Second quantization. The fields turn into field operators with quanta=particles. Coordinates are labels, not operators.

$$\text{Classical Poisson bracket } \{\cdot,\cdot\} \quad \longrightarrow \quad \text{Quantum commutator } i[\cdot,\cdot].$$

Scalar field example:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

In classical terms, the momentum is $\pi = \pi(\vec{x}) = \partial_0\phi$ and the Hamiltonian density is

$$\mathcal{H} = \pi\partial_0\phi - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}\partial_i\phi\partial^i\phi + \frac{1}{2}m^2\phi^2.$$

It can be re-written as

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}\phi(-\Delta + m^2)\phi + \text{total derivative}.$$

The plane waves $e^{\pm i\vec{p}\cdot\vec{x}}$ diagonalize the operator $-\Delta + m^2$. To *promote* ϕ to an operator $\widehat{\phi}$, keeping the analogy with the harmonic oscillator, one takes:

$$\widehat{\phi}(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2E(\vec{p})}}\big(e^{i\vec{p}\cdot\vec{x}}\widehat{a}(\vec{p}) + e^{-i\vec{p}\cdot\vec{x}}\widehat{a}^\dagger(\vec{p})\big) \quad \text{and} \quad \widehat{\pi}(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^{3/2}} i\sqrt{\frac{E(\vec{p})}{2}}\big(-e^{i\vec{p}\cdot\vec{x}}\widehat{a}(\vec{p}) + e^{-i\vec{p}\cdot\vec{x}}\widehat{a}^\dagger(\vec{p})\big)$$

with $E(\vec{p}) = (\vec{p}^2 + m^2)^{1/2}$. The following commutation rules for a (no hats onwards) imply the canonical relations in the quantum setting

$$\begin{array}{lcl} [a(\vec{p}), a^\dagger(\vec{q})] = \delta(\vec{p}-\vec{q}) & & i[\pi(\vec{x}), \phi(\vec{y})] = \delta(\vec{x}-\vec{y}) \\ \text{rest zero} & \Rightarrow & \text{rest zero} \end{array}$$

Substituting the Hamiltonian becomes

$$H = \int d^3\vec{x}\mathcal{H} = \int d^3\vec{p}\frac{E(\vec{p})}{2}\big(a^\dagger(\vec{p})a(\vec{p}) + a(\vec{p})a^\dagger(\vec{p})\big) = \int d^3\vec{p}E(\vec{p})a^\dagger(\vec{p})a(\vec{p}) + \infty.$$

The number operator $a^\dagger(\vec{p})a(\vec{p})$ counts the particles of momentum $\vec{p}$. Using normal ordering $\frac{1}{2} : a^\dagger a + aa^\dagger : = a^\dagger a$ and ∞ does not appear. It is considered a non-observable ground energy anyway.

Free scalar field. In the Heisenberg picture $a^\dagger \mapsto e^{iEt}a^\dagger$. The field operator becomes

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E(\vec{p})}} \big(e^{-ip\cdot x} a(\vec{p}) + e^{ip\cdot x} a^\dagger(\vec{p}) \big)$$

where $x = (t, \vec{x})$ and Minkowski scalar product is assumed. The moment is

$$\pi(x) = \int \frac{d^3\vec{p}}{(2\pi)^{3/2}} i \sqrt{\frac{E(\vec{p})}{2}} \big(-e^{-ip\cdot x} a(\vec{p}) + e^{ip\cdot x} a^\dagger(\vec{p}) \big)$$

that is coherent with $\pi = \partial_0 \phi$. Now we have

$$i[\phi(x), \phi(y)] = D(x - y)$$

where D vanishes at time zero (equal times), $\partial_0 D(x^0 = 0, \vec{x}) = \delta(\vec{x})$ and $(\Box + m^2)D = 0$. It is the Pauli-Jordan function[1].

