Non-homeomorphic Galois conjugate Beauville structures on $\mathrm{PSL}(2,p)$

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Abstract

Catanese's rigidity results for surfaces isogenous to a product of curves indicate that Beauville surfaces should provide a fertile source of examples of Galois conjugate varieties that are not homeomorphic, a phenomenon discovered by J. P. Serre in the sixties.

In this paper, we construct Beauville surfaces $S = (C_1 \times C_2)/G$ with group $G = \mathrm{PSL}(2,p)$ for $p \geq 7$, and curves C_1, C_2 such that the orbit of S under the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ contains non-homeomorphic conjugate surfaces. When $p = 7$ the orbit consists exactly of two surfaces that have non-isomorphic fundamental groups, and the curves C_1, C_2 have genera 8 and 49, which is shown to be the minimum for which there is a pair of non-homeomorphic Galois conjugate Beauville surfaces. As p grows the orbits contain an arbitrarily large number of non-homeomorphic surfaces.

Along the way we prove a metric rigidity theorem for Beauville surfaces which provides an elementary proof of the part of Catanese's theory needed to prove our results.

Keywords: Beauville surfaces, Galois action, Fundamental group, Triangle curves, Triangle Fuchsian groups
2010 MSC: 14J25, 14H30, 20H10

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1. Introduction and statement of results

A complex algebraic curve C will be termed *triangle curve* if it admits a finite group of automorphisms $G < \mathrm{Aut}(C)$ so that $C/G \cong \mathbb{P}^1$ and the natural projection $C \to C/G$ ramifies over three values, say 0, 1 and ∞. If the branching orders at these points are l, m and n we will say that C/G is an orbifold of type (l, m, n). Due to Belyi's Theorem ([9]) triangle curves are defined over the field

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[1] Partially supported by a MICINN grant MTM2009-11848.
[2] Partially supported by an FPU grant of the MICINN.

$\overline{\mathbb{Q}}$ of algebraic numbers, and provide a geometric action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, namely: if C is defined by a polynomial $F(X,Y) \in \overline{\mathbb{Q}}[X,Y]$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then C^σ is defined by $F^\sigma(X,Y)$, the polynomial obtained by applying σ to the coefficients of F.

For complex surfaces S an analogous criterion in which Belyi functions are replaced by Lefschetz functions is given in [24]. Among the complex surfaces defined over a number field an important class is that of Beauville surfaces defined as follows.

A *Beauville surface* (of unmixed type) is a compact complex surface S satisfying the following properties:

(1) It is isogenous to a higher product, that is $S \cong C_1 \times C_2/G$, where C_i $(i = 1,2)$ are curves of genera $g_i \geq 2$ and G is a finite group acting freely on $C_1 \times C_2$ by holomorphic transformations.
(2) The group G acts effectively on each curve C_i so that $C_i/G \cong \mathbb{P}^1$ and the covering $C_i \to C_i/G$ ramifies over three points.

Beauville surfaces were introduced by F. Catanese in [11] generalizing a construction by A. Beauville which appears as exercise number 4 in page 159 of [8], and have since been studied by several authors. The relevance of Beauville surfaces lies mainly on the fact that they are the rigid ones among the surfaces isogenous to a higher product. In fact, Catanese proved that if $S = C_1 \times C_2/G$ and $S' = C_1' \times C_2'/G'$ are homeomorphic Beauville surfaces then $G \cong G'$ and, perhaps after interchanging factors, $C_i' \cong C_i$ or $\overline{C_i}$ ([11], [4]).

This result suggests that Beauville surfaces should provide a fertile source of examples of Galois conjugate varieties that are not homeomorphic. Indeed any Beauville surface $S = C_1 \times C_2/G$, where C_1, C_2 are curves of genera $g_1 \neq g_2$ such that there is a $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ so that C_1^σ is not isomorphic to C_1 or $\overline{C_1}$ will be not homeomorphic to S^σ. The problem is that, as far as we know, the only examples of Beauville surfaces in which the algebraic equations of the curves C_i are explicitly given are Beauville's own examples, in which $C_1 = C_2$ is a Fermat curve $F_n : x^n + y^n + z^n = 0$ and it is easy to see that in that case $S^\sigma = S$ for every Galois element σ ([25]). Rather, the construction of Beauville surfaces with Beauville group G is usually achieved by choosing a pair of triples of generators (a_i, b_i, c_i) of G satisfying certain properties (see section 5) and in general there is no way to figure out what the action of σ on these generators looks like.

To explain the relevance of these examples we recall that, by Hodge's Theorem, the dimensions of the cohomology groups $H^i(X,\mathbb{C})$ of a complex projective variety X can be expressed in terms of the Hodge numbers $h^{p,q}(X) = \dim H^p(X,\Omega^q)$ which, by Serre's GAGA principle, remain invariant under Galois conjugation. It follows that the most standard topological invariants, namely the Betti numbers and the signature of a complex projective surface are Galois invariant (see e.g. [36] Th. 6.33). Nevertheless in 1964 J. P. Serre ([32]) gave an example of a complex projective surface possessing non-homeomorphic Galois conjugates. Several instances of this or similar phenomena have been found since then (see e.g. [1], [2], [15], [12], [30], [6], [33], [16]).

Another important property of our examples is that, while the fundamental groups $\pi_1(S)$ and $\pi_1(S^\sigma)$ are not isomorphic, their profinite completions are. This will be a direct consequence of Grothendieck's theory of the algebraic fundamental group of algebraic varieties.

The main results of our paper are

(1) For each prime number $p \geq 7$ and each natural number $n > 6$ such that n divides either $(p-1)/2$ or $(p+1)/2$ we construct a Beauville surface $X = (E_1 \times E)/G$ with group $G = \mathrm{PSL}(2,p)$ satisfying the following properties (Theorems 2, 3 and 8):

   (i) E_1 is a curve of genus $g = \frac{1}{24n}(n-6)p(p-1)(p+1)+1$ defined over $\mathbb{Q}(\cos \pi/n)$ with automorphism group $\mathrm{Aut}(E_1) \cong G$;

   (ii) E is a curve of genus $g = \frac{1}{4}(p+1)(p-1)(p-3)+1$ defined over $\mathbb{Q}$ such that $\mathrm{Aut}(E) = \mathrm{PSL}(2,p) \times \mathfrak{S}_3$;

   (iii) The orbit of X under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ contains $\phi(n)/2$ surfaces which are pairwise homotopically non-equivalent, thus providing an infinite family of explicit examples of Serre's type.

(2) We construct a Beauville surface $S = (D_1 \times D)/G$ with group $G = \mathrm{PSL}(2,7)$ such that (Theorems 4, 3 and 7):

   (i) D_1 is a curve of genus $g = 8$ defined over $\mathbb{Q}(\sqrt{2})$ with automorphism group $\mathrm{Aut}(D_1) \cong \mathrm{PGL}(2,7)$;

   (ii) D is a curve of genus $g = 49$ defined over $\mathbb{Q}$ such that $\mathrm{Aut}(D) = \mathrm{PSL}(2,7) \times \mathfrak{S}_3$;

   (iii) The orbit of S under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ consists of two surfaces with non-isomorphic fundamental groups, hence non-homeomorphic.

   (iv) This pair of genera is the minimum for which there is a pair of non-homeomorphic conjugate Beauville surfaces.

(3) We provide an alternative approach to the part of Catanese's rigidity theory for Beauville surfaces that is needed to detect when two Beauville surfaces have different fundamental groups. More precisely we show (Theorem 5):

   Two Beauville surfaces S and S' are isometric if and only if $\pi_1(S) \cong \pi_1(S')$

   This result implies that the fundamental group of a Beauville surface $S = C_1 \times C_2/G$ determines the curves C_1 and C_2 up to complex conjugation (Theorem 6), a theorem due originally to Catanese. The proof of Theorem 5 only depends on the rigidity of triangle groups and other basic facts of Fuchsian group theory, thereby making the paper self-contained and the theory accessible to a wider readership.

The authors are grateful to Marston Conder, Andrei Jaikin, Gareth Jones and Jürgen Wolfart for their generous advice and very helpful ideas on several points of this paper. They would also like to thank the referee for many comments that helped to improve significantly the exposition of the paper.

2. Triangle curves, triangle groups and rotation numbers

The content of this section is well known. It mostly amounts to the statement that, via uniformization, triangle curves correspond to normal subgroups of Fuchsian triangle groups. However in order to get some insight of the meaning of the Galois action at the Fuchsian group level we will need to make this correspondence very precise, and the existing references do not always fit suitably in our approach.

A hyperbolic triangle group is a Fuchsian group – i.e. a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ – that arises as follows. Let l, m and n be positive integers such that $1/l + 1/m + 1/n < 1$. Consider a hyperbolic triangle in the hyperbolic plane, with vertices v_0, v_1 and v_∞ and angles π/l, π/m and π/n respectively. Let us denote by R_i the reflection over the edge opposite to v_i. These three transformations generate a group of isometries of the hyperbolic plane, and the index two subgroup consisting of the conformal elements is a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ that is called a triangle group of signature (l,m,n). Elementary hyperbolic geometry ensures that the triangle and hence the associated triangle group described above are unique up to conjugation in $\mathrm{PSL}(2,\mathbb{R})$ ([7], §7.12). In this article we will reserve the notation $T = T(l,m,n)$ for the triangle in the upper-half plane $\mathbb{H}$ which is the image under $M(\mathrm{w}) = \frac{i(1+\mathrm{w})}{1-\mathrm{w}}$ of the triangle depicted in Figure 1 inside the unit disc $\mathbb{D}$, i.e. the only triangle with $v_0 = 0$, $v_\infty \in \mathbb{R}^+$ and $v_1 \in \mathbb{H}^-$. The corresponding triangle group will be denoted by $\Gamma = \Gamma(l,m,n)$.

It is a classical fact (see [28], Appendix 2) that this is a Fuchsian group with presentation

$$\Gamma(l,m,n) = \langle x, y, z \ : \ x^l = y^m = z^n = xyz = 1 \rangle, \tag{1}$$

where

$$x = R_1 R_\infty, \quad y = R_\infty R_0, \quad z = R_0 R_1 \tag{2}$$

are the positive rotations around the points v_0, v_1 and v_∞ through angles $2\pi/l$, $2\pi/m$ and $2\pi/n$ respectively. Note that the quadrilateral consisting of the union of T and one of its reflections $R_i(T)$ (shaded triangle in Fig. 1) serves as a fundamental domain for $\Gamma(l,m,n)$. Thus, the quotient $\mathbb{H}/\Gamma$ is an orbifold of genus zero with three cone points $[v_0]_\Gamma$, $[v_1]_\Gamma$ and $[v_\infty]_\Gamma$ of orders l, m and n respectively. For later use we emphasize that the elements x, y and z thus defined are positive rotations of angle precisely $2\pi/l$, $2\pi/m$ and $2\pi/n$ around the vertices v_0, v_1 and v_∞ respectively. It is also classical that any other finite order element of $\Gamma(l,m,n)$ is conjugate to a power of x, y or z and that these account for all elements in Γ that fix points (see for example [22], section 2.4.3). In the rest of the paper we identify $\mathbb{H}/\Gamma$ with $\mathbb{P}^1$ via the isomorphism $\Phi : \mathbb{H}/\Gamma \longrightarrow \mathbb{P}^1$ uniquely determined by the conditions

$$\begin{array}{rccc} \Phi : & \mathbb{H}/\Gamma & \longrightarrow & \mathbb{P}^1 \\ & [v_0]_\Gamma & \longmapsto & 0 \\ & [v_1]_\Gamma & \longmapsto & 1 \\ & [v_\infty]_\Gamma & \longmapsto & \infty \end{array} \tag{3}$$

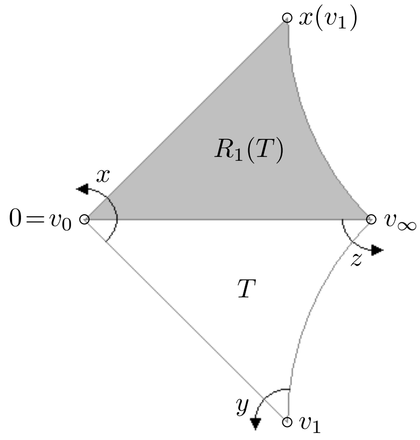

Figure 1: Generators x, y and z together with a fundamental domain of $\Gamma(l, m, n)$ (depicted inside the unit disk model of the hyperbolic plane).

