Post-Newtonian approximations

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Abstract

After a very basic introduction about the geometric view of gravitation, we deal with Newtonian and post-Newtonian approximations giving an overview of the parametrized post-Newtonian formalism. In the final part, we focus on the classical experimental tests. This is a bibliographic work in which an effort has been made to explore the historical timeline and motivation, managing the original references.

Contents

1 General relativity as a metric theory

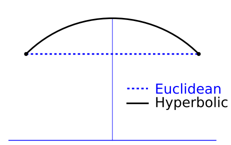
Summary. Basic general comments about the geometry underlying a gravitational metric theory and the equivalence principle.

Albert Einstein was working during years on a metric theory of gravity. The main basic idea is that gravitation is not due to a force or field, with the classical meaning, gravitation is itself a deformation in the structure of spacetime, a geometric phenomenon. In his own words [Ein16] (the translation is taken from [LEMWed]) "gravitation occupies an exceptional position with regard to other forces, particularly the electromagnetic forces, since the ten functions representing the gravitational field at the same time define the metrical properties of the space measured".

It is important to emphasize that Einstein's theory is a particular metric theory of gravitation but it is possible to construct alternative theories. The post-Newtonian approximations and formalism studied in this work provides a method to compare different metric theories and to study its effects in the weak field and slow motion setting.

Roughly speaking, a metric is a way of specifying the way o measuring lengths at each point. In a more formal way, it defines a scalar product (disregarding positive definiteness) at each point. It is usually represented by $g_{\mu\nu}dx^\mu dx^\nu$ (summation convention is assumed) meaning that the scalar product of the basis vectors ∂_μ and ∂_ν is $g_{\mu\nu}$.

For instance, in $\mathbb{R}\times\mathbb{R}^+$ the metric dx^2+dy^2 is the usual Euclidean one in which the squared length of the vector $a\partial_x + b\partial_y$ is a^2+b^2. If we employ the hyperbolic metric $y^{-2}\big(dx^2+dy^2\big)$ introduced by H. Poincaré, we get $(a/b)^2+1$. The length of a curve is given by $\int\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$. The shortest path joining two points, say $P_1=(-a,b)$ and $P_2=(a,b)$, in the Euclidean case is a straight line, but if we employ the hyperbolic metric, lengths become shorter at higher points at it can be proved that the shortest path is actually an arc of circle with a complicate parametrization.



dx^2+dy^2	geodesic = straight line
$\dfrac{1}{y^2}\big(dx^2+dy^2\big)$	geodesic = arc of circle

The shortest paths are geodesics and the idea to keep in mind is that *non-standard metrics bend the natural straight geodesics.*

The metrics appeared in relativity thanks to H. Minkowski (a former teacher of Einstein), who introduced the metric

$$c^2dt^2-dx^2-dy^2-dz^2$$

and proved that the Lorentz transformations are isometries, i.e. they preserve it. In this way, special relativity becomes geometric. In his address "Space and Time" [LEMWed] Minkowski told "space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality". He had invented the spacetime. This is very often attributed to Einstein, who, ironically, did not praise this mathematical approach (he said "Since the mathematicians have invaded the theory of relativity, I do not understand it myself anymore" [Sch49, p.102]). Later it was fundamental for general relativity (his regret in 1912 was "I have been instilled with great reverence for mathematics, the subtler parts of which I naively used to regard as luxury!"[1])

The geodesics for the Minkowski metric are straight lines and can be interpreted as a manifestation of the inertia principle. As in the previous example, a modification of the Minkowski metric bends the geodesics. The idea of the metric gravitational theories is to deal with metrics in which these geodesics give the equation of motion corresponding to the effects of gravitation. The context of Riemannian geometry gives also full covariance (in this sense, general relativity is *general*) and we can use any coordinate system (x^0, x^1, x^2, x^3) even without any physical meaning.

General relativity, as a metric theory, establishes a geometric interpretation of gravitation following the following dictionary:

Relativity		**Geometry**
spacetime	$\longrightarrow$	Lorentzian manifold M of dimension 4
gravity	$\longrightarrow$	metric on M with signature -2
rays of light	$\longrightarrow$	null geodesics
massive particles	$\longrightarrow$	timelike geodesics
proper time	$\longrightarrow$	parameter of the normalized timelike geodesics

The second point assures that the metric is a perturbation of Minkowski metric. The spacelike geodesics are less important because their tangent vectors have negative square length, according to the metric. In part of the literature (e.g. [Wei72]) the metric is conserved with opposite sign.

