$$

and

$$e := c_1a_1 = \left(1, n, 2, n-1, 3, \ldots, \frac{n}{2}-3, \frac{n}{2}-2, \frac{n}{2}, n-2, \frac{n}{2}+1, \frac{n}{2}+3, \ldots, n-3\right).$$

Then,

$$e^{-1}d := (1,2,n-1,n,n-3)\left(\frac{n}{2}-3, \frac{n}{2}-1, \frac{n}{2}, \frac{n}{2}+2, \frac{n}{2}+1, n-2, \frac{n}{2}-2\right).$$

Therefore, the subgroup H generated by a_1 and c_1 contains a 5−cycle. Moreover, since each of the three elements fixed by c_1 is permuted by a_1 with an element not fixed by c_1, it is clear that H acts primitively on $\{1,2,\ldots,n\}$. By Jordan's symmetric group theorem it then follows that H is either A_n or S_n. Now, since $a_1 \notin A_n$ we finally conclude that $H = S_n$. □

Proposition 13. *S_n has a strongly real unmixed Beauville structure for $n > 10$ even.*

Proof. Let us consider the following two triples of hyperbolic generators

$$a_1,\ b_1 := a_1^{-1}c_1^{-1},\ c_1 \quad \text{as in Lemma 12 above}$$
$$a_2 = (n,n-1),\ b_2 = (1,n,n-1,\ldots,2),\ c_2 = (1,2,\ldots,n-1)$$

We observe that b_1 is a $(n-2)$-cycle; in fact the inverse of the permutation e occurring in Lemma 12 above which fixes $\frac{n}{2}-1$ and $\frac{n}{2}+2$. From here we deduce that the two triples are compatible in the sense of Criterion A. As for the question of reality, one easily checks that Criterion B of section 2.2 is satisfied if we choose as δ_1, δ_2 the elements

$$\delta_1 = 1, \text{ and } \delta_2 = (n,n-1)(1,n-3,n-4,\ldots,3,2,n-2) \in A_n$$

and as $\psi = \psi_\tau$ the automorphism consisting in conjugation by the permutation $\tau = (1,2)(n,n-1)(3,n-3)\cdots(k,n-k)\cdots\left(\frac{n}{2}-1,\frac{n}{2}+1\right)\left(\frac{n}{2}-2,\frac{n}{2}+2\right) \in S_n$. □

Corollary 14. *A_n admits a strongly real unmixed Beauville structure for $n > 10$ even.*

Proof. This clearly follows from Corollary 11. □

5.3. **Real unmixed Beauville structures on A_n and S_n, n odd.**

Lemma 15. *For $n > 10$ odd,*

$$a_1 = (1, n-1)(2, n)\left(\frac{n-5}{2}, \frac{n+7}{2}\right)\left(\frac{n-3}{2}, \frac{n-1}{2}\right)\left(\frac{n+3}{2}, \frac{n+5}{2}\right)$$

and $c_1 = (1, \dots, n-2)$ generate S_n.

Proof. Consider the permutations

$$d := a_1c_1 = \left(1, n, 2, 3, \dots, \frac{n-7}{2}, \frac{n+7}{2}, \frac{n+9}{2}, \dots, n-2, n-1\right) \cdot \left(\frac{n-5}{2}, \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+5}{2}\right)\left(\frac{n-3}{2}\right)\left(\frac{n+3}{2}\right)$$

and

$$b_1^{-1} = c_1a_1 = \left(1, n-1, 2, n, 3, 4, \dots, \frac{n-5}{2}, \frac{n+9}{2}, \frac{n+11}{2}, \dots, n-2\right) \cdot \left(\frac{n-3}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+7}{2}\right)\left(\frac{n-1}{2}\right)\left(\frac{n+5}{2}\right)$$

Then,

$$d^{-1}b_1^{-1} = (1, n-2, n-1, n, 2)\left(\frac{n-7}{2}, \frac{n+5}{2}, \frac{n+1}{2}, \frac{n+3}{2}\right)\left(\frac{n-5}{2}, \frac{n+7}{2}, \frac{n-3}{2}, \frac{n-1}{2}\right).$$

From here the conclusion follows arguing in exactly the same way as we did in the proof of Lemma 12. □

Proposition 16. *For $n > 10$ odd, S_n has a strongly real unmixed Beauville structure.*

Proof. Let us consider the following two triples of hyperbolic generators

$$a_1,\ b_1 := a_1^{-1}c_1^{-1},\ c_1 \quad \text{as in Lemma 15 above}$$

$$a_2 = (1, 2),\quad b_2 = a_2^{-1}c_2^{-1},\quad c_2 = (1, 2, \dots, n)$$

We now observe that b_1 is a product of a $(n-6)$-cycle which fixes the elements $\frac{n-3}{2}$, $\frac{n-1}{2}$, $\frac{n+1}{2}$, $\frac{n+3}{2}$, $\frac{n+5}{2}$ and $\frac{n+7}{2}$ and a 4-cycle, while b_2 is a $(n-1)$-cycle which fixes the element 2. From this we deduce that the two triples are compatible in the sense of Criterion A.

As for the question of reality, one easily checks that Criterion B of section 2.2 is satisfied if we choose as δ_1, δ_2 the elements

$$\delta_1 = 1, \text{ and } \delta_2 = (3, 5, \dots, n, 4, 6, 8, \dots, n-1) \in A_n$$

and as $\psi = \psi_\tau$ the automorphism consisting in conjugation by the permutation

$$\tau = (1,2)(n,n-1)(3,n-2)\cdots(k,n-k+1)\ldots\left(\frac{n-3}{2},\frac{n+5}{2}\right)\left(\frac{n+3}{2},\frac{n-1}{2}\right)$$

□

Applying Corollary 11 we now get

Corollary 17. *For $n > 10$ odd, A_n admits a strongly real unmixed Beauville structure.*

6. Conclusion

Finally assembling together the results of section 4 and corollaries 14 and 17 we obtain the proof of theorems 1, 2 and 3.

References

[1] Beauville, A. *Surfaces algébriques complexes* Astérisque, **54**, SMF, Paris (1978).
[2] Bauer, I., Catanese, F. and Grunewald, F. *Beauville surfaces without real structures I.* Geometric methods in algebra and number theory, 1-42, Progr. Math. **235**, Birkhaüser Boston, Boston, MA (2005).
[3] Bauer, I., Catanese, F. and Grunewald, F. *Chebycheff and Belyi polynomials, dessins d'enfants, Beauville surfaces and group theory.* Mediterr. J. Math. **3** (2006), no. 2, 121-146.
[4] Catanese, F. *Fibred surfaces, varieties isogenous to a product and related moduli spaces.* Amer. J. Math. **122** (2000), no. 1, 1-44.
[5] Catanese, F. *Moduli Spaces of surfaces and real structures.* Ann. Math. **158** (2003), no. 12, 577-592.
[6] Fuertes, Y, González-Diez, G. and Jaikin-Zapirain, A. *On Beauville surfaces*, preprint.
[7] Singerman, D. *Finitely Maximal Fuchsian Groups.* J. London Math. Soc. (2), **6** (1972), 29-38.
[8] Wielandt, H. *Finite Permutation Groups.* Academic Press, 1964.