Now let G be a finite group, C a complex algebraic curve and $\mathrm{Aut}(C)$ its automorphism group. By a *$G-$covering of type* (l, m, n) we shall understand a Galois covering $f : C \longrightarrow \mathbb{P}^1$ ramified over 0, 1 and ∞ with orders l, m and n respectively, endowed with a monomorphism $\mathfrak{i} : G \longrightarrow \mathrm{Aut}(C)$ such that the covering group $\mathrm{Aut}(C, f)$ agrees with $\mathfrak{i}(G)$. Such an object we shall denote by $(C, f) \equiv (C, f, \mathfrak{i})$. We will regard two such covers $(C_1, f_1, \mathfrak{i}_1)$ and $(C_2, f_2, \mathfrak{i}_2)$ as equivalent if there is an isomorphism $\tau : C_2 \longrightarrow C_1$ such that $f_2 = f_1 \circ \tau$ and $\mathfrak{i}_2(G) = \tau^{-1}\mathfrak{i}_1(G)\tau$. We will say that a $G-$covering as above is *hyperbolic* if the genus of C is ≥ 2.

Now let G be a finite group and a, b, c three generators. We shall say that (a, b, c) is a *hyperbolic triple of generators* of type (l, m, n) if the following conditions hold

(i) $abc = 1$

(ii) $\mathrm{ord}(a) = l$, $\mathrm{ord}(b) = m$ and $\mathrm{ord}(c) = n$

(iii) $\dfrac{1}{l} + \dfrac{1}{m} + \dfrac{1}{n} < 1$.

To any given triple of hyperbolic generators (a, b, c) of G we can associate an equivalence class of $G-$coverings of type (l, m, n) as follows.

Since any finite order element of $\Gamma(l, m, n)$ is conjugate to a power of x, y or z, it is obvious that the kernel K of the epimorphism

$$\begin{array}{rccc} \rho : & \Gamma(l, m, n) & \longrightarrow & G \\ & x & \longmapsto & a \\ & y & \longmapsto & b \\ & z & \longmapsto & c \end{array}$$

is a torsion-free Fuchsian group. As a consequence there is an isomorphism $\widetilde{\Phi} : \mathbb{H}/K \to C$ from the quotient Riemann surface $\mathbb{H}/K$ to an algebraic curve C on which the group G acts by the rule

$$\mathfrak{i}(g)(\widetilde{\Phi}([\mathrm{w}]_K)) = \widetilde{\Phi}\left([\gamma(\mathrm{w})]_K\right), \text{ for any choice of } \gamma \in \Gamma \text{ such that } \rho(\gamma) = g \quad (4)$$

Since the natural projection $\pi : \mathbb{H}/K \longrightarrow \mathbb{H}/\Gamma$ ramifies over three points, we have a $G-$covering (C, f) of type (l, m, n) defined by the commutative diagram

$$\begin{array}{ccc} \mathbb{H}/K & \xrightarrow{\widetilde{\Phi}} & C \\ \big\downarrow & & \big\downarrow f \\ \mathbb{H}/\Gamma & \xrightarrow{\Phi} & \mathbb{P}^1 \end{array} \quad (5)$$

where Φ is the isomorphism defined in (3).

Clearly a triple of generators which differs from (a, b, c) by the action of an automorphism of G gives rise to the same $G-$cover. Also clear is that a different choice of isomorphism $\widetilde{\Phi}' : \mathbb{H}/K \longrightarrow C'$ between $\mathbb{H}/K$ and another curve C' gives rise to an isomorphic $G-$covering (C', f'), where in fact $f' = f \circ (\widetilde{\Phi} \circ \widetilde{\Phi}'^{-1})$.

The covering is hyperbolic precisely because the orders l, m and n satisfy condition (iii), as by the Riemann-Hurwitz formula (see for example [22], Lemma 2.39) the genus g of C is given by

$$2g - 2 = |G| \left(1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)\right)$$

Conversely a hyperbolic $G-$covering (C, f) of type (l, m, n) determines a triple of generators of G, defined up to an element of $\mathrm{Aut}(G)$, in the following manner. Uniformization theory tells us that there is a torsion free Fuchsian group K uniformizing C whose normalizer $N(K)$ contains $\Gamma = \Gamma(l, m, n)$ and there is an isomorphism of coverings of the form

$$\begin{array}{ccc} \mathbb{H}/K & \xrightarrow{\widetilde{u}} & C \\ \big\downarrow & & \big\downarrow f \\ \mathbb{H}/\Gamma & \xrightarrow{u} & \mathbb{P}^1 \end{array}$$

If the orders l, m and n are all distinct then necessarily $u = \Phi$. Otherwise note that any element of $N(\Gamma)$ induces an automorphism of $\mathbb{H}/\Gamma$ which permutes the points $[v_0]_\Gamma$, $[v_1]_\Gamma$ and $[v_\infty]_\Gamma$. It is known that $N(\Gamma)/\Gamma$ is isomorphic to the symmetric group $\mathfrak{S}_2$ if $l = m \neq n$ and to $\mathfrak{S}_3$ if $l = m = n$ ([34]). So, in any case, there is an element $\delta \in N(\Gamma)$ producing the following commutative diagram

$$\begin{array}{ccccc} \mathbb{H}/\delta^{-1}K\delta & \xrightarrow{\delta} & \mathbb{H}/K & \xrightarrow{\widetilde{u}} & C \\ \big\downarrow & & \big\downarrow & & \big\downarrow f \\ \mathbb{H}/\Gamma & \xrightarrow{\delta} & \mathbb{H}/\Gamma & \xrightarrow{u} & \mathbb{P}^1 \end{array}$$

where $u \circ \delta$ equals Φ. Accordingly we will simply write $\widetilde{\Phi}$ for $\widetilde{u} \circ \delta$.

Since any element of G is determined by its action on C, the identity

$$\widetilde{\Phi}([\gamma(\mathrm{w})]) = \mathfrak{i}\,(\rho(\gamma))\,\widetilde{\Phi}([\mathrm{w}]), \quad \text{for all } \gamma \in \Gamma$$

defines an epimorphism $\rho : \Gamma \longrightarrow G$ (which in turn induces an isomorphism $\overline{\rho} : \Gamma/K \longrightarrow G$) and hence a hyperbolic triple of generators

$$(a, b, c) = (\rho(x), \rho(y), \rho(z))\,.$$

If we start with an equivalent $G-$covering $\tau : (C, f) \longrightarrow (C', f')$ and choose a corresponding Fuchsian group representation we get a diagram of the form

$$\begin{array}{ccccccc}
\mathbb{H}/K & \xrightarrow{\widetilde{\Phi}} & C & \xrightarrow{\tau} & C' & \xleftarrow{\widetilde{\Phi'}} & \mathbb{H}/K' \\
\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow \\
\mathbb{H}/\Gamma & \xrightarrow{\Phi} & \mathbb{P}^1 & \xrightarrow{\mathrm{Id}} & \mathbb{P}^1 & \xleftarrow{\Phi} & \mathbb{H}/\Gamma
\end{array}$$

We note that for this diagram to be commutative the corresponding isomorphism $\mathbb{H}/K \longrightarrow \mathbb{H}/K'$ must be induced by a transformation $\lambda \in \Gamma$, so that $\widetilde{\Phi'} = \tau \circ \widetilde{\Phi} \circ \lambda^{-1}$. Now, plugging this information into the identity $\widetilde{\Phi'}([\gamma(\mathrm{w})]) = \mathfrak{i}'\,(\rho'(\gamma))\,\widetilde{\Phi'}([\mathrm{w}])$ defining the epimorphism $\rho' : \Gamma \longrightarrow G$ corresponding to the $G-$covering (C', f'), we get the identity

$$\tau \circ \mathfrak{i}\left(\rho(\lambda^{-1}\gamma)\right) = \mathfrak{i}'\,(\rho'(\gamma)) \circ \tau \circ \mathfrak{i}\left(\rho(\lambda^{-1})\right)$$

It follows that $\rho'(\gamma) = \varphi \circ \rho(\gamma)$ for all $\gamma \in \Gamma$, where $\varphi \in \mathrm{Aut}(G)$ is defined by $\varphi(g) = (\mathfrak{i}')^{-1}(g_0 \cdot \mathfrak{i}(g) \cdot g_0^{-1})$ with $g_0 = \tau \circ \mathfrak{i}\left(\rho(\lambda^{-1})\right) \in \mathrm{Aut}(C)$. As a consequence $(a', b', c') = (\varphi(a), \varphi(b), \varphi(c))$.

Summarising we have:

Proposition 1. *There is a bijection between*

$$\left\{ \begin{array}{c} \textit{Hyperbolic triangle} \\ G-\textit{coverings } (C,f) \end{array} \right\} \Big/ \sim \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \textit{Hyperbolic triples} \\ \textit{of generators of } G \end{array} \right\} \Big/ \mathrm{Aut}(G)$$

where $\sim$ *stands for the equivalence of* $G-$*coverings defined above.*

Example 1. We consider the canonical triple of generators of the (additive) group $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$:

$$a = (1, 0), \quad b = (0, 1), \quad c = (n-1, n-1).$$

Any other triple of generators $a' = (t, u)$, $b' = (v, w)$, $c' = (-t-v, -u-w)$ is obtained from (a, b, c) by applying the automorphism of G induced by the matrix $\left(\begin{smallmatrix} t & v \\ u & w \end{smallmatrix}\right) \in \mathrm{GL}(2, n)$. As a consequence there is only one class of hyperbolic triples of generators of G of type (n,n,n), hence only one equivalence class of $G-$coverings of type (n,n,n), this being given by the natural projection $\mathbb{H}/K \longrightarrow \mathbb{H}/\Gamma$, where K is the kernel of the epimorphism $\rho : \Gamma(n,n,n) \longrightarrow G$ sending the generators x, y and z of $\Gamma = \Gamma(n,n,n)$ to a, b and c respectively. Clearly $K = [\Gamma,\Gamma]$, and therefore $G = \Gamma/[\Gamma,\Gamma]$. Now, by uniqueness, this covering must be equivalent to the Fermat covering (F_n, f) provided by the curve $F_n : \mathrm{w}_1^n + \mathrm{w}_2^n + \mathrm{w}_3^n = 0$ and the group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ acting by

$$\zeta_{(\alpha,\beta)}([\mathrm{w}_1,\mathrm{w}_2,\mathrm{w}_3]) = [\xi_5^\alpha \mathrm{w}_1, \xi_5^\beta \mathrm{w}_2, \mathrm{w}_3],$$

where $\xi_5 = e^{2\pi i/5}$.

An important observation for the purpose of this paper is now in order. Let $P \in C$ be a point fixed by an automorphism $\tau \in \mathrm{Aut}(C)$ of order r. If ψ is a local parameter around P such that $\psi(P) = 0$ then

$$\psi \circ \tau \circ \psi^{-1}(\mathrm{w}) = \xi_r^k \mathrm{w}, \text{ where } \xi_r = e^{2\pi i/r} \text{ and } k \in \mathbb{Z}$$

As $\xi_r^k = (\psi \circ \tau \circ \psi^{-1})'(0)$ it is clear that this root of unity does not depend on the choice of the local coordinate ψ. One says that τ rotates through angle $2\pi k/r$ at P or that ξ_r^k (or simply k) is the *rotation number* of τ at P. Note that the rotation number k is defined only modulo r.

