Where is my spiral?

Rutgers experimental math seminar

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1. Introduction. This is a joint work with D. Raboso (Van der Corput method and optical illusions. *Indag. Math.*, **26**:723–735, 2015). Having in mind the nature of this seminar, two assets of this humble work are that some questions in it remain open and its origin is fully experimental. It stems from lecture notes I was writing for graduate students about the van der Corput method and the stationary phase approximation (employed in mathematical physics and analytic number theory). I got a theoretical result about the oscillatory sum

$$\sum_{n=1}^{N} e\big(\sqrt{n}\big) \qquad \text{Notation:} \quad e(t) := e^{2\pi i t} \tag{1}$$

and my computer contradicted my claim and, as usual, she is always right! In fact the situation was weird because the computer provided a *numerical* confirmation and a *visual* disproof.

2. Oscillatory sums and integrals. In general terms oscillatory sums are *difficult* to estimate and oscillatory integrals are *simple*. For instance, optimal uniform bounds for $\sum_{n=1}^{N} e(t\log n)$, mainly in the range $t \le N^{1/2}$, would give fundamental advances in our understanding of the Riemann ζ function with consequences in the spacing between primes. On the other hand computing even explicitly $\int_1^N e(t\log x)\,dx$ belongs to undergraduate level.

Rougly speaking, van der Corput lemma and stationary phase approximation say that there are two models for an oscillatory integral

$$I = \int e\big(f(x)\big)\,dx \qquad \text{with } f \text{ convex.} \tag{2}$$

If the derivative is $|f'| > \lambda$ then I can be *bounded* by a linear model $\int e(\lambda x)\, dx$ getting λ^{-1}. On the other hand, if $f'(x_0) = 0$ this stationary point gives a fundamental contribution and I can be *approximated* by a quadratic model $\int e\big(\frac{1}{2}\lambda(x-x_0)^2\big)\, dx$ with $\lambda = f''(x_0)$.

Is it possible to replace oscillatory sums by oscillatory integrals? No:

$$\sum_{n=1}^{N} e(n) = N \qquad \leftrightarrow \qquad \int_1^N e(x)\, dx = 0. \tag{3}$$

Second opinion. Yes if you admit sums of integrals. Essentially

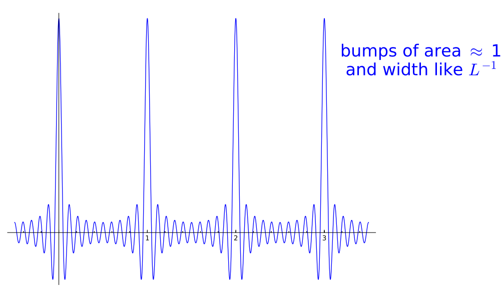
$$\sum_{n=a}^{b} e\big(f(n)\big) = \sum_{f'(a)\le n\le f'(b)} \int_a^b e\big(f(t)-nt\big)\, dt + \text{admissible error}. \tag{4}$$

The same formula works for concave functions swapping $f'(a)$ and $f'(b)$.

If $\Delta f'$ is large the stationary phase approximation of the integral gives a longer exponential sum to approximate (this is very bad) but even in this case the van der Corput method and the Vinogradov method can squeeze valuable nontrivial information.

Idea of the proof of (4). With increasing approximation on L we have

$$\sum_{n=-L}^{L} e(-nx) \quad \approx \tag{5}$$



Then

$$\sum_{n=a}^{b} e\big(f(n)\big) \approx \sum_{n=-L}^{L} \int_a^b e\big(f(x)-nx\big)\, dx \qquad \text{for } L \text{ large}. \tag{6}$$

If $n \gg' (b)$ or $n \ll f'(a)$ this is negligible by the first model.

3. The theoretical-experimental paradox. Let us consider the oscillatory sum for the phase function $f(x) = \alpha\sqrt{x}$ for $\alpha > 0$ a fixed constant (originally $\alpha = 1$),

$$S_\alpha(N) = \sum_{n=1}^{N} e\big(\alpha\sqrt{n}\big). \tag{7}$$

Separating a finite number of initial terms, we have $|f'| \ll 1$ and the only integral corresponding to $n = 0$ appears in (4). This means that after some special behavior for N small, when N is much larger than α^2 we get and approximation of the form

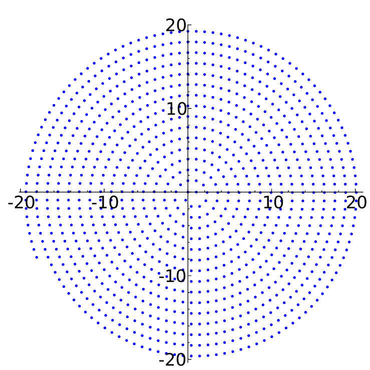
$$S_\alpha(N) \approx \text{constant} + \int_{C_0}^{N} e\big(\alpha\sqrt{x}\big)\, dx. \tag{8}$$