There is a fundamental missing point here and it is a way to infer the metric. This is the difference between different metric theories. In the case of general relativity, the admissible metrics are the solution of the so-called *field equations*. Very recently we celebrated the 100th anniversary of its publication (although they were published twice: by Einstein and by D. Hilbert [CRS97]) that is considered the birth of general relativity (although the theory was not presented as a whole until [Ein16]).

Before any consideration about the origin of the field equations one should wonder about the motivation for a metric theory of gravitation or even for any new theory of gravitation.

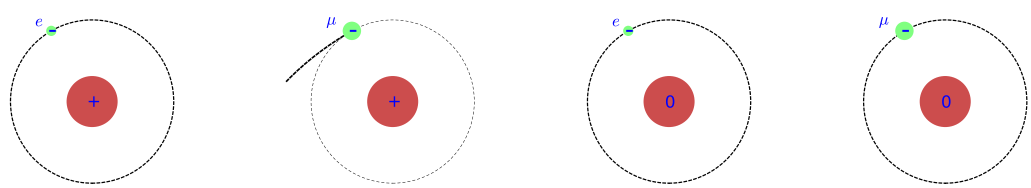
Special relativity was born by a clear reason: to conciliate kinematics and Maxwell equations. It was "needed" in the physics mainstream of the early years of XX century. In fact

[1] A facsimile of the whole letter to A. Sommerfeld can be found here:
http://einstein-virtuell.mpiwg-berlin.mpg.de/VEA/SC95647728_MOD129969944_SEQ-1877920857_SL-1688271241_de.html.

some of the fundamental formulas, the Lorentz transformations, were known before the seminal paper [Ein05] of Einstein, with a different meaning, and even a form (with a wrong factor [FLS64]) of the most famous formula in Science, $E = mc^2$, was already considered.

The situation with general relativity is completely different. The theory created by I. Newton worked perfectly and the unexplained perihelion shift of Mercury (see the last section) was so tiny that did not constitute a major problem.

In this context, the main motivation for a geometric theory of gravitation was of theoretical nature: the equivalence principle. There are several ways to state it. One of the most common is that, in few words, gravitation is indistinguishable from acceleration without gravitation. It implies that the inertial mass and the gravitational mass are identical. We are so used to this idea that very often we do not realize how singular is gravitation in comparison, for instance, with electrostatic fields, although the formulas for the forces are similar[2]. Let us exemplify it briefly imagining a very heavy atom obeying classical mechanics (of course such atom does not exist) and neglecting magnetic forces. An electron to a certain distance needs a certain initial velocity v_0 to keep a circular orbit. If we repeat the experiment with a muon μ^-, then the same initial velocity v_0 would give a very flat arc of a conic. This is natural, because a muon is a "fat electron" (around 200 times fatter) and with the same electric force it is more difficult to perturb its natural rectilinear movement. It is in our everyday experience that more massive implies more difficult to move or to stop.



But if we consider an hypothetical "gravitational atom" with a (very heavy) neutral nucleus that attracts gravitationally to an electron, there is no difference between putting an electron or a muon. In some way is like the nucleus knows that has to apply more force to heavy particles than to light particles to compensate the inertia, driving to the same trajectories.

A solution to avoid this perplexing situation is to assume that gravitation is something that is in the space(time). All the particles follow "straight lines" as in the case without gravity but now the definition of the straight lines is different from usual because the rule to measure lengths and angles has changed and is represented by a metric. It is like observing the planar projections of the trajectory of a marble when a child throws it close to a hole. The deformation with respect to the linear trajectories is due to the geometry of the hole, it is not attracting in any way the marble. But then one can wonder what is the role of the neutral nucleus, in the

[2]The memory of Einstein about the moment in which he thought about it is (according to [Pai82, Ch.9]) "I was sitting in a chair in the patent office at Bern when all of a sudden a thought occurred to me: 'If a person falls freely he will not feel his own weight.' I was startled. This simple thought made a deep impression upon me. It impelled me toward a theory of gravitation."

above example, if gravitation is a property of spacetime. In words of J.A. Wheeler "matter tells spacetime how to curve, and curved spacetime tells matter how to move".