Since by formula (4) the action of the element $a \in G$ (resp. b, resp. c) at the point $P_0 = \widetilde{\Phi}([v_0]_K)$ (resp. $P_1 = \widetilde{\Phi}([v_1]_K)$, resp. $P_\infty = \widetilde{\Phi}([v_\infty]_K)$) is locally described by the action of the element $x \in \Gamma$ (resp. y, resp. z), we may conclude that the element a (resp. b, resp. c) possesses one fixed point in the fibre of 0 (resp. of 1, resp. of ∞) with rotation number ξ_l (resp. ξ_m, resp. ξ_n).

The relevance of the rotation numbers relies on the fact that if $\tau : C \to C$ is a finite order automorphism fixing a point P with rotation angle ξ and σ is a field automorphism of $\mathbb{C}$ (or any field of definition for C and τ) then $\tau^\sigma : C^\sigma \to C^\sigma$ is a finite order automorphism fixing P^σ with rotation angle $\sigma(\xi)$. We recall that this is so because if $\tau^* : H^0(C,\Omega) \to H^0(C,\Omega)$ is the $\mathbb{C}-$linear automorphism induced by τ on the space of regular $1-$forms and ω is an eigenvector such that $\omega(P) \neq 0$, then a straightforward local computation shows that the rotation number agrees with the eigenvalue of ω, and this is an algebraically defined object.

3. Galois conjugation of triangle curves

By Belyi's theorem ([9]) $G-$covers can be defined over $\overline{\mathbb{Q}}$. It is also known (see [23]) that the automorphisms of any hyperbolic triangle curve are defined over $\overline{\mathbb{Q}}$. This permits an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on equivalence classes of $G-$coverings (C,f). For an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ one simply defines $(C,f,\mathfrak{i})^\sigma := (C^\sigma, f^\sigma, \mathfrak{i}^\sigma)$, where $f^\sigma : C^\sigma \longrightarrow \mathbb{P}^1$ is obtained by applying σ to the coefficients defining the covering $f : C \longrightarrow \mathbb{P}^1$ and $\mathfrak{i}^\sigma : G \longrightarrow \mathrm{Aut}(C^\sigma)$ is defined by $\mathfrak{i}^\sigma(h) = (\mathfrak{i}(h))^\sigma$.

This rather canonical action of the absolute Galois group on $G-$covers turns out to be very mysterious at the level of triples of generators, their equivalent counterparts in Proposition 1. One way to gain some insight on it is by relating the rotation numbers of these generators at certain points of C to their rotation numbers at the corresponding points of C^σ. As far as we know this approach was first used by M. Streit in [35].

Proposition 2. *Let (a,b,c) be a hyperbolic triple of generators of G of type (l,m,n) defining a $G-$covering (C,f). Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be such that $\sigma(\xi_l) = \xi_l^\alpha$, $\sigma(\xi_m) = \xi_m^\beta$ and $\sigma(\xi_n) = \xi_n^\gamma$, where $\xi_k = e^{2\pi i/k}$.*

Then the $G-$covering (C^σ, f^σ) corresponds to a hyperbolic triple of generators $(a_\sigma, b_\sigma, c_\sigma)$ of G of the form

$$\begin{array}{rcl} a_\sigma & = & h_a a^{\alpha'} h_a^{-1} \\ b_\sigma & = & h_b b^{\beta'} h_b^{-1} \\ c_\sigma & = & h_c c^{\gamma'} h_c^{-1} \end{array} \tag{6}$$

where $\alpha\alpha' \equiv 1 \bmod l$, $\beta\beta' \equiv 1 \bmod m$, $\gamma\gamma' \equiv 1 \bmod n$ and $h_a, h_b, h_c \in G$.

Proof. Suppose that $(a_\sigma, b_\sigma, c_\sigma)$ is a hyperbolic triple of generators of G defining the $G-$covering $(C^\sigma, f^\sigma, \mathfrak{i}^\sigma)$. This means that if K_σ is the kernel of the epimorphism

$$\begin{array}{rccc} \rho_\sigma : & \Gamma(l,m,n) & \longrightarrow & G \\ & x & \longmapsto & a_\sigma \\ & y & \longmapsto & b_\sigma \\ & z & \longmapsto & c_\sigma \end{array}$$

there is a commutative diagram

$$\begin{tikzcd} \mathbb{H}/K_\sigma \arrow[r, "\widetilde{\Phi}"] \arrow[d] & C^\sigma \arrow[d, "f^\sigma"] \\ \mathbb{H}/\Gamma \arrow[r, "\Phi"] & \mathbb{P}^1 \end{tikzcd}$$

such that a_σ (resp. b_σ, resp. c_σ) fixes a point $P_{0,\sigma} \in (f^\sigma)^{-1}(0)$ (resp. $P_{1,\sigma} \in (f^\sigma)^{-1}(1)$, resp. $P_{\infty,\sigma} \in (f^\sigma)^{-1}(\infty)$) with rotation angle ξ_l (resp. ξ_m, resp. ξ_n).

On the other hand, since a fixes the point $P_0 \in f^{-1}(0)$ with rotation number ξ_l then, by definition of the action of G on C^σ, a fixes the point $P_0^\sigma \in (f^\sigma)^{-1}(0)$ with rotation number $\sigma(\xi_l)$. Since $P_{0,\sigma}$ and P_0^σ belong to the same fiber $(f^\sigma)^{-1}(0)$, there must be an element $h_a^{-1} \in G$ such that $\mathfrak{i}^\sigma(h_a^{-1})(P_{0,\sigma}) = P_0^\sigma$. Therefore $h_a a h_a^{-1}$ fixes the point $P_{0,\sigma}$ with rotation angle ξ_l^α and so $a_\sigma^\alpha = h_a a h_a^{-1}$. We can proceed in the same way with the other two generators and write

$$\begin{array}{rcl} a_\sigma^\alpha & = & h_a a h_a^{-1} \\ b_\sigma^\beta & = & h_b b h_b^{-1} \\ c_\sigma^\gamma & = & h_c c h_c^{-1} \end{array}$$

Raising these elements to the α'-th, β'-th and γ'-th power respectively one gets the result. $\square$

Remark 1. (i) Note that through conjugation by an element of G, e.g. h_c^{-1}, we can always normalize the second triple so that for instance, $c_\sigma = c^{\gamma'}$.

(ii) The exponents $\alpha', \beta', \gamma' \in \mathbb{N}$ occurring in formulae (6) can be chosen to be equal. This is because if r is the least common multiple of the integers l, m, n and $\sigma(\xi_r) = \xi_r^\delta$ then one also has $\sigma(\xi_l) = \xi_l^\delta, \sigma(\xi_m) = \xi_m^\delta$ and $\sigma(\xi_n) = \xi_n^\delta$.

In the special case where σ is complex conjugation there is a precise formula for the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on triples

Proposition 3. *Let (a, b, c) be a hyperbolic triple of generators of G defining a $G-$covering (C, f). Then the complex conjugate $G-$covering $(\overline{C}, \overline{f})$ is defined by the triple $(a^{-1}, ab^{-1}a^{-1}, c^{-1})$.*

Proof. We will work here with the unit disc $\mathbb{D}$ instead of the upper half-plane.

We observe that if

$$\begin{array}{ccc} \mathbb{D}/K & \xrightarrow{\widetilde{\Phi}} & C \\ \downarrow & & \downarrow f \\ \mathbb{D}/\Gamma & \xrightarrow{\Phi} & \mathbb{P}^1 \end{array}$$

is the commutative diagram (5) defining (C, f) then the covering $(\overline{C}, \overline{f})$ is defined by the diagram

$$\begin{array}{ccc} \mathbb{D}/\overline{K} & \xrightarrow{\widetilde{\Phi}_1} & \overline{C} \\ \downarrow & & \downarrow \overline{f} \\ \mathbb{D}/\Gamma & \xrightarrow{\Phi} & \mathbb{P}^1 \end{array}$$

where for a subgroup H of $\mathrm{Aut}(\mathbb{D})$ we put $\overline{H} = \{\overline{h} : h \in H\}$ and $\widetilde{\Phi}_1(\mathrm{w}) = \overline{\widetilde{\Phi}}(\overline{\mathrm{w}})$.

Note that the function $\Phi_1(\mathrm{w}) = \overline{\Phi(\overline{\mathrm{w}})} = \overline{\Phi}(\overline{\mathrm{w}})$ induces the same isomorphism $\mathbb{D}/\Gamma \simeq \mathbb{P}^1$ as Φ. Moreover, since $x(\mathrm{w}) = \xi_l \cdot \mathrm{w}$ and z is conjugate to $\mathrm{w} \mapsto \xi_n \cdot \mathrm{w}$ by means of a real Möbius transformation (see Figure 1) we see that $\overline{x} = x^{-1}$ and $\overline{z} = z^{-1}$. It follows that $\overline{\Gamma} = \Gamma$ and that the epimorphisms

$$\begin{array}{rccc} \rho: & \Gamma(l,m,n) & \longrightarrow & G \\ & x & \longmapsto & a \\ & y & \longmapsto & b \\ & z & \longmapsto & c \end{array} \qquad \begin{array}{rccc} \overline{\rho}: & \Gamma(l,m,n) & \longrightarrow & G \\ & x & \longmapsto & a^{-1} \\ & y & \longmapsto & ab^{-1}a^{-1} \\ & z & \longmapsto & c^{-1} \end{array}$$

are related by $\overline{\rho}(\gamma) = \rho(\overline{\gamma})$. We see that $\overline{K} = \ker(\overline{\rho})$ and the hyperbolic triple $(a^{-1}, ab^{-1}a^{-1}, c^{-1})$ defines the $G-$covering $(\overline{C}, \overline{f})$. □

Two important notions regarding Galois action on algebraic varieties are those of field of moduli and field of definition. A field k is a *field of definition* of an algebraic variety V if V is isomorphic to an algebraic variety defined by a finite number of polynomials with coefficients in k. The *field of moduli* of an algebraic variety V defined over $\overline{\mathbb{Q}}$ is the subfield of $\overline{\mathbb{Q}}$ consisting of all elements fixed by the group $G_V = \{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : V^\sigma \cong V\}$. Note that this is the inertia group at V of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of isomorphism classes of algebraic varieties defined over $\overline{\mathbb{Q}}$. In particular the index of G_V agrees with the cardinality of the orbit of V. An obvious adaptation of this definition to the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on isomorphism classes of coverings (C, f) leads to the concepts of fields of moduli and definition of a covering.

The field of moduli is contained in any field of definition, but in general they do not coincide. However, triangle curves and $G-$coverings are known to be defined over their fields of moduli ([37]).

Proposition 2 immediately implies the following

Corollary 1. *Abelian $G-$coverings are defined over $\mathbb{Q}$.*

Proof. Proposition 2 together with the second part of Remark 1 imply that if (a, b, c) is the triple defining an abelian $G-$covering (C, f) then, for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the triple defining the covering (C^σ, f^σ) is of the form (a^k, b^k, c^k). Now these two triples differ by the automorphism ψ of G defined by $\psi(u) = u^k$. Hence, for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the $G-$coverings (C, f) and (C^σ, f^σ) are equivalent. $\square$

Alternative proofs of this fact have been found by R. Hidalgo ([26]) and B. Mühlbauer (forthcoming PhD thesis). See also the article by I. Bauer and F. Catanese [3].

4. Triangle curves with covering group $G = \mathrm{PSL}(2, p)$

In this section we find hyperbolic triangle curves with covering group $G = \mathrm{PSL}(2, p) = \mathrm{SL}(2, p)/\{\pm \mathrm{Id}\}$ via their corresponding triples of generators. More precisely, we study triples of generators of types (p, p, p) and $(2, 3, n)$, for certain integers n, in $\mathrm{PSL}(2, p)$ and triples of type $(3, 3, 4)$ in $\mathrm{PSL}(2, 7)$. These triples will be used later, in section 6, in the construction of our Beauville surfaces.