The integral can be explicitly computed and implies that

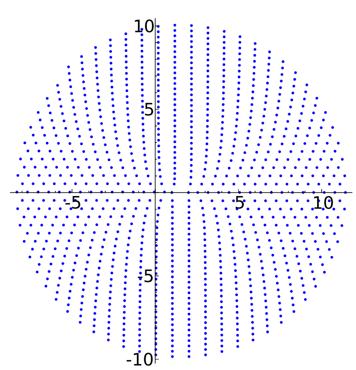
$$\begin{array}{l}\text{Plot of the partial}\\ \text{sums } S_\alpha\end{array} \approx \begin{array}{l}\text{Off centered plot of } \mathcal{A}_\alpha(x) = \dfrac{\sqrt{x}}{\pi i \alpha} e\big(\alpha\sqrt{x}\big)\\ \text{at integer values.}\end{array} \tag{9}$$

When writing the lecture notes I checked numerically this for $\alpha = 1$ and the computer confirmed the very good approximation predicted by the theory. Clearly, $\mathcal{A}_\alpha(x)$ defines an Archimedean spiral of width $1/\pi\alpha^2$ then I decided to include a figure for illustration and a quite different paradoxical truth appeared. Instead of a spiral I saw a pattern composed by vertical branches. The obvious question is the title of this talk: *Where is my spiral?*

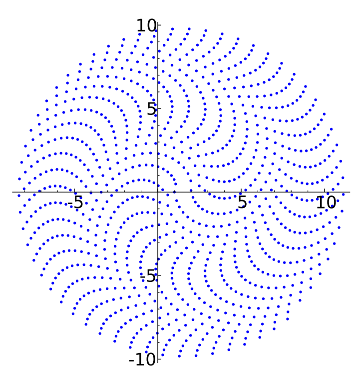
I played with the parameter α. Summing up, for $\alpha < 1$ one gets the expected spiral, for $\alpha^2 \in \mathbb{Z}^+$ one gets branches in two flavors depending on the parity of n and for $1 < \alpha^2 \notin \mathbb{Z}$ one gets in general patterns with appealing aesthetic structure which we do not fully understand.



$\alpha = 1/2$

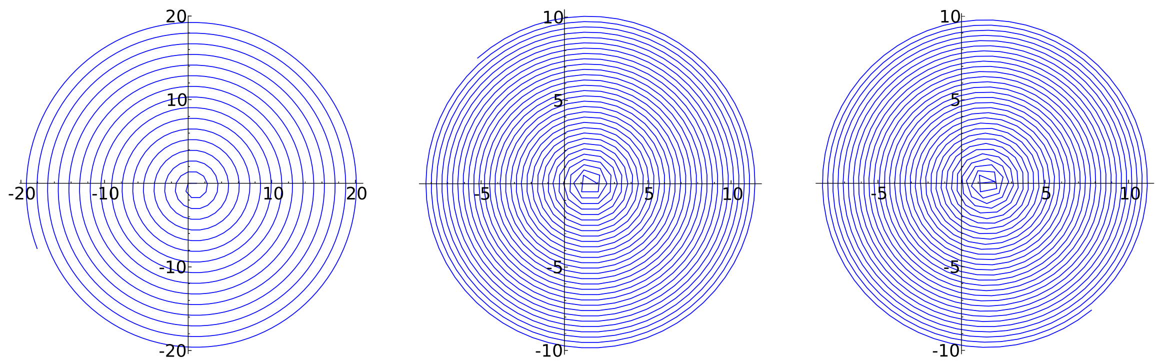


$\alpha = 1$



$\alpha = 65/64$

4. A mathematical model. The previous plots constitute in some sense an optical illusion or a kind of Moire effect because the individual numerical values fit perfectly the predictions of theory. Our sight tends to connect close points in successive turnings to form in the case $\alpha = 1$ the vertical branches. Recall that the width of the spiral is $1/\pi\alpha^2$ then it becomes natural that for α small the points are close only when they are in the same turning and no confusion is possible. If instead of plotting individual points we plot the segment joining them, the spiral is always there as the following figures show.



$\alpha = 1/2$ $\qquad$ $\alpha = 1$ $\qquad$ $\alpha = 65/64$

My guess is that for the most of the people it is hard to believe that we can actually get Archimedean spirals joining the points with segments.