2 The Newtonian limit

Summary. The Newtonian limit for metric theories and the possibility of recovering general relativity from a geometric statement of Newtonian theory based on the idea of geodesic deviation.

A valid metric theory has to have the Newtonian theory as a limit. It is possible to do simple calculations to get a basic property of the metric from this requirement. This was already done by Einstein in §E of [Ein16] (translated in [LEMWed]).

In a weak gravitational field, the metric $g_{\alpha\beta}$ is nearly Minkowskian, i.e.

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon h_{\alpha\beta}$$

with ϵ a small positive number. We assume as a part of the weak field hypotheses that the field is quasi-static with controlled spatial variations, meaning

$$h_{\alpha\beta,0} \approx 0 \qquad \text{and} \qquad |h_{\alpha\beta,i}| < C.$$

For the first condition, keep in mind the case the static case $h_{\alpha\beta,0} = 0$. One has to assume that $h_{\alpha\beta,0}$ is very small in comparison with ϵ anyway. The second condition is actually relevant for $\alpha = 0$.

The equations of motion are the equations of the geodesics and the latter require the computation of the Christoffel symbols, that involve the inverse matrix of $(g_{\alpha\beta})$. Let H be the matrix $(h_{\alpha\beta})$ and η the (diagonal) matrix of the Minkowski metric. As η^2 is the identity (in all of this work we assume relativistic units $c = 1$ if the contrary is not explicitly stated), we can approximate the inverse matrix of $(g_{\alpha\beta})$ as

$$(g^{\alpha\beta}) = (\eta + \epsilon H)^{-1} = (\eta(I + \epsilon\eta H))^{-1} = (I + \epsilon\eta H)^{-1}\eta \approx \eta - \epsilon\eta H\eta.$$

In fact the error term depends on ϵ^2. With this approximation

$$\Gamma^{\nu}_{\alpha\beta} = \frac{1}{2}g^{\nu\mu}(g_{\alpha\mu,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}) \approx \frac{1}{2}\epsilon\,\eta^{\nu\mu}(h_{\alpha\mu,\beta} + h_{\mu\beta,\alpha} - h_{\alpha\beta,\mu}). \tag{2.1}$$

The worldline of a massive particle with proper time τ parametrization is given by a timelike geodesic obeying the formula

$$\frac{d^2x^{\nu}}{d\tau^2} + \Gamma^{\nu}_{\alpha\beta}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau} = 0. \tag{2.2}$$

In the slow motion approximation (and implicitly assuming accelerations under control), we have

$$\frac{dx^0}{d\tau} \approx 1 \qquad \text{and} \qquad \frac{d^2x^i}{d\tau^2} \approx \Big(\frac{dx^0}{d\tau}\Big)^{-1} \frac{d}{d\tau}\Big(\frac{dx^i}{dx^0}\frac{dx^0}{d\tau}\Big) \approx \frac{d^2x^i}{dt^2}.$$

Substituting in (2.2)

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{00} \approx 0.$$

Using (2.1) and $h_{\alpha\beta,0} \approx 0$, it follows

$$\frac{d^2x^i}{dt^2} + \frac{1}{2}\epsilon h_{00,i} \approx 0 \qquad \text{that can be rephrased as} \qquad \Big(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\Big) \approx -\nabla\frac{g_{00}-1}{2}.$$

This is to be compared with the classic formula involving the potential Φ

$$\Big(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\Big) = \nabla\mathcal{U} \quad \text{with } \mathcal{U} = -\Phi \qquad \text{that leads to} \quad g_{00} \approx 1 - 2\mathcal{U}.$$

Then the Newtonian gravitation is represented by a metric of the form $(1-2\mathcal{U})dt^2 + \dots$ and the dots can be replaced by anything giving the Euclidean metric, with minus sign, in first approximation. The simplest is

$$(1-2\mathcal{U})dt^2 - dx^2 - dy^2 - dz^2. \tag{2.3}$$

With the previous arguments we have obtained the Newtonian limit of any hypothetical metric theory and we have obtained an approximation of the metric tensor. We have a lot of freedom to construct a theoretical theory of gravitation (of course, the experiments impose restrictions). Part of the formalism introduced latter is devoted to classify and interpret this freedom.