Recall that if $p > 2$ is a prime, G is a group of order $p(p-1)(p+1)/2$, and observe that this expression already shows that it always has elements of orders 2, 3 and p. Conjugacy classes of elements and subgroups of $\mathrm{PSL}(2, p)$ are very well known. They can be found in almost any introductory book on linear groups (see e.g. [27] or [19] for an exhaustive exposition).

Throughout this section we will repeatedly use the following known result, which can be found e.g. in [19], §5.2. If $p \geq 5$ is a prime, then the conjugacy class of an element of $\mathrm{PSL}(2, p)$ is determined by its trace, except for elements of order p which lie in two different classes and always have trace ± 2.

Now by the results of section 2, the study of $G-$coverings is equivalent to the study of triples of generators of $G = \mathrm{PSL}(2, p)$. These were studied by Macbeath in [29]. In order to present the results we need, we consider for any triple $(\alpha, \beta, \gamma) \in \mathbb{F}_p^*$ the set $E(\alpha, \beta, \gamma)$ that consists of all triples of elements (A, B, C) of $\mathrm{SL}(2, p)$ with traces α, β and γ respectively, such that their product is the identity. Consequently we write $\overline{E}(\alpha, \beta, \gamma)$ for the image of $E(\alpha, \beta, \gamma)$ in $\mathrm{PSL}(2, p)$.

A triple (α, β, γ) is called singular if its *discriminant* $\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4$ vanishes, and exceptional if the orders of the elements in the triples of $\overline{E}(\alpha, \beta, \gamma)$ are one of the following

$$(2,2,n), (2,3,3), (3,3,3), (3,4,4), (2,3,4),$$
$$(2,5,5), (5,5,5), (3,3,5), (3,5,5), (2,3,5)$$

Then Theorems 2 and 3 in [29] can be summarized as follows.

Theorem 1 (Macbeath)**.** *A triple in* $E(\alpha, \beta, \gamma)$ *generates the whole group* $\mathrm{PSL}(2,p)$ *if and only if* (α, β, γ) *is neither singular nor exceptional. In this case*

(1) *There are two conjugacy classes of triples in* $E(\alpha, \beta, \gamma)$ *modulo* $\mathrm{SL}(2,p)$*;*
(2) *There is one conjugacy class of triples in* $E(\alpha, \beta, \gamma)$ *modulo* $\mathrm{Aut}(\mathrm{SL}(2,p))$.

To count the effective number of corresponding triples in $\mathrm{PSL}(2,p)$ we will use the following obvious observation.

Lemma 1. *Let* (α, β, γ) *and* $E(\alpha, \beta, \gamma)$ *be as above. Then in* $\mathrm{PSL}(2,p)$

$$\overline{E}(\alpha, \beta, \gamma) = \overline{E}(-\alpha, -\beta, \gamma) = \overline{E}(-\alpha, \beta, -\gamma) = \overline{E}(\alpha, -\beta, -\gamma)$$

Proof. If we write (A, B, C) for a triple in $E(\alpha, \beta, \gamma)$, then clearly

$$(A,B,C) \in E(\alpha,\beta,\gamma) \iff (-A,-B,C) \in E(-\alpha,-\beta,\gamma) \iff$$
$$\iff (-A,B,-C) \in E(-\alpha,\beta,-\gamma) \iff (A,-B,-C) \in E(\alpha,-\beta,-\gamma)$$

and these four triples project in $\mathrm{PSL}(2,p)$ to the same element. □

4.1. Type $(2,3,n)$

We will look first for triangle curves – or equivalently, triples of generators – of type $(2,3,n)$.

Lemma 2. *Let* p *be a prime number* $p \geq 5$ *and* n *any natural number dividing either* $(p-1)/2$ *or* $(p+1)/2$*. Then*

(1) *There are* $\phi(n)/2$ *conjugacy classes of elements of order* n *in* $\mathrm{PSL}(2,p)$*;*
(2) *These are characterized by the trace of any of its elements;*
(3) *In fact for every* $\mathfrak{c} \in \mathrm{PSL}(2,p)$ *of order* n*, the elements* $\mathfrak{c}^i$ *with* $\gcd(i,n) = 1$ *and* $0 < i < n$*, provide representatives for all these conjugacy classes; the elements* $\mathfrak{c}^i$ *and* $\mathfrak{c}^{n-i}$ *lying in the same class.*

Proof. The group $\mathrm{PSL}(2,p)$ contains two cyclic subgroups of order $(p-1)/2$ and $(p+1)/2$, namely the projective images of

$$H_- = \left\{ M_\lambda \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \ : \ \lambda \in \mathbb{F}_p^* \right\} \cong \mathbb{F}_p^*$$

and

$$H_+ = \left\{ M_{(x,y)} \equiv \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} \ : \ x, y \in \mathbb{F}_p^*, \ x^2 - \varepsilon y^2 = 1 \right\}$$

where ε is a generator of the cyclic group $\mathbb{F}_p^*$ (see for instance [19], §5.2).

Now, every element of $\mathrm{PSL}(2,p)$ of order n dividing $(p-1)/2$ (resp. $(p+1)/2$) is conjugate to an element of H_- (resp. H_+), which contains $\phi(n)$ such elements of order n. All these matrices have different traces $\lambda_i + \lambda_i^{-1}$ (resp. $2x$) except for mutually inverse elements M_{λ_i} and $M_{\lambda_i^{-1}}$ (resp. $M_{(x,y)}$ and $M_{(x,-y)}$), which are therefore conjugate. It follows that there are $\phi(n)/2$ conjugacy classes of elements of order n in $\mathrm{PSL}(2,p)$.

Point *(3)* follows from the fact that H_- (resp. H_+) is cyclic. □

We are now interested in the number of classes of triples of generators of $G = \mathrm{PSL}(2,p)$ of type $(2,3,n)$ under the action $\mathrm{Aut}(G)$. Recall that elements of order 2 and 3 in $\mathrm{PSL}(2,p)$ have trace 0 and ±1 respectively (see for example [29]).

Lemma 3. *Let p be a prime number $p \geq 5$ and $n > 6$ any natural number dividing either $(p-1)/2$ or $(p+1)/2$.*

(1) *There are $\phi(n)$ classes of triples of generators of type $(2,3,n)$ modulo G.*
(2) *There are $\phi(n)/2$ classes of triples of generators of type $(2,3,n)$ modulo* $\mathrm{Aut}(G)$.
(3) *The conjugacy class of the element of order n characterizes the conjugacy class of the triple modulo* $\mathrm{Aut}(G)$.

Proof. We know that there are $\phi(n)/2$ conjugacy classes of elements of order n. For each class $\mathcal{C}$ let $t \in \mathbb{F}_p$ be the trace of any element $\mathfrak{c} \in \mathcal{C}$, which is defined up to multiplication by ±1. The possible traces of triples of type $(2,3,n)$ are therefore $(0, \pm1, \pm t)$. For all of them the discriminant $t^2 - 3$ is different from zero, since otherwise the order of $\mathfrak{c}$ would be less than or equal to 6. Indeed, by the Cayley–Hamilton theorem $\mathfrak{c}^2 - t\mathfrak{c} + \mathrm{Id} = 0$, and therefore we would have

$$0 = (\mathfrak{c}^2 - t\mathfrak{c} + \mathrm{Id})^2 - (2 + 2\mathfrak{c}^2)(\mathfrak{c}^2 - t\mathfrak{c} + \mathrm{Id}) = -\mathfrak{c}^4 + \mathfrak{c}^2 - \mathrm{Id}$$

which implies

$$0 = \mathfrak{c}^2(-\mathfrak{c}^4 + \mathfrak{c}^2 - \mathrm{Id}) + (-\mathfrak{c}^4 + \mathfrak{c}^2 - \mathrm{Id}) = -\mathfrak{c}^6 + \mathfrak{c}^4 - \mathfrak{c}^2 - \mathfrak{c}^4 + \mathfrak{c}^2 - \mathrm{Id} = -\mathfrak{c}^6 - \mathrm{Id}$$

so in $\mathrm{PSL}(2,p)$ we would have $\mathfrak{c}^6 = \mathrm{Id}$.

Now by Lemma 1 it is enough to study $E(0,1,t)$ and, since $(0,1,t)$ is neither singular nor exceptional, the result follows from Theorem 1. □

By the previous two lemmas, for any element $\mathfrak{c}$ of order n the $\phi(n)/2$ conjugacy classes of triples of type $(2,3,n)$ have representatives $(\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}^i)$, where $1 \leq i < n/2$ with $\gcd(i,n) = 1$. Let us denote by (E_i, f_i) the corresponding $G-$covers. The curves E_i are pairwise non-isomorphic. This can be seen as follows: suppose that we had $E_i \cong E_j$ and set $\Gamma = \Gamma(2,3,n)$. Then their uniformizing groups $K_i \lhd \Gamma$ and $K_j \lhd \Gamma$ would be conjugate by an element of $\mathrm{PSL}(2,\mathbb{R})$, say $K_j = \alpha K_i \alpha^{-1}$. Note that α does not belong to $\Gamma(2,3,n)$ since the triples defining the $G-$coverings (E_i, f_i) and (E_j, j_j) are not equivalent modulo G. Conjugating now the inclusion $K_j \lhd \Gamma$ by α^{-1} we get $\alpha^{-1} K_j \alpha = K_i \lhd \alpha^{-1} \Gamma \alpha$.

But then K_i is normal in both Γ and $\alpha^{-1}\Gamma\alpha$. Since $\Gamma(2,3,n)$ is a maximal Fuchsian group (see [34]) this is impossible unless $\alpha \in \Gamma(2,3,n)$, which is a contradiction.

We claim now that for any k with $\gcd(n,k) = 1$, the curves E_1 and E_k are Galois conjugate. The idea of the proof is contained in the case $n = 7$, proved by M. Streit in [35].

Let us consider the action on (E_1, f_1) of an element $\sigma_k \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma_k(\xi_n) = \xi_n^{k^{-1}}$. By Proposition 2 the $G-$covering (E_1^σ, f_1^σ) must correspond to a triple $(h_{\mathfrak{a}}\mathfrak{a}^{\alpha'}h_{\mathfrak{a}}^{-1}, h_{\mathfrak{b}}\mathfrak{b}^{\beta'}h_{\mathfrak{b}}^{-1}, \mathfrak{c}^{\gamma'})$, with $\gamma' \equiv k \pmod{n}$. By the previous lemma this triple is equivalent to $(\mathfrak{a}_k, \mathfrak{b}_k, \mathfrak{c}^k)$, and so $(E_1^\sigma, f_1^\sigma) = (E_k, f_k)$. There are therefore $\phi(n)$ options for k, yielding $\phi(n)/2$ different curves Galois conjugate to E_1. This is because for each such k the curves $E_1^{\sigma_k}$ and $E_1^{\sigma_{n-k}}$ are isomorphic since, $\mathfrak{c}^k$ and $\mathfrak{c}^{n-k}$ being conjugate, they correspond to equivalent triples.

Hence the $G-$coverings (E_i, f_i) form a complete orbit under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Finally note that if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the complex conjugation $\sigma(\mathrm{w}) = \overline{\mathrm{w}}$ then $\sigma(\xi_n) = \xi_n^{-1}$, and since $\mathfrak{c}_1$ and $\mathfrak{c}_1^{-1}$ are conjugate in $\mathrm{PSL}(2,p)$ then $E_1^\sigma \cong E_1$. From this fact one can conclude that $\mathbb{Q}(\xi_n) \cap \mathbb{R} = \mathbb{Q}(\cos \pi/n)$ is the field of moduli of these curves, and hence a field of definition ([35]).