With the notation introduced in (9), given two points $\mathcal{A}_\alpha(k_1)$ and $\mathcal{A}_\alpha(k_2)$ on the spiral if they have angles differing by a quantity close to 2π they become close in successive turnings. It requires $2\pi\alpha\sqrt{m_2} \approx 2\pi\alpha\sqrt{m_1} + 2\pi$. Consequently, given m_1 the best approximating m_2 is

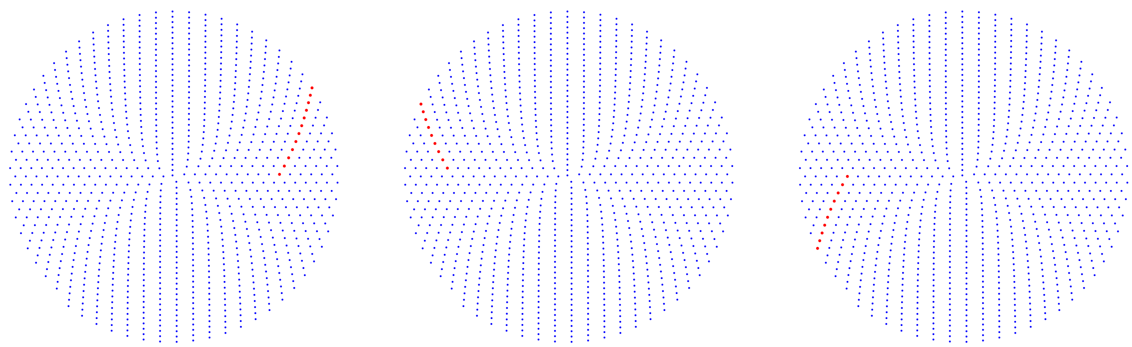
$$m_2 = \operatorname{round}\Big(\big(\sqrt{m_1} + \alpha^{-1}\big)^2\Big) = m_1 + \big\lfloor 2\alpha^{-1}\sqrt{m_1} + \alpha^{-2} + 1/2\big\rfloor. \tag{10}$$

It suggests that we observe *branches* $\big\{\mathcal{A}_\alpha(t_k)\big\}_k$ with t_k given by the recurrence

$$t_{k+1} = t_k + \big\lfloor 2\alpha^{-1}\sqrt{t_k} + \alpha^{-2} + 1/2\big\rfloor. \tag{11}$$

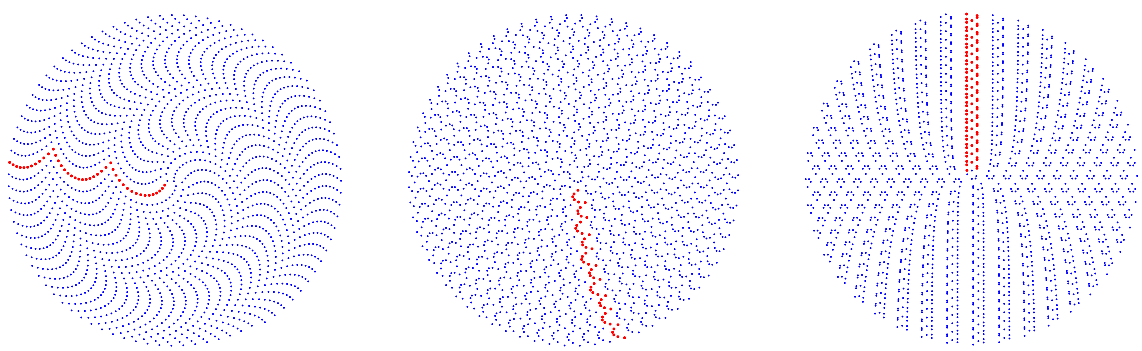
From this point on we quit the original exponential sum and we stick to the model embodied in this recurrence.

The following figures illustrate the validity of this model for $\alpha = 1$. For each t_0 the branch $\big\{\mathcal{A}_\alpha(t_k)\big\}_k$ is actually a geometrical branch in the figures. The model connects each point with the closest point in the *next* turn then for a given t_0 the branch always move away from the origin. The maximal branches for $\alpha = 1$ correspond to t_0 of the form $4m^2+m$ (1st quadrant), $4(m+2)^2-5(m+2)+2$ (2nd quadrant), $4(m+2)^2-(m+1)$ (3rd quadrant), $4m^2-5m+2$ (4th quadrant). This is an exercise!



$\mathsf{t}_0 = 4 \cdot 10^2 + 10$ $\qquad$ $\mathsf{t}_0 = 4 \cdot 12^2 - 5 \cdot 12 + 2$ $\qquad$ $\mathsf{t}_0 = 4 \cdot 11^2 - 11$

This also works for general values of α



$\alpha = 65/64,\ \mathsf{t}_0 = 7$ $\qquad$ $\alpha = 13/10,\ \mathsf{t}_0 = 7$ $\qquad$ $\alpha = \sqrt{5},\ \mathsf{t}_0 = 7$

The last example may seem strange because the branch is rather a band. This is due to the fact that the closest point in the next turn can be not so close. Anyway, the branches explain the structure in bands and a finer study of t_k would give the inner structure of each band.