Dirac equation and Dirac matrices. The Dirac equation is

$$(i\partial\!\!\!/ - m)\Psi = 0 \qquad \text{with} \qquad \partial\!\!\!/ = \gamma^\mu \partial_\mu$$

and γ^μ are 4×4 (complex) matrices satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{I}, \qquad \mu, \nu = 0, 1, 2, 3.$$

It is also defined $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ that anti-commutes with the rest. A possible choice are the Dirac matrices defined in 2×2 blocks

$$\gamma^0 = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad \text{and} \quad \gamma^5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}.$$

They are associated to a representation $D = D(\Lambda)$ of the Lorentz group[2] in the following way: If we want $x \mapsto \Lambda x$, $\Psi \mapsto D\Psi$ to preserve Dirac's equation, we need

$$D^{-1}\gamma^\mu D = \Lambda^\mu_\nu \gamma^\nu$$

because $(i\partial\!\!\!/ - m)\Psi = 0 \mapsto i(\Lambda^{-1})^\mu_\nu \partial_\mu \gamma^\nu D\Psi - mD\Psi = 0$ that is $i(\Lambda^{-1})^\mu_\nu D^{-1}\gamma^\nu D \partial_\mu \Psi - m\Psi = 0$. The matrix γ^5 commutes with products of two γ^μ and it proves that D is reducible giving rise to the projectors $P_\pm = (\mathbf{I} \pm \gamma_5)/2$. We have the relations

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger \qquad \text{and} \qquad \gamma^0 D^\dagger \gamma^0 = D^{-1}.$$

[1]It seems that a explicit formula is $D(x) = -i \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\sin(p_\mu x^\mu)}{E(\vec{p})} = i \int \frac{d^4\vec{p}}{(2\pi)^3} e^{-ip\cdot x} \operatorname{sgn}(p_0) \delta(p^2 - m^2)$.

[2]It seems that $D(\Lambda) = \exp\big(\frac{1}{2}\lambda_{\mu\nu}\frac{1}{4}[\gamma^\mu, \gamma^\nu]\big)$ where $\lambda_{\mu\nu}$ are the coefficients for the usual generators.

The second follows from $\gamma^0[\gamma^\mu,\gamma^\nu]^\dagger\gamma^0 = [\gamma^\nu,\gamma^\mu]$.

It is convenient to define

$$\overline{\Psi} = \Psi^\dagger\gamma^0.$$

With this definition

$\overline{\Psi}\Psi$	is a scalar	$\overline{\Psi}\gamma^5\Psi$	is a pseudo-scalar
$\overline{\Psi}\gamma^\mu\Psi$	is a vector	$\overline{\Psi}\gamma^\mu\gamma^5\Psi$	is an axial vector

The first quantity is a scalar because $\overline{\Psi}\Psi = \Psi^\dagger\gamma^0\Psi \mapsto \Psi^\dagger D^\dagger\gamma^0 D\Psi = \overline{\Psi}D^{-1}\gamma^0 D\Psi$. For the second in the first column, use $D^{-1}\gamma^\mu D = \Lambda^\mu_\nu\gamma^\nu$. In the second column, pseudo-scalar and axial vector refer to the fact that they change sign under the parity transformation $\Psi(\vec{x},t)\mapsto \gamma^0\Psi(-\vec{x},t)$ (probably this only makes sense in the chiral representation.

Free spinor field. The natural Lagrangian leading to Dirac's equation is

$$\mathcal{L} = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi.$$

Up to a total derivative, it can be written as

$$\mathcal{L} = \frac{1}{2}i\overline{\Psi}\overleftrightarrow{\partial}_\mu\gamma^\mu\Psi - m\overline{\Psi}\Psi \qquad \text{where} \quad A\overleftrightarrow{\partial}_\mu B = A\partial_\mu B - (\partial_\mu A)B.$$

This Lagrangian has the internal symmetry $\Psi\mapsto e^{-i\alpha}\Psi$, $\overline{\Psi}\mapsto e^{-i\alpha}\overline{\Psi}$. the corresponding current is (waving hands about the 4 components in the calculation)

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\frac{d}{d\alpha}\Big|_{\alpha=0} e^{-i\alpha}\Psi = i\overline{\Psi}\gamma^\mu(-i)\Psi = \overline{\Psi}\gamma^\mu\Psi$$

and the conserved charge is $Q = \int d^3\vec{x}\,\overline{\Psi}\gamma^0\Psi = \int d^3\vec{x}\,\Psi^\dagger\Psi$ (conservation of probability).