We have proved the following theorem.

Theorem 2. *Let p be a prime number $p \geq 5$ and $n > 6$ any natural number dividing either $(p-1)/2$ or $(p+1)/2$. Then:*

(1) *The $\phi(n)/2$ covers (E_i, f_i), for $1 \leq i < n/2$ and $\gcd(i,n) = 1$, are the only $G-$coverings with covering group $G = \mathrm{PSL}(2,p)$ and type $(2,3,n)$;*
(2) *They correspond to the triples $(\mathfrak{a}_i, \mathfrak{b}_i, \mathfrak{c}^i)$;*
(3) *They form a complete orbit under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$;*
(4) *The curves E_i have genus $g = \frac{1}{24n}(n-6)p(p-1)(p+1)+1$ and they are pairwise non-isomorphic. They can all be defined over $\mathbb{Q}(\cos(\pi/n))$ and have automorphism group $\mathrm{Aut}(E_i) \cong G$.*

The expression for the genus is a consequence of the Riemann–Hurwitz formula and the claim about the automorphism group follows from the fact that $\Gamma(2,3,n)$ is a maximal Fuchsian group ([34]).

Example 2. For $p = 13$ and $n = 7$ the following triples define three Galois conjugate curves of type $(2,3,7)$:

$$
\begin{aligned}
(a_1, b_1, c) &= \left(\begin{pmatrix} 8 & 3 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 8 \\ 8 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 12 & 6 \end{pmatrix} \right) \\
(a_2, b_2, c^2) &= \left(\begin{pmatrix} 0 & 12 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 12 \\ 4 & 6 \end{pmatrix}, \begin{pmatrix} 12 & 6 \\ 7 & 9 \end{pmatrix} \right) \\
(a_3, b_3, c^3) &= \left(\begin{pmatrix} 12 & 1 \\ 11 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 10 \\ 9 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 9 \\ 4 & 9 \end{pmatrix} \right)
\end{aligned}
$$

Any other triple $(\mathfrak{a}', \mathfrak{b}', \mathfrak{c}')$ of type $(2,3,7)$ can be mapped by an automorphism of $\mathrm{PSL}(2,13)$ to one of these, depending on the conjugacy class of $\mathfrak{c}'$. These three curves are Hurwitz curves of genus 14, i.e. curves C whose automorphism group reaches the Hurwitz bound $|\mathrm{Aut}(C)| \leq 84(g-1)$. They are defined over the number field $\mathbb{Q}(\cos(\pi/7))$ and they are Galois conjugate under the action of any Galois element satisfying $\xi_7 \mapsto \xi_7^2$ and $\xi_7 \mapsto \xi_7^3$ respectively ([35]).

4.2. Type (p,p,p)

We now focus now on triples of type (p,p,p) in $G = \mathrm{PSL}(2,p)$ for $p > 5$.

Lemma 4. *Let $p > 5$ be a prime number. Then*

(1) *There is only one class of triples of generators of type (p,p,p) modulo $\mathrm{Aut}(G)$, which is represented by*

$$u = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}, \quad v = \begin{pmatrix} 3 & -4 \\ 4 & -5 \end{pmatrix}, \quad w = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(2) *There are two classes of triples of generators of type (p,p,p) modulo G, the second one being represented by a triple of the form (u', v', w^k) for suitable u', v', where $(\frac{k}{p}) = -1$, i.e. k is not a quadratic residue modulo p.*

Proof. It can be easily checked that u, v, w are elements of order p whose product is the identity. Moreover, recall that, as mentioned in the introduction to this section, all triples of type (p,p,p) have traces of the form $(\pm 2, \pm 2, \pm 2)$. By Lemma 1 we can consider just the cases $(2,2,2)$ and $(2,-2,2)$, but only the latter is neither singular nor exceptional, and therefore it follows from Theorem 1 that (u,v,w) is the only triple of generators of type (p,p,p) modulo $\mathrm{Aut}(G)$.

It also follows from the same theorem that there are two such triples of generators modulo G and, since for any k which is not a quadratic residue modulo p the element w^k is not conjugate to w (see for example [19], §5.2), we can suppose that these two classes of triples of generators are represented by (u,v,w) and (u', v', w^k). $\square$

Now take the $G-$covering (E,f) corresponding to the triple of generators (u,v,w) above. Lemma 4 implies that $(E,f) \cong (E^\sigma, f^\sigma)$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This means that the field of moduli of E is $\mathbb{Q}$, and since E is a triangle curve, $\mathbb{Q}$ is a field of definition as well.

Theorem 3. *For each prime number $p > 5$ there is a unique $G-$covering (E,f) of type (p,p,p) with $G = \mathrm{PSL}(2,p)$. Moreover the following properties hold*

(1) *The $G-$covering (E,f) can be defined over $\mathbb{Q}$;*
(2) *E has genus $g = \frac{1}{4}(p+1)(p-1)(p-3)+1$;*
(3) *The automorphism group $\mathrm{Aut}(E)$ is isomorphic to $\mathrm{PSL}(2,p) \times \mathfrak{S}_3$.*

Proof. The formula for the genus is a consequence of the Riemann–Hurwitz formula. After the comment preceding the statement of the theorem the only part left to prove is the one regarding the automorphism group. Let K be the

Fuchsian group uniformizing the curve E, i.e. the kernel of the epimorphism $\rho : \Gamma(p,p,p) \longrightarrow \mathrm{PSL}(2,p)$ defined by

$$\begin{array}{rccc} \rho : & \Gamma(p,p,p) & \longrightarrow & \mathrm{PSL}(2,p) \\ & x & \longmapsto & u \\ & y & \longmapsto & v \\ & z & \longmapsto & w \end{array}$$

where x, y, z are the generators of $\Gamma(p,p,p)$ chosen in formula (2) in section 2, and u, v, w are as in Lemma 4. We recall that the automorphism group of E is given by $\mathrm{Aut}(E) \cong N(K)/K$.

It is well known that the group $\Gamma(p,p,p)$ injects into the maximal triangle group $\Gamma(2,3,2p)$ as a normal subgroup of index 6 ([34]). This injection can be realized geometrically as the inclusion map of $\Gamma(p,p,p)$ in the triangle group $\widetilde{\Gamma}(2,3,2p)$ associated to one of the six triangles $\widetilde{T} = \widetilde{T}(2,3,2p)$ of angles $\pi/2$, $\pi/3$, $\pi/2p$ in which $T(p,p,p)$ is naturally subdivided (see Figure 2). Note that $\widetilde{T} = \alpha(T)$, and hence $\widetilde{\Gamma}(2,3,2p) = \alpha\Gamma(2,3,2p)\alpha^{-1}$, for some $\alpha \in \mathrm{PSL}(2,\mathbb{R})$.

Now we consider the group homomorphism defined by

$$\begin{array}{rccl} \widetilde{\rho} : & \widetilde{\Gamma}(2,3,2p) & \longrightarrow & \mathrm{PSL}(2,p) \times \mathfrak{S}_3 \\ & \tilde{x} & \longmapsto & x' = (X,\mu) \\ & \tilde{y} & \longmapsto & y' = (Y,\nu) \\ & \tilde{z} & \longmapsto & z' = (Z,\mu\nu) \end{array}$$

where:

- $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ are the generators of $\widetilde{\Gamma}(2,3,2p)$ of orders 2, 3 and $2p$ depicted in Figure 2,
- $X = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$, $Y = \begin{pmatrix} -1 & \frac{p+3}{2} \\ -2 & 2 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & \frac{p+1}{2} \\ 0 & 1 \end{pmatrix}$, and
- μ, ν are generators of $\mathfrak{S}_3$ such that $\mu^2 = \nu^3 = (\mu\nu)^2 = 1$.

Notice that the generators x, y, z of $\Gamma(p,p,p)$ are related to the generators $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ of $\widetilde{\Gamma}(2,3,2p)$ by

$$\begin{array}{rcl} x & = & \tilde{y}\tilde{z}^2\tilde{y}^{-1} = \tilde{x}\tilde{z}^2\tilde{x}^{-1} \\ y & = & \tilde{y}^{-1}\tilde{z}^2\tilde{y} \\ z & = & \tilde{z}^2 \end{array}$$

This can be seen by checking that the fixed points of $\tilde{z}^2$, $\tilde{y}\tilde{z}\tilde{y}^{-1}$ and $\tilde{y}^{-1}\tilde{z}^2\tilde{y}$ are v_∞, $\tilde{y}(v_\infty) = v_0$ and $\tilde{y}^{-1}(v_\infty) = v_1$ respectively (see Figure 2).

Now we point out the following facts:

- The rule $\widetilde{\rho}$ certainly defines a homomorphism, since $\mathrm{ord}(x') = 2$, $\mathrm{ord}(y') = 3$, $\mathrm{ord}(z') = 2p$ and $x'y'z' = \mathrm{Id}$.

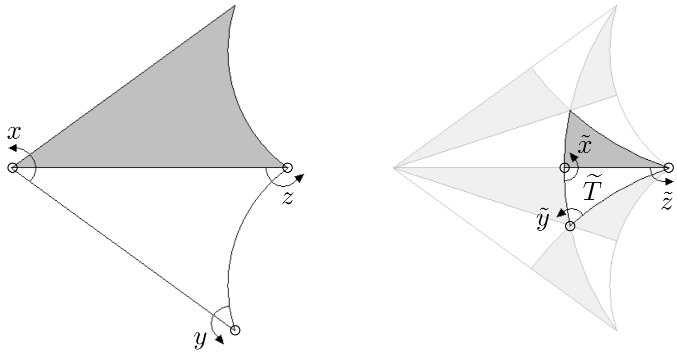

Figure 2: Fundamental domains and generators for the groups $\Gamma(p,p,p)$ and $\widetilde{\Gamma}(2,3,2p)$ $(p=5)$.

- The restriction of $\widetilde{\rho}$ to $\Gamma = \Gamma(p,p,p)$ coincides with ρ. This is because of the following identities:

$$\begin{aligned}\widetilde{\rho}(x) &= y'z'^2y'^{-1} = x'z'^2x'^{-1} = (u,\mathrm{Id})\\ \widetilde{\rho}(y) &= y'^{-1}z'^2y' = (v,\mathrm{Id})\\ \widetilde{\rho}(z) &= z'^2 = (w,\mathrm{Id})\end{aligned}$$

  As a consequence $\widetilde{\rho}$ is an epimorphism. In fact it is easy to see that the subgroup $\widetilde{\rho}(\Gamma(p,p,p)) = G$ together with the elements $\widetilde{\rho}(\tilde{x}) = x'$ and $\widetilde{\rho}(\tilde{y}) = y'$ already generate a group in which G has index at least 6.

- In particular $K < \ker(\widetilde{\rho})$ and since

$$[\widetilde{\Gamma}(2,3,2p) : \Gamma(p,p,p)] = [\mathrm{PSL}(2,p)\times\mathfrak{S}_3 : \mathrm{PSL}(2,p)]$$

  it follows that $K = \ker(\widetilde{\rho})$. Moreover, since $\widetilde{\Gamma}(2,3,2p)$ is a maximal triangle group it also follows that $\widetilde{\Gamma}(2,3,2p)$ equals $N(K)$, the normalizer of K in $\mathrm{PSL}(2,\mathbb{R})$.

We conclude that $\mathrm{Aut}(E) \cong N(K)/K \cong \mathrm{PSL}(2,p)\times\mathfrak{S}_3$. $\square$

The general study of the extendability of the automorphism group of triangle curves has been considered by Bujalance, Cirre and Conder (see [10], Thm. 5.2).