5. Solving the recurrence. If we omit the integral part in (11) and we subtract 1/2 we get $t_{k+1} = t_k + 2\alpha^{-1}\sqrt{t_k} + \alpha^{-2}$ which has a general solution of the form $\alpha^{-2}(k+k_0)^2$. Then we expect a quadratic growth. In fact it matches the parabolic branches observed for $\alpha = 1$. On the other hand the actual form of (11) gives in principle little hope for an explicit solution. Curiously the case $\alpha^2 \in \mathbb{Z}^+$ can be fully solved. We came to it thanks to the

Experimental fact. For each $\alpha^2 = n \in \mathbb{Z}^+$ the second finite differences of t_k have period n' with $n' = n/2$ if n is even and $n' = n$ if n is odd.

Once this is mathematical proved one arrives to

Result. For each $\alpha^2 = n \in \mathbb{Z}^+$ the solution of (11) is of the form $t_k = f_r(k)$ where $f_0, f_1, \dots, f_{n'}$ are certain quadratic polynomial and r is the residue of k modulo n'.

Alternatively, one can write $t_k = f(k)$ with f a quadratic polynomial with coefficients depending on r. It turn out that for each given t_0 these coefficients can be completely determined. The dependence on t_0 is not very simple and it is connected to a certain arithmetical representation

Fact. Given $n \in \mathbb{Z}^+$ any $t_0 \in \mathbb{Z}_{\geq 0}$ admits a unique representation of the form $t_0 = ni^2 - i + j$ with $i \in \mathbb{Z}^+$ and $|j| < ni$.

The proof is very easy. Essentially checking that the polynomial $P(x,y) = nx^2 - x + y$ satisfies $P\big(x+1, 1-n(x+1)\big) - P(x, nx-1) = 1$.

For t_0 we get a pair (i,j) and with some arithmetical operations involving the parity of $\lfloor (n+12)/8 \rfloor$ and the size of j we get another integral pair (c_0, c_1). Instead of writing the actual formula (which is rather ugly) I will just mention an example: For n and $\lfloor (n+12)/8 \rfloor$ even and $j = 0$ it is deduce $c_0 = 2i$ and $c_1 = n/2 - \lfloor n/4 \rfloor$. Taking as granted that we know the formula for c_0 and c_1 in terms of t_0 then we get a perfectly explicit solution of (11).

More precise result. If $\alpha^2 = n \in \mathbb{Z}^+$ with $n > 2$ even, the solution of (11) for $k \geq 1$ is

$$t_k = \frac{(k+c_1)^2 - (r+1)^2}{n} + \frac{r+1-k-c_1}{2} + c_0 k + t_0 \tag{12}$$

where r is the remainder of $k + c_1 - 1$ when divided by n.

There is something similar but slightly more complicated for the odd case.

The cases $n = 1$ and $n = 2$ are somewhat special and the theory collapses to produce extremely simple solutions given respectively

$$t_k = k^2 + k\big\lfloor 2\sqrt{t_0} + \frac{1}{2}\big\rfloor + t_0 \qquad \text{and} \qquad t_k = \frac{k(k+1)}{2} + k\big\lfloor 2\sqrt{t_0}\big\rfloor + t_0. \tag{13}$$

The geometric interpretation is that in these cases we do not observe subpatterns due to the residues modulo n'.

6. Closing remarks and questions. For simplification I have not given the full experimental fact observed when computing values of t_k

> Enhanced experimental fact. For each $\alpha^2 = n \in \mathbb{Z}^+$ the second finite differences of t_k have period n' with $n' = n/2$ if n is even and $n' = n$ if n is odd. And on each block of length n they are zero with exactly two exceptions.

In some sense, c_0 and c_1 embody the information about these exceptions. Our proof gives these constants but it is not very pretty, even for us! Perhaps this is unavoidable because the formulas for c_0 and c_1 are complicate. If we forget about the constants, a natural question is

> Challenge. Find a short elegant proof of the experimental fact.

This is a warming up for the real natural problem here: The extension to every value of α. When $\alpha^2 \notin \mathbb{Z}^+$, the second finite differences of t_k seems to be quasiperiodic.

> Dream. It is possible to give a general explicit solution of the recurrence (11) in terms of continued fractions associated to α.

Feel free to disagree, different persons have different dreams. The good ones are those that become true.