In this context, let us throw an apparently crazy idea: Can we obtain general relativity as a geometrization of Newton theory without extra ingredients? It seems impossible because the field equations are very complicate. We follow in the subsequent argument the spirit of the nice paper [BB05].

$$\begin{array}{ccc}
\boxed{\text{General relativity}} & & \boxed{\text{Newtonian theory}} \\
R_{\alpha\beta} = 0 & \xleftarrow{\text{geod. deviat.}} & \Delta\Phi = 0 \\
\dfrac{Du}{d\tau} = 0 & \xrightarrow{v/c\to 0} & \dfrac{d\vec{v}}{dt} = -\nabla\Phi \\
\Big\downarrow G \to 0 & & \Big\downarrow G \to 0 \\
\dfrac{du}{d\tau} = 0 & \xrightarrow{v/c\to 0} & \dfrac{d\vec{v}}{dt} = 0 \\
\text{(relativistic kinematics)} & & \text{(Newtonian kinematics)}
\end{array}$$

In the Newtonian theory, the gravitational potential Φ is the solution of the Poisson equation

$$\Delta\Phi = 4\pi G\rho \qquad \text{with} \quad \Delta = \delta^{ij}\partial_i\partial_j. \tag{2.4}$$

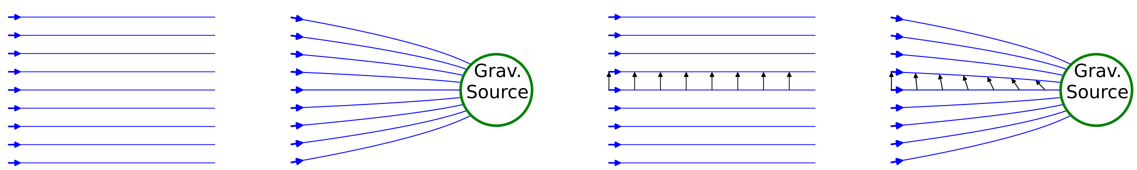
Instead of considering the equations of motion for a test particle, $d^2\vec{x}/dt^2 = -\nabla\Phi$, we are going to study how Φ affects to the relative motion of nearby particles.

If we throw parallel test particles when $\Phi = 0$, they continue being parallel. On the other hand, under the effect of a gravitational source, $\Phi \neq$ constant, the trajectories lose this property.

Let us parametrize the trajectories as $\alpha(t,s) = \big(x(t,s), y(t,s), z(t,s)\big)$ where t is the time and s selects the particle and let us say that $s = 0$ is the "central" particle. The vector field $\vec{V}$ along its trajectory that points to the "adjacent" particle is

$$\vec{V} = \frac{\partial\alpha}{\partial t}(t,0).$$

This vector field indicates a relative position.



The, so to speak, relative acceleration of the adjacent particle is

$$\frac{d^2V^i}{dt^2} = \frac{d^2}{dt^2}\frac{\partial\alpha^i}{\partial s}\Big|_{s=0} = \frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial^2\alpha^i}{\partial t^2} = -\delta^{ij}\frac{\partial}{\partial s}\Big|_{s=0}\partial_j\Phi\big(\alpha(t,s)\big) = -\delta^{ij}\big(\partial_k\partial_j\Phi\big)V^k$$

where we have employed the equations of motion. If we define $A^i_k = \delta^{ij}\partial_k\partial_j\Phi$ and we recall the Poisson equation (2.4), we can write this formula as

$$\frac{d^2V^i}{dt^2} + A^i_kV^k = 0 \qquad \text{with} \quad A^i_i = 4\pi G\rho. \tag{2.5}$$

In a metric theory, the trajectories are postulated to be geodesics. This identification gives for free, on purely geometric grounds, the geodesic deviation formula [dC92] [O'N83]

$$\frac{D^2V^\alpha}{dt^2} + \mathcal{A}^\alpha_\gamma V^\gamma \qquad \text{where} \quad \mathcal{A}^\alpha_\gamma = R^\alpha_{\beta\gamma\delta}\dot{x}^\beta\dot{x}^\delta \tag{2.6}$$

(where $R^\alpha_{\beta\gamma\delta}$ is the Riemann tensor) that is completely analogous to the previous physical equation.