Example 3. In the particular case $p=7$ the two conjugacy classes of triples of type $(7,7,7)$ in $G = \mathrm{PSL}(2,p)$ are represented by

$$u = \begin{pmatrix} 6 & 1\\ 3 & 3\end{pmatrix},\quad v = \begin{pmatrix} 3 & 3\\ 4 & 2\end{pmatrix},\quad w = \begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}$$

and

$$u^{-1} = \begin{pmatrix} 3 & 6\\ 4 & 6\end{pmatrix},\quad v' = \begin{pmatrix} 6 & 0\\ 3 & 6\end{pmatrix},\quad w^{-1} = \begin{pmatrix} 1 & 6\\ 0 & 1\end{pmatrix}$$

which are conjugate under the element $\alpha = \left(\begin{smallmatrix} 6 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{PGL}(2,7) \cong \mathrm{Aut}(G)$.

We will write (D, f) for the corresponding $G-$covering.

4.3. Type $(3,3,4)$ *in* $\mathrm{PSL}(2,7)$

We will focus our attention now on triples of type $(3,3,4)$ in $G = \mathrm{PSL}(2,7)$. It can be found by computational means (e.g. with MAGMA) that up to conjugation in $\mathrm{PSL}(2,7)$ there are four such triples, namely

$$\begin{aligned}
(a_1, b_1, c) &= \left(\begin{pmatrix} 1 & 5 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} \right) \\
(a_2, b_2, c) &= \left(\begin{pmatrix} 0 & 6 \\ 1 & 6 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} \right) \\
(a_1', b_1', c) &= \left(\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} \right) \\
(a_2', b_2', c) &= \left(\begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 6 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} \right)
\end{aligned}$$

Moreover, in Theorem 4 we will use the fact that in $\mathrm{PGL}(2,7)$ there are two non-equivalent triples of type $(2,3,8)$, namely

$$\begin{aligned}
(r_1, s_1, t_1) &= \left(\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 5 & 5 \\ 2 & 6 \end{pmatrix} \right) \\
(r_2, s_2, t_2 = t_1^5) &= \left(\begin{pmatrix} 2 & 0 \\ 4 & 5 \end{pmatrix}, \begin{pmatrix} 6 & 4 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 6 & 1 \\ 6 & 2 \end{pmatrix} \right)
\end{aligned}$$

Parts (3), (4) and (5) of the following theorem are contained in a forthcoming paper by M. Conder, G. Jones, M. Streit and J. Wolfart ([14]) and the two remaining ones could be easily deduced from them. Since they consider a wide range of groups and types, their methods are much more sophisticated than ours, so we provide here an *ad hoc* proof for the case we are interested in.

Theorem 4. *The following statements hold:*

(1) *The $G-$coverings (D_1, f_1) and (D_2, f_2), defined by the triples (a_1, b_1, c) and (a_2, b_2, c) respectively, are the only two $G-$coverings of type $(3,3,4)$ and covering group $\mathrm{PSL}(2,7)$, up to isomorphism.*
(2) *The $G-$coverings (D_1', h_1) and (D_2', h_2), defined by the triples (r_1, s_1, t_1) and (r_2, s_2, t_2) respectively, are the only two $G-$coverings of type $(2,3,8)$ and covering group $\mathrm{PGL}(2,7)$, up to isomorphism. Moreover, D_1' and D_2' are non-isomorphic curves.*
(3) *$D_1 \cong D_1'$ and $D_2 \cong D_2'$. In particular D_1 and D_2 are not isomorphic. Both curves have genus 49.*
(4) *Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ satisfy $\sigma(\xi_8) = \xi_8^5$. Then $D_1^\sigma = D_2$.*
(5) *D_1 and D_2 are defined over $\mathbb{Q}(\sqrt{2})$. In particular $\overline{D}_1 \cong D_1$ and $\overline{D}_2 \cong D_2$.*

Proof. *(1)* The triples (a_1, b_1, c) and (a_1', b_1', c) (resp. (a_2, b_2, c) and (a_2', b_2', c)) are conjugate by the element $\left(\begin{smallmatrix}4 & 5\\ 2 & 5\end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix}1 & 6\\ 1 & 5\end{smallmatrix}\right)$), so they are equivalent under the action of $\mathrm{PGL}(2,7) \cong \mathrm{Aut}(\mathrm{PSL}(2,7))$. However (a_1, b_1, c) and (a_2, b_2, c) are not conjugate in $\mathrm{PGL}(2,7)$.

(2) The $G-$coverings D_1' and D_2' correspond to the inclusion of certain torsion-free normal subgroups $K_1, K_2 < \Gamma(2,3,8)$. We claim that not even the curves D_1' and D_2' are isomorphic. If they were there would exist an $\alpha \in \mathrm{PSL}(2,\mathbb{R})$ such that $K_2 = \alpha K_1 \alpha^{-1}$. But then K_2 would be normal both in $\Gamma(2,3,8)$ and $\alpha\Gamma(2,3,8)\alpha^{-1}$, and since $\Gamma(2,3,8)$ is a maximal Fuchsian group ([34]) this can only occur if $\alpha \in \Gamma(2,3,8)$. But K_1 and K_2 are not conjugate in $\Gamma(2,3,8)$ because their corresponding defining triples are not equivalent.

(3) In a way similar to the case of $\Gamma(p,p,p) < \widetilde{\Gamma}(2,3,2p)$ in the proof of Theorem 3, the group $\Gamma(3,3,4)$ is included in the triangle group $\widetilde{\Gamma}(2,3,8)$ associated to the triangle $\widetilde{T} = \widetilde{T}(2,3,8)$ in Figure 3. Again $\widetilde{T} = \alpha(T)$ for some $\alpha \in \mathrm{PSL}(2,\mathbb{R})$, and hence $\widetilde{\Gamma}(2,3,8) = \alpha\Gamma(2,3,8)\alpha^{-1}$.

Now consider the following diagram

$$\begin{array}{ccccc}
\widetilde{K} & \hookrightarrow & \widetilde{\Gamma}(2,3,8) & \xrightarrow{\ \widetilde{\rho}\ } & \mathrm{PGL}(2,7) \\
 & & \uparrow & & \uparrow \\
K & \hookrightarrow & \Gamma(3,3,4) & \xrightarrow{\ \rho\ } & \mathrm{PSL}(2,7)
\end{array}$$

where the vertical arrows are the natural inclusions, $\widetilde{K} = \ker \widetilde{\rho}$, $K = \ker \rho$ and the two epimorphisms ρ and $\widetilde{\rho}$ are given by

$$\begin{array}{cccc}
\rho: & \Gamma(3,3,4) & \longrightarrow & \mathrm{PSL}(2,7) \\
 & x & \longmapsto & a_1 \\
 & y & \longmapsto & b_1 \\
 & z & \longmapsto & c
\end{array}
\qquad\qquad
\begin{array}{cccc}
\widetilde{\rho}: & \widetilde{\Gamma}(2,3,8) & \longrightarrow & \mathrm{PGL}(2,7) \\
 & \tilde{x} & \longmapsto & r_1 \\
 & \tilde{y} & \longmapsto & s_1 \\
 & \tilde{z} & \longmapsto & t_1
\end{array}$$

where x, y, z and $\tilde{x}, \tilde{y}, \tilde{z}$ are the generators of $\Gamma(3,3,4)$ and $\widetilde{\Gamma}(2,3,8)$ respectively provided by the rotations depicted in Figure 3 below.

The following obvious identities show that this is a commutative diagram

$$\begin{array}{llll}
y = \tilde{y} & , \quad z = \tilde{z}^2 & \text{(see Figure 3)} & \text{and} \\
b_1 = s_1 & , \quad c = t_1^2 & \text{in } \mathrm{PGL}(2,7) &
\end{array}$$

Therefore it is clear that $K = \widetilde{K} \cap \Gamma(3,3,4)$. Now since $[\widetilde{\Gamma}(2,3,8) : \Gamma(3,3,4)]$ equals $[\mathrm{PGL}(2,7) : \mathrm{PSL}(2,7)]$ it follows that $\widetilde{K} = K$ and $D_1 \cong D_1'$. It can be argued in the same way to deduce that $D_2 \cong D_2'$. Since we have already proved that $D_1' \not\cong D_2'$, this implies $D_1 \not\cong D_2$.

The statement about the genus follows from the Riemann–Hurwitz formula.

(4) We note now that the conjugacy classes of (r_1, s_1, t_1) and (r_2, s_2, t_2) in $\mathrm{PGL}(2,7)$ are determined by the conjugacy classes in $\mathrm{PGL}(2,7)$ of their elements

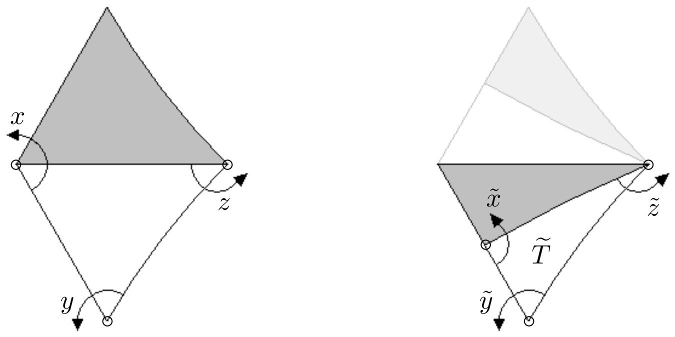

Figure 3: Fundamental domains and generators for the groups $\Gamma(3,3,4)$ and $\widetilde{\Gamma}(2,3,8)$.

of order 8 (t_1 and $t_2 = t_1^5$ respectively). Therefore applying Proposition 2 with an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma(\xi_8) = \xi_8^5$ we can conclude that $D_1'^{\sigma} \cong D_2'$, and therefore $D_1^{\sigma} \cong D_2$.

(5) Triangle curves are known to be defined over their field of moduli. By the comments in the proof of the previous point, any Galois element fixing the field $\mathbb{Q}(\xi_8)$ belongs to the inertia groups G_{D_1} and G_{D_2}. Moreover, by Proposition 3 the curve $\overline{D'}_1 = \overline{D}_1$ (resp. $\overline{D'}_2 = \overline{D}_2$) is defined by the triple $(r_1^{-1}, r_1 s_1^{-1} r_1^{-1}, t_1^{-1})$ (resp. $(r_2^{-1}, r_2 s_2^{-1} r_2^{-1}, t_2^{-1})$). Since t_1 and t_1^{-1} (resp. t_2 and t_2^{-1}) lie in the same conjugacy class, we deduce that $\overline{D}_1 \cong D_1$ (resp. $\overline{D}_2 \cong D_2$), and so complex conjugation belongs to both inertia groups too.

As a consequence the field of moduli of both D_1 and D_2 is contained in $\mathbb{Q}(\xi_8) \cap \mathbb{R} = \mathbb{Q}(\sqrt{2})$. Since by points (3) and (4) this field must be a non-trivial extension of $\mathbb{Q}$, we deduce that $\mathbb{Q}(\sqrt{2})$ is the field of moduli, hence the minimum field of definition of both D_1 and D_2. $\square$

Remark 2. Point (5) explains why, although the curves D_1 and D_2 are determined by $(3,3,4)$ triples, in order to distinguish them one needs to work with triples of type $(2,3,8)$. Since the action of a Galois element σ on ξ_3 and ξ_4 does not determine $\sigma(\sqrt{2})$, the effect of Galois conjugation could not be seen in the $(3,3,4)$ triples.

5. Catanese's theory of Beauville surfaces via uniformization

In this section we collect some results about Beauville surfaces with the aim of applying the knowledge we have acquired about the action of the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the curves of the previous section to the understanding of the Galois action on certain Beauville surfaces isogenous to products of pairs of these curves. We must formulate them in the language used in the previous sections, that is in terms of Fuchsian groups. Once this is done, we will see that a couple of elementary observations allow us to obtain a metric rigidity theorem which implies some striking properties of Beauville surfaces, originally proved by Catanese, which will be essential in the last section. This will make the paper self-contained and the theory of Beauville surfaces accessible to a wider readership.