The generalized moments are $\pi_\alpha = \frac{i}{2}(\overline{\Psi}\gamma^0)_\alpha$ and $\overline{\pi}_\alpha = -\frac{i}{2}(\gamma^0\Psi)_\alpha$, then

$$\mathcal{H} = -\frac{i}{2}\overline{\Psi}\overleftrightarrow{\partial}_i\gamma^i\Psi + m\overline{\Psi}\Psi \qquad \text{or} \qquad \mathcal{H} = -i\overline{\Psi}\partial_i\gamma^i\Psi + m\overline{\Psi}\Psi$$

using the first form of the Lagrangian. The Hamiltonian is

$$H = \int d^3\vec{x}\,\Psi^\dagger\gamma^0(-i\partial_i\gamma^i + m)\Psi.$$

Looking for plane waves $\Psi(\vec{x}) = e^{i\vec{p}\cdot\vec{x}}C(\vec{p})$ that diagonalize $\gamma^0(-i\partial_i\gamma^i + m)$ we have that $C(\vec{p})$ is an eigenvector of $H(\vec{p}) = p^i\gamma^0\gamma^i + m\gamma^0$ which is Hermitian with 2 double eigenvalues $\pm E(\vec{p}) = \pm\sqrt{\vec{p}^2 + m^2}$. They give solutions of the Dirac equation depending on the spin σ

$$\sqrt{\frac{m}{E(\vec{p})}}u(\vec{p},\sigma)e^{-ip\cdot x} \qquad \text{and} \qquad \sqrt{\frac{m}{E(\vec{p})}}v(\vec{p},\sigma)e^{ip\cdot x}.$$

The first solutions correspond to particles (electrons) and the second correspond to antiparticles (positrons). The normalization is

$$u^\dagger u = v^\dagger v = \frac{E(\vec{p})}{m}\delta_{\sigma\sigma'} \qquad \text{and} \qquad \overline{u}u = -\overline{v}v = \delta_{\sigma\sigma'}$$

where the first u or v is evaluated at $(\vec{p},\sigma)$and the second at $(\vec{p},\sigma')$. We have also

$$\sum_\sigma u_\alpha(\vec{p},\sigma)\overline{u}_\beta(\vec{p},\sigma) = \frac{(\not{p}+m)_{\alpha\beta}}{2m} \qquad \text{and} \qquad \sum_\sigma v_\alpha(\vec{p},\sigma)\overline{v}_\beta(\vec{p},\sigma) = \frac{(\not{p}-m)_{\alpha\beta}}{2m}.$$

In the Heisenberg picture the second quantization of the spinor field is the following formal solution of the Dirac equation

$$\Psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^{3/2}}\sqrt{\frac{m}{E(\vec{p})}}\sum_\sigma \big(e^{-i\vec{p}\cdot\vec{x}}u(\vec{p},\sigma)b_\sigma(\vec{p}) + e^{i\vec{p}\cdot\vec{x}}v(\vec{p},\sigma)d^\dagger_\sigma(\vec{p})\big).$$

Here $b_\sigma(\vec{p})$ annihilates a particle and $d^\dagger_\sigma(\vec{p})$ creates an antiparticle and they satisfy the relations

$$\{b_\sigma(\vec{p}), b^\dagger_{\sigma'}(\vec{q})\} = \{d_\sigma(\vec{p}), d^\dagger_{\sigma'}(\vec{q})\} = \delta_{\sigma\sigma'}\delta(\vec{p}-\vec{q})$$

and the rest of the anti-commutators are 0. For instance, b_σ and d_σ anti-commute. Replacing commutators by anti-commutators $\{\cdot,\cdot\} = [\cdot,\cdot]_+$ is required to keep the statistics of the fermions. In analogy with the free scalar field, we have

$$H = \int d^3\vec{x}\mathcal{H} = \int d^3\vec{p}E(\vec{p})\big(b^\dagger_\sigma(\vec{p})b_\sigma(\vec{p}) + d^\dagger_\sigma(\vec{p})d_\sigma(\vec{p})\big) + \infty.$$

As before, one can avoid the infinity using normal ordering that for fermions satisfies $:bb^\dagger: = -b^\dagger b$ and the same for d.