Let $S = C_1 \times C_2/G$ be a Beauville surface. Clearly its holomorphic universal cover is the bidisc $\mathbb{H} \times \mathbb{H}$ and the covering group is a subgroup of $\mathrm{Aut}(\mathbb{H} \times \mathbb{H})$. Let us denote it by Γ_{12}, so that $S = \mathbb{H} \times \mathbb{H}/\Gamma_{12}$ with $\Gamma_{12} \cong \pi_1(S)$. It is easy to see that the two conditions in the definition of Beauville surface introduced in section 1 are equivalent to the following three properties of Γ_{12}

(1) $\Gamma_{12} < \mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{H})$, the index 2 subgroup of $\mathrm{Aut}(\mathbb{H} \times \mathbb{H})$ consisting of factor preserving elements ([31]).

(2) There are exact sequences

$$\begin{array}{llllllllll} \text{i)} & 1 & \longrightarrow & K_1 \times K_2 & \longrightarrow & \Gamma_{12} & \xrightarrow{\rho} & G & \longrightarrow & 1 \\ \text{ii)} & 1 & \longrightarrow & K_i & \longrightarrow & \Gamma_i & \xrightarrow{\rho_i} & G & \longrightarrow & 1 \qquad (i = 1, 2) \end{array}$$

where each of the groups Γ_i is a triangle group that defines a $G-$covering $f_i : C_i \cong \mathbb{H}/K_i \longrightarrow \mathbb{P}^1 \cong \mathbb{H}/\Gamma_i$ and Γ_{12} is defined by

$$\Gamma_{12} = \{(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2 \ : \ \rho_1(\gamma_1) = \rho_2(\gamma_2)\} < \Gamma_1 \times \Gamma_2$$

so that $\rho(\gamma_1, \gamma_2) = \rho_1(\gamma_1) = \rho_2(\gamma_2)$. The action of an element $g \in G$ on points $[\mathrm{w}_1, \mathrm{w}_2] \in \mathbb{H} \times \mathbb{H}/K_1 \times K_2$ is given by $g([\mathrm{w}_1, \mathrm{w}_2]) = [\gamma_1(\mathrm{w}_1), \gamma_2(\mathrm{w}_2)]$ where $g = \rho_1(\gamma_1) = \rho_2(\gamma_2)$.

(3) Let (a_i, b_i, c_i) be the generating triple defining the $G-$cover (C_i, f_i); then the subsets of G

$$\Sigma(a_i, b_i, c_i) := \bigcup_{g \in G} \bigcup_{j=1}^{\infty} \{g a_i^j g^{-1}, g b_i^j g^{-1}, g c_i^j g^{-1}\}, \quad (i = 1, 2)$$

consisting of the elements of G that fix points on C_1 and C_2 respectively, have trivial intersection, that is

$$\Sigma(a_1, b_1, c_1) \cap \Sigma(a_2, b_2, c_2) = \{1_G\} \tag{7}$$

This ensures that G acts freely on $C_1 \times C_2$.

Since clearly any pair of triples satisfying condition (7) automatically defines a group Γ_{12} uniformizing a Beauville surface, one has the following criterion for a finite group G to arise in the construction of Beauville surfaces.

Criterion. *([11]) The group G admits an unmixed Beauville structure if and only if it has two hyperbolic triples of generators (a_i, b_i, c_i) of order (l_i, m_i, n_i), $i = 1, 2$, satisfying the compatibility condition* (7)*.*

This is a useful tool, since it permits to check through a computer program whether or not a group (of not very large order) admits Beauville structure. For instance the following result can be checked by these means

Proposition 4. *Let $S = (C_1 \times C_2)/G$ be a Beauville surface such that the pair of genera (g_1, g_2) of the curves C_1 and C_2 is at most $(8, 49)$ (in the lexicographic order). If G is non-abelian then $G \cong \mathrm{PSL}(2, 7)$.*

Proof. It is known that the minimum possible genus of a curve occurring in the construction of a Beauville surface is 6 ([17]). It is also known that the symmetric group on 5 elements $\mathfrak{S}_5$ is the only non-abelian group up to order 128 admitting a Beauville structure ([4]). The corresponding pair of genera is $(19, 21)$ (see [17]). A list of all groups G acting on a curve C of small genus so that C/G is an orbifold of genus zero with three branching values is given in [13]. There are only six such groups of orders $|G| \geq 128$ acting on Riemann surfaces of genus 6 to 8. A computation carried out with MAGMA for these six groups shows that the only one admitting a Beauville structure is $G = \mathrm{PSL}(2, 7)$ (with pair of genera $(8, 49)$). □

Example 4. Consider the following triples of generators of type $(5, 5, 5)$ of the (additive) group $G = \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$:

$$a_1 = (1, 0), \quad b_1 = (0, 1), \quad c_1 = (4, 4),$$
$$a_2 = (3, 1), \quad b_2 = (4, 2), \quad c_2 = (3, 2).$$

By Example 1 the $G-$covering associated to both triples is the Fermat cover (F_5, f) described there. One can easily check that these triples satisfy the compatibility condition (7), hence they define a Beauville surface $X = (F_5 \times F_5)/G$. Since clearly the epimorphisms $\rho_1, \rho_2 : \Gamma(5, 5, 5) \longrightarrow G$ corresponding to the triples (a_1, b_1, c_1) and (a_2, b_2, c_2) are related by $\rho_2 = A \circ \rho_1$, where $A = \left(\begin{smallmatrix} 3 & 4 \\ 1 & 2 \end{smallmatrix}\right)$, we see that $(\alpha, \beta) = \rho_1(x^\alpha y^\beta) = \rho_2(x^{\alpha+3\beta} y^{2\alpha+4\beta})$. Therefore, according to point (2) above, the action of the element $(\alpha, \beta) \in G$ on the product $F_5 \times F_5$ is given by

$$\zeta_{(\alpha,\beta)}([u_1, u_2, u_3], [v_1, v_2, v_3]) = ([\xi_5^\alpha u_1, \xi_5^\beta u_2, u_3], [\xi^{\alpha+3\beta} v_1, \xi^{2\alpha+4\beta} v_2, v_3]).$$

This surface is, in fact, Beauville's original example in [8].

We point out here that the group G and the curves C_1, C_2 intervening in the definition of a Beauville surface $S = C_1 \times C_2/G$ are invariants of the isomorphism class of S ([11]); in fact a stronger result holds (see Remark 4).

Remark 3. Let $q = (a_1, b_1, c_1\,;\, a_2, b_2, c_2)$ and $q' = (a_1', b_1', c_1'\,;\, a_2', b_2', c_2')$ be two pairs of triples defining Beauville surfaces $S = \mathbb{H} \times \mathbb{H}/\Gamma_{12}$ and $S' = \mathbb{H} \times \mathbb{H}/\Gamma_{12}'$.

(1) If q and q' differ by $\psi \in \mathrm{Aut}(G)$, then $\Gamma_{12}' = \Gamma_{12}$ and therefore $S' = S$.
(2) If the triple (a_i', b_i', c_i') is conjugate to (a_i, b_i, c_i) by an element $g_i = \rho_i(\gamma_i)$, $i = 1, 2$ then $\Gamma_{12}' = (\gamma_1, \gamma_2)\, \Gamma_{12}\, (\gamma_1, \gamma_2)^{-1}$ and therefore $S' \cong S$.
(3) Some other simple modifications of each of the two triples give also rise to isomorphic Beauville surfaces. For instance, if $\mathrm{ord}(a_1) = \mathrm{ord}(b_1)$, the triples

$$(a_1, b_1, c_1\,;\, a_2, b_2, c_2) \quad \text{and} \quad (a_1 b_1 a_1^{-1}, a_1, c_1\,;\, a_2, b_2, c_2)$$

define isomorphic Beauville surfaces. To see this, consider $\delta = x_1\widetilde{x}_1$, where $\widetilde{x}_1$ is the rotation through angle $\pi/2$ around the midpoint of the edge $[v_0, v_1]$. Then a glance at Figure 3 shows the following relations

$$\begin{aligned} \delta x_1 \delta^{-1} &= x_1 y_1 x_1^{-1} \\ \delta y_1 \delta^{-1} &= x_1 \\ \delta z_1 \delta^{-1} &= z_1 \end{aligned}$$

which imply that $\rho_1'(\gamma) = \rho_1(\delta\gamma\delta^{-1})$, and therefore $\Gamma_{12}' = (\delta, 1)^{-1}\,\Gamma_{12}\,(\delta, 1)$. Note that now δ lies in $N(\Gamma_1)$, the normalizer of Γ_1 in $\mathrm{PSL}(2, \mathbb{R})$, but not in Γ_1 itself.

A complete characterisation of isomorphism classes of Beauville surfaces in terms of pairs of triples is given in [4].

5.1. Metric rigidity of Beauville surfaces

We recall that the group of factor-preserving isometries of $\mathbb{H}\times\mathbb{H}$ agrees with $\mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{H})$, which contains the uniformizing group Γ_{12}. Therefore any Beauville surface carries a canonical metric induced by the product metric on $\mathbb{H} \times \mathbb{H}$.

The rigidity of triangle groups implies the following rigidity theorem for Beauville surfaces

Theorem 5. *Two Beauville surfaces S and S' are isometric if and only if $\pi_1(S) \cong \pi_1(S')$.*

Proof. Let us identify the fundamental groups of S and S' with their corresponding uniformizing groups $\Gamma_{12}, \Gamma_{12}' < \mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{H})$ and let $\Phi : \Gamma_{12} \longrightarrow \Gamma_{12}'$ be a group isomorphism. First we claim that, up to renumbering, $\Phi(K_1) = K_1'$ and $\Phi(K_2) = K_2'$ so that, in particular, $\Phi(K_1 \times K_2) = K_1' \times K_2'$. Clearly the commutator $\mathrm{Comm}_{\Gamma_{12}}((\gamma_1, \gamma_2))$ of an element $(\gamma_1, \gamma_2) \in \Gamma_{12}$ agrees with $(\mathrm{Comm}_{\Gamma_1}(\gamma_1) \times \mathrm{Comm}_{\Gamma_2}(\gamma_2)) \cap \Gamma_{12}$, and it is known that $\mathrm{Comm}_{\Gamma_i}(\gamma_i)$ is abelian if $\gamma_i \neq 1$ (see e.g. [22], Remark 2.3). Therefore the group $\mathrm{Comm}_{\Gamma_{12}}((\gamma_1, \gamma_2))$ is abelian when $\gamma_i \neq 1$ for $i = 1, 2$, and non-abelian otherwise, for if, say, $\gamma_1 = 1$, then obviously $\mathrm{Comm}_{\Gamma_{12}}((\gamma_1, \gamma_2))$ contains the subgroup K_1, which is already non-abelian. This implies that any element in the group $\Phi(K_1)$ is either of the form $(k_1', 1) \in K_1'$ or of the form $(1, k_2') \in K_2'$, and the result follows.

Moreover, since clearly $\Gamma_1 \cong \Gamma_{12}/K_2$ and $\Gamma_2 \cong \Gamma_{12}/K_1$, it further follows that Φ induces isomorphisms $\Phi_i : \Gamma_i \longrightarrow \Gamma_i'$ defined by

$$\Phi_1(\gamma_1) = p_1 \circ \Phi(\gamma_1, \gamma_2)$$

where p_1 stands for the first projection and γ_2 is any element of Γ_2 so that $(\gamma_1, \gamma_2) \in \Gamma_{12}$. In other words the isomorphism $\Phi : \Gamma_{12} \longrightarrow \Gamma_{12}'$ extends to an isomorphism $\Phi_1 \times \Phi_2 : \Gamma_1 \times \Gamma_2 \longrightarrow \Gamma_1' \times \Gamma_2'$.

Now it is a well-known and elementary fact that any group isomorphism between triangle groups is induced by an isometry of $\mathbb{H}$ (the trivial case of Teichmüller theory, see [34]) and therefore the product of the isometries δ_1, δ_2 corresponding to Φ_1, Φ_2 induces the required isometry $\delta_1 \times \delta_2 : S \longrightarrow S'$. $\square$

As a corollary we obtain

Theorem 6 (Catanese [11], [5]). *Let $S = C_1 \times C_2/G$ be a Beauville surface. Then*

(1) *If $S' = C_1' \times C_2'/G'$ is another Beauville surface such that $\pi_1(S) = \pi_1(S')$ then, up to renumbering, $C_i' \cong C_i$ or $\overline{C_i}$ for $i = 1, 2$.*
(2) *There are at most four non-isomorphic Beauville surfaces with fundamental group isomorphic to $\pi_1(S)$.*

Proof. (1) The isomorphisms between K_i and K_i' in the previous proof are induced by isometries δ_i. Thus, depending on whether these are orientation-preserving or orientation-reversing, we have $C_i' \cong C_i$ or $C_i' \cong \overline{C_i}$.

(2) Let $\delta_1 \times \delta_2 : S \longrightarrow S'$ be an isometry between S and any other Beauville surface S' with same fundamental group. If both isometries δ_i are simultaneously orientation-preserving then $\delta_1 \times \delta_2 : S \longrightarrow S'$ is a holomorphic isomorphism. This clearly leaves at most four possibilities for the isomorphism class of S'. $\square$

Remark 4. We observe that the group G is an invariant of the homotopy class of S, and so are the curves C_i, up to complex conjugacy, and their types.

In particular any holomorphic isomorphism between Beauville surfaces S and S' induces an isomorphism between the corresponding curves C_i and C_i'. Thus the group G, the curves C_i and the types of the orbifolds C_i/G are invariants of the isomorphism class of S.

6. Non-homeomorphic conjugate Beauville structures on $\mathrm{PSL}(2, p)$

It was proved by Bauer, Catanese and Grunewald in [4] that $\mathrm{PSL}(2, p)$ admits Beauville structure for every prime $p > 5$, a result later generalized to $\mathrm{PSL}(2, q)$ for prime powers $q > 5$ by Fuertes and Jones [18] and Garion [21] (see also [20]). In this section we will construct Beauville surfaces with group $\mathrm{PSL}(2, p)$ whose Galois orbits contain surfaces with non-isomorphic fundamental group.

First we consider Beauville surfaces with group $\mathrm{PSL}(2, 7)$ and pair of genera $(8, 49)$, which turns out to be the minimum for which this phenomenon occurs. We find that there are only two of them, that they form a complete orbit under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and that they are not homeomorphic to each other.

Then for $p > 7$ we construct Beauville surfaces with group $\mathrm{PSL}(2, p)$, whose Galois orbits contain an arbitrarily large number of pairwise non-homeomorphic Beauville surfaces.

Let us stress here the fact that the Beauville surface defined on Example 4 can be defined over $\mathbb{Q}$, and therefore the absolute Galois group acts trivially on it.

6.1. The case PSL$(2,7)$

We will deal with Beauville structures of type $((3,3,4),(7,7,7))$ in the group PSL$(2,7)$. Let (a_1,b_1,c), (a'_1,b'_1,c), (a_2,b_2,c) and (a'_2,b'_2,c) be the $(3,3,4)$ triples of generators of PSL$(2,7)$ in section 4.3 and (u,v,w) and (u^{-1},v',w^{-1}) be the $(7,7,7)$ triples introduced in Example 3. Thanks to the Criterion in section 5 we can introduce the following Beauville surfaces

- S_1 defined by the pairs of triples (a_1,b_1,c) and (u,v,w);
- S_2 defined by the pairs of triples (a_2,b_2,c) and (u,v,w);

With the notation of section 4 these surfaces can be written as

$$S_1 = \frac{D_1 \times D}{G_1}, \qquad S_2 = \frac{D_2 \times D}{G_2}$$

where $G_1 \cong \mathrm{PSL}(2,7)$ (resp. $G_2 \cong \mathrm{PSL}(2,7)$) is a subgroup of $\mathrm{Aut}(D_1 \times D)$ (resp. a subgroup of $\mathrm{Aut}(D_2 \times D)$).

Note that the compatibility condition (7) in the Criterion is automatically satisfied, since the orders involved in each of the two triples are coprime. We have the following

Theorem 7. *For the surfaces S_1 and S_2 constructed above the following statements hold*

(1) *They are the only Beauville surfaces with group $G = \mathrm{PSL}(2,7)$ and curves of genera 8 and 49;*
(2) *They constitute a complete orbit for the action of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*;*
(3) *They have non-isomorphic fundamental groups, hence they are not homeomorphic to each other;*
(4) *Their pair of genera $(8,49)$ is the minimum (in the lexicographic order) for which non-homeomorphic Galois conjugate Beauville surfaces exist.*

Proof. (1) It can be seen that any pair of triples of PSL$(2,7)$ producing a Beauville surface with curves of genera 8 and 49 have to have type $(3,3,4)$ and $(7,7,7)$ respectively (see e.g. [17], Theorem 13). By Remark 3, when defining Beauville surfaces we can consider triples of generators up to conjugacy in G. Therefore the surfaces defined by the following pairs of triples account for all the Beauville surfaces of this type:

I.	$(a_1,b_1,c\,;\,u,v,w)$	V.	$(a_2,b_2,c\,;\,u,v,w)$
II.	$(a_1,b_1,c\,;\,u^{-1},v',w^{-1})$	VI.	$(a_2,b_2,c\,;\,u^{-1},v',w^{-1})$
III.	$(a'_1,b'_1,c\,;\,u^{-1},v',w^{-1})$	VII.	$(a'_2,b'_2,c\,;\,u^{-1},v',w^{-1})$
IV.	$(a'_1,b'_1,c\,;\,u,v,w)$	VIII.	$(a'_2,b'_2,c\,;\,u,v,w)$

Note that S_1 and S_2 are defined by the pairs of triples I and V respectively.

Now I and III define the same Beauville surface. In fact by the results in section 4 the triples (a_1,b_1,c) and (a'_1,b'_1,c) are related by an element $\phi_1 \in \mathrm{Aut}(G) \setminus G$ and similarly there exists $\phi_2 \in \mathrm{Aut}(G) \setminus G$ relating (u,v,w) and

(u^{-1}, v', w^{-1}). Since $[\mathrm{Aut}(G) : G] = 2$ we know that $\phi_2 = \phi_1\varphi$ for some inner automorphism φ. Therefore both triples are related by the diagonal action of ϕ_1 composed with the action of $\mathrm{Id} \times \varphi$ and so our claim follows from Remark 3. An analogous argument shows that the Beauvile surfaces defined by II and IV, by V and VII and by VI and VIII are pairwise isomorphic too.

We now claim that II defines the same surface as I (resp. VI defines the same surface as V). In order to prove it, we first note that the pairs of triples $(a_i, b_i, c\,;\, u, v, w)$ and $(a_i b_i a_i^{-1}, a_i, c\,;\, u, v, w)$, for $i = 1, 2$, define isomorphic Beauville surfaces by point *(3)* in Remark 3. Now if we denote by ψ conjugation by $\left(\begin{smallmatrix} 5 & 5 \\ 2 & 6 \end{smallmatrix}\right) \in \mathrm{PGL}(2,7)$ (resp. conjugation by $\left(\begin{smallmatrix} 4 & 3 \\ 4 & 6 \end{smallmatrix}\right) \in \mathrm{PGL}(2,7)$) and by φ conjugation by $\left(\begin{smallmatrix} 6 & 6 \\ 5 & 4 \end{smallmatrix}\right) \in G$ (resp. conjugation by $\left(\begin{smallmatrix} 2 & 6 \\ 1 & 0 \end{smallmatrix}\right) \in G$) we see that the element ψ acting diagonally, composed with $\mathrm{Id} \times \varphi$ interchanges the triples $(a_1 b_1 a_1^{-1}, a_1, c\,;\, u, v, w)$ and $(a_1, b_1, c\,;\, u^{-1}, v', w^{-1})$ (resp. interchanges the triples $(a_2 b_2 a_2^{-1}, a_2, c\,;\, u, v, w)$ and $(a_2, b_2, c\,;\, u^{-1}, v', w^{-1})$).

(2) The curve D^σ is isomorphic to D for each $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Now, by Theorem 4, if $\sigma(\xi_8) = \xi_8^5$ the curves D_1^σ and D_2 are isomorphic and therefore, by Remark 4, for any such σ we have $S_1^\sigma \cong S_2$.

(3) If $\pi_1(S_1) \cong \pi_1(S_2)$, then Theorem 6 would imply that D_1 would be isomorphic either to D_2 or to $\overline{D}_2$ which, by parts 3 and 5 of Theorem 4, is not the case.

(4) To see the minimality of the pair $(g_1, g_2) = (8, 49)$, first let us note that all Beauville surfaces with abelian Beauville group are of the form $F_n \times F_n / G_A$, where F_n is the Fermat curve of degree n, hence defined over $\mathbb{Q}$, and that the action of G_A is also Galois invariant (see Corollary 1 in [25]). It follows that all such surfaces are defined over $\mathbb{Q}$. Now the result is a consequence of Proposition 4. $\square$

Theorem 7 also implies the following

Corollary 2. *The field of moduli of the Beauville surfaces S_1 and S_2 is $\mathbb{Q}(\sqrt{2})$.*

Proof. It is obvious that the inertia groups G_{S_1} and G_{D_1} (resp. G_{S_2} and G_{D_2}) coincide, and then the corollary follows from part (5) of Theorem 4. $\square$

6.2. Arbitrarily large Galois orbits of Beauville surfaces with group $\mathrm{PSL}(2,p)$

We consider now Beauville structures of type $((2,3,n),(p,p,p))$ in the group $G = \mathrm{PSL}(2,p)$, where $n > 6$ divides either $(p-1)/2$ or $(p+1)/2$.

Let $(\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{c})$ be one of the triples of type $(2,3,n)$ in Theorem 2 and (u, v, w) be the (p,p,p) triple introduced in Lemma 4. The compatibility condition (7) is trivially satisfied again for pairs of triples of type (p,p,p) and $(2,3,n)$, so let us denote by $X = E_1 \times E/G$ the Beauville surface defined by these triples.

Since, by Theorem 2, for any Galois element σ such that $\sigma(\xi_n) \neq \xi_n^{\pm 1}$ we have $E_1^\sigma \ncong E_1$ and $E_1^\sigma \ncong \overline{E_1}$, we have at least $\phi(n)/2$ non-homeomorphic conjugate

Beauville surfaces

$$X^{\sigma_i} = \frac{E_i \times E}{\mathrm{PSL}(2,p)},$$

where σ_i are Galois elements satisfying $\sigma_i(\xi_n) = \xi_n^j$ with $ij \equiv 1 \mod n$.

As a consequence we have the following.

Theorem 8. *For each prime number $p > 7$ and each integer $n > 6$ dividing either $(p-1)/2$ or $(p+1)/2$ there exist a Beauville surface $X = E_1 \times E/G$ with $G = \mathrm{PSL}(2,p)$ such that the following statements hold*

(1) *E_1 and E are $G-$coverings of type $(2,3,n)$ and (p,p,p) respectively;*
(2) *The orbit of X under the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ contains at least $\phi(n)/2$ surfaces which are pairwise non-isomorphic.*
(3) *In fact, they have pairwise non-isomorphic fundamental groups, hence they are not homeomorphic to each other.*

In respect to the question of determining the fields of definition of Beauville surfaces, raised by Bauer, Catanese and Grunewald (see [5]), the above theorem shows that minimal fields of definition of Beauville surfaces can have arbitrarily large degree over the rationals.

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