Propagators. The propagators appear in the theory as the expectation of a time ordered product and give a probability amplitude to travel from a point to another. Mathematically they are fundamental solutions of the "equations of motion". The propagators can also be considered in momentum space. From the mathematical point of view this is applying the Fourier transform. In the following cases it reduces to cross out the integral and $d^4p/(2\pi)^4$.

For the scalar field, $G_F(x-y) = \langle 0|T\{\phi(x)\phi(y)\}|0\rangle$, it solves $(\Box + m^2)G_F(x) = -i\delta(x)$ and is given by

$$G_F(x) = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{ipx}}{p^2-m^2+i\epsilon}.$$

For the spinor field, $G_{\alpha\beta}(x-y) = \langle 0|T\{\Psi_\alpha(x)\overline{\Psi}_\beta(y)\}|0\rangle$, it solves $(i\partial\!\!\!/ - m)G_{\alpha\beta}(x) = i\delta(x)\mathbf{I}$ and is given by (possible problem with a sign)

$$G_{\alpha\beta}(x) = (i\partial\!\!\!/ + m)G_F(x) = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ipx}}{p^2 - m^2 + i\epsilon}(p\!\!\!/ + m)_{\alpha\beta} = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ipx}}{p\!\!\!/ - m + i\epsilon}.$$

The last equality is just formal.

For photons with the Feynman gauge

$$G^{\mu\nu}(x) = -i\int \frac{d^4p}{(2\pi)^4}\frac{e^{ipx}}{p^2 + i\epsilon}\eta^{\mu\nu}.$$

Some Lagrangians. A recollection of some common Lagrangians.

Free scalar field in the real and complex cases

$$\mathcal{L}_r = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \qquad \mathcal{L}_c = \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi.$$

The second corresponds to the first for $N = 2$ fields taking $\varphi = \frac{1}{2}\big(\phi^{(1)} + i\phi^{(2)}\big)$.

Phi-4 theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

Free spinor field, two forms differing in a total derivative,

$$\mathcal{L} = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi, \qquad \mathcal{L} = \frac{1}{2}i\overline{\Psi}\overset{\leftrightarrow}{\partial}_\mu\gamma^\mu\Psi - m\overline{\Psi}\Psi.$$

Yukawa, the coupling of a scalar field a a spinor field

$$\mathcal{L} = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 - \lambda_Y\overline{\Psi}\Gamma\Psi\phi$$

where $\Gamma = \mathbf{I}$ or $\Gamma = \gamma^5$.

QED, coupling the spinor field and the electromagnetic field,

$$\mathcal{L} = \frac{1}{2}i\overline{\Psi}\overset{\leftrightarrow}{D\!\!\!/}\gamma^\mu\Psi - m\overline{\Psi}\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \qquad \text{where} \qquad D_\mu = \partial_\mu + ieA_\mu.$$

It has local gauge invariance given by $\Psi \mapsto e^{-i\theta(x)e}\Psi$ and $A_\mu \mapsto A_\mu + \partial_\mu\theta(x)$. This Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}i\overline{\Psi}\overset{\leftrightarrow}{\partial\!\!\!/}\gamma^\mu\Psi - m\overline{\Psi}\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - ej^\mu A_\mu$$

for the electromagnetic current $j^\mu = e\overline{\Psi}\gamma^\mu\Psi$. Fixing the gauge of the electromagnetic field may require a term $-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2$ with α a constant. For Feynman gauge, $\alpha = 1$.

QCD, it has the form

$$\mathcal{L} = \frac{1}{2} i \underset{\sim}{\overline{\Psi}} \overset{\leftrightarrow}{\not{D}} \underset{\sim}{\Psi} - m \underset{\sim}{\overline{\Psi}} \underset{\sim}{\Psi} - \frac{1}{2g^2} \mathrm{Tr}\big(\underset{\sim}{F}^{\mu\nu} \underset{\sim}{F}_{\mu\nu} \big)$$

where g is a coupling constant, $\underset{\sim}{\Psi}$ is given by three gluon fields, $D_\mu = \partial_\mu - iA_\mu(x)$ with $A_\mu(x)$ is a 3×3 (block) matrix and $\underset{\sim}{F}_{\mu\nu}$ is its curvature form $i[D_\mu, D_\nu]$.

Interacting theories. The general scheme is $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}$ where $\mathcal{H}_0$ the free Hamiltonian density and $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$ with $\mathcal{L}_{\text{int}}$ the interaction part of the Lagrangian. For instance, $\mathcal{H}_{\text{int}} = \lambda\phi^4/4!$ in the phi-4 theory.

It is convenient to consider the *interaction picture* in which the operators evolve in time with the unperturbed Hamiltonian, $\mathcal{O}(t) = e^{itH_0}\mathcal{O}e^{-itH_0}$ and, consequently, the states with $\Psi(t) = e^{itH_0}e^{-itH}\Psi$. If we define $U(t, t_0)$ such that $U(t, t_0)e^{iH_0 t_0}\Psi(t_0) = e^{iH_0 t}\Psi(t)$ then $\langle e^{+iH_0\infty}\Psi_1|\mathcal{O}|e^{-iH_0\infty}\Psi_2\rangle$ gives the same result for $\mathcal{O} = S$, the corresponding S-matrix, and for $\mathcal{O} = U(+\infty, -\infty)$, then both operators coincide. A direct calculation from the definition of $U(t, t_0)$ using the Schrödinger equation proves

$$\frac{\partial}{\partial t} U(t, t_0) = -iH_{\text{int}}^{(0)}(t)U(t, t_0) \qquad \text{with} \quad H_{\text{int}}^{(0)}(t) = e^{itH_0}H_{\text{int}}e^{-itH_0}.$$

Then

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \mathrm{T}\Big\{ \Big(\int_{-\infty}^{\infty} d\tau H_{\text{int}}^{(0)}(\tau) \Big)^n \Big\} = \mathrm{Texp}\Big(-i \int_{-\infty}^{\infty} d\tau H_{\text{int}}^{(0)}(\tau) \Big).$$

If we want to study $\vec{p} + \vec{p}\,' \to \vec{q} + \vec{q}\,'$ in the phi-4 theory where $|\vec{p}\rangle = a_c^\dagger(\vec{p})|0\rangle$, etc. with the covariant creation operator $a_c(\vec{p}) = (2\pi)^{3/2}\sqrt{2E(\vec{p})}a(\vec{p})$, we have to compute

$$\langle \vec{q}\vec{q}\,'|S|\vec{p}\vec{p}\,'\rangle = \langle 0|a_c(\vec{q})a_c(\vec{q}\,')\mathrm{Texp}\Big(-i\frac{\lambda}{4!} \int_{-\infty}^{\infty} \int d^4x \; : \big(\phi^{(0)}(x)\big)^4 : \Big) a_c^\dagger(\vec{p})a_c^\dagger(\vec{p}\,')|0\rangle$$

with $\phi^{(0)}(t, \vec{x}) = e^{itH_0}\phi(\vec{x})e^{-itH_0}$. With the change in the creation operator, we have

$$\phi^{(0)}(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2E(\vec{k})} \big(e^{-ik\cdot x}a_c(\vec{k}) + e^{ik\cdot x}a_c^\dagger(\vec{k}) \big)$$

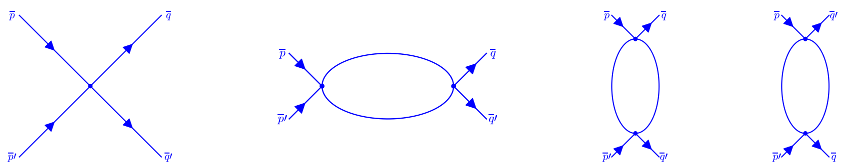
where $d^3\vec{k}/E(\vec{k})$ is Lorentz invariant. The new commutation rule is

$$[a_c(\vec{p}), a_c^\dagger(\vec{q})] = (2\pi)^3 2E(\vec{p})\delta(\vec{p} - \vec{q}).$$

When expanding the formula for S it is very convenient to apply *Wick's theorem* that allows to write $\mathrm{T}\{A_1A_2\cdots A_n\}$, with A_i sums of creation and annihilation operators, as the sum of contractions of disjoint pairs of operators multiplied by the normal ordered product of the remaining operators. The contraction of A and B is a scalar G, the propagator in the theory, such that $G\mathbf{I} = \mathrm{T}\{AB\} - :AB:$. For instance, $\mathrm{T}\{\phi^{(0)}(x_1)\phi^{(0)}(x_2)\phi^{(0)}(x_3)\phi^{(0)}(x_4)\}$ is the sum of $:\phi^{(0)}(x_1)\phi^{(0)}(x_2)\phi^{(0)}(x_3)\phi^{(0)}(x_4):$, six terms of the form $:\phi^{(0)}(x_i)\phi^{(0)}(x_j): G(x_i - x_j)$ and three terms of the form $G(x_i - x_j)G(x_k - x_l)\mathbf{I}$.

Feynman rules. The complication in the notation and computation can be circumvent using a pictorial way of representing the interactions called *Feynman diagrams*.

In the phi-4 theory the interaction vertexes involve 4 legs (edges). Let us say that we want to study $\vec{p} + \vec{p}\,' \to \vec{q} + \vec{q}\,'$. Then the simplest diagrams are



The loop number is $L - V + 1$ where L is the number of internal lines and V is the number of vertexes. The loop number is 0 for the first diagram and 1 for the rest.

The contribution to the S matrix is specified for the Feynman rules (in momentum space): A moment k is assigned to the internal lines (with an arbitrary orientation) and the propagator $i/(k^2 - m^2 + i\epsilon)$. For each (inner) vertex we put a term of the form

$$(-i\lambda)(2\pi)^4\delta\big(\sum_{\text{in}} p - \sum_{\text{out}} p\big).$$

We multiply all of these terms and integrate $d^4k/(2\pi)^4$ with k the momenta associated to the internal lines. Finally, we divide by the symmetries of the diagram, meaning the order of the group permuting vertexes and internal lines leaving the diagram topologically invariant.

For instance, in the first diagram there are not internal lines nor symmetries, then it contributes

$$S_0 = (-i\lambda)(2\pi)^4\delta(p + p' - q - q').$$

The second diagram has symmetry factor $1/2$ because we can change the upper and the lower edges (but not the vertexes preserving the names of the particles). If we assign to the inner lines momenta k and l to the right, its contribution is

$$\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\int \frac{d^4l}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon}\frac{i}{l^2 - m^2 + i\epsilon}(-i\lambda)(2\pi)^4\delta(p+p'-k-l)(-i\lambda)(2\pi)^4\delta(k+l-q-q')$$

and changing variables $k + l \mapsto k$ and performing the integration in k, the result is

$$\frac{1}{2}(-i\lambda)^2(2\pi)^4\delta(p+p'-q-q')F(p+p') \quad \text{with} \quad F(s) = \int \frac{d^4l}{(2\pi)^4}\frac{i}{(s-l)^2 - m^2 + i\epsilon}\frac{i}{l^2 - m^2 + i\epsilon}$$

The third and fourth diagrams gives the same contribution changing the variables. The one loop contribution then amounts

$$S_1 = \frac{1}{2}(-i\lambda)^2(2\pi)^4\delta(p+p'-q-q')\big(\widetilde{F}(s)+\widetilde{F}(t)+\widetilde{F}(u)\big)$$

where $s=(p+p')^2$, $s=(p-q)^2$, $t=(p-q')^2$ are the Mandelstam variables. The problem is that the integrals defining $\widetilde{F}$ are divergent. The solution (renormalization) involves to consider m and λ non-observable and related to the cut-off to regularize these integrals.