Let us focus on the case $T_{\alpha\beta} = 0$ (see [BB05] for $T_{\alpha\beta} \neq 0$) that corresponds to $\rho = 0$ in the Newtonian case. Then the analog of $A^i_i = 0$ in (2.5) is $\mathcal{A}^\alpha_\alpha = 0$ in (2.6) that implies that $R^\alpha_{\beta\alpha\delta} = R_{\beta\delta}$ vanishes for general velocities. These are the right and exact general relativity field equations obtained from Newtonian grounds!

3 Post-Newtonian approximations

Summary. General ideas about post-Newtonian gravitation and the needed orders of approximation. It is exemplified with Schwarzschild's solution and illustrated with some historical comments.

Before suggesting any metric theory one must keep in mind the enormous success and accuracy of the Newtonian theory. Then any realistic theory would be capable of post-Newtonian approximations, i.e. the possibility of recovering the effects of the Newtonian theory in the case of slow motion (of the gravitational sources) and weak field and of predicting post-Newtonian corrections. In these and other names, "post" refers to go beyond the Newtonian theory.

There are two basic issues to settle when carrying out post-Newtonian approximations. First of all what are the quantities that participate in the approximation (physical observables that can be measured in an experiment). Secondly, to determine the degree of approximation to obtain post-Newtonian corrections to overcome Newtonian theory.

There is a certain subjectivity in the first issue that also affects to the second one because different quantities can be estimated with different precision. Then we should specify not only the quantities but also some hierarchy or comparison among them.

Consider the energy-momentum tensor[3] of a perfect fluid, which plays an important role in post-Newtonian approximations:

$$T_{\alpha\beta} = (\rho + \rho\Pi + p)u_\alpha u_\beta - p g_{\alpha\beta}. \tag{3.1}$$

Here ρ is the density of total (rest) mass-energy, $\rho\Pi$ takes into account the internal energy (being Π the internal energy per unit of ρ) that embodies non-gravitational energy, for instance coming from thermodynamics, p is the pressure and u_α is the 4-velocity.

As we have mentioned before, in the post-Newtonian approach we focus on slow motion (although its effectiveness is sometimes broader [Wil11]) then the last three components of the velocity are essentially $v^i = dx^i/dt$ with (t, x^1, x^2, x^3) a nearly globally Minkowskian coordinate system. To compare the size of the different matter variables appearing in the post-Newtonian formalism, we say that the velocity v is $O[1]$ and in general v^n is $O[n]$. Matching units in (3.1),

[3] The choice of an energy-momentum tensor disturbed Einstein (see for instance [EIH38]) who wrote in [Ein36] about the field equations: "it is similar to a building, one wing of which is made of fine marble (left part of the equation), but the other wing of which is built of low grade wood (right side of equation). The phenomenological representation of matter is, in fact, only a crude substitute for a representation which would correspond to all known properties of matter".

we have that Π and p/ρ are $O[2]$. The modern view of the Newtonian model is based on the gravitational potential $\mathcal{U}$ and then it is important to determine its comparative size. Recall that in simplified situations (a particle under a conservative force$\sim r^{-n}$) virial theorem assures that the time average of kinetic energy is proportional to the time average of the potential energy. Canceling the mass the potential and the squared velocity are comparable. In fact the Newtonian setting suggests $2\mathcal{U}$ to be added to Π in (3.1).

Summing up, in the post-Newtonian formalism, matter is mechanically described by the *matter variables* ρ, Π, v and p. The Newtonian force is represented by its corresponding potential $\mathcal{U}$ and it is assumed

$$\mathcal{U} = O[2], \qquad p/\rho = O[2], \qquad \Pi = O[2] \tag{3.2}$$

where $O[n]$ means a (small) upper limit that controls the nth power of the components of the velocity. Just to give a rough idea, the orbital speed of the Earth is about $1.01 \cdot 10^{-4}$ as a "dimensionless" quantity (corresponding to relativistic units $c = 1$). Then to study the orbit of the Earth around the Sun, we can think that $O[2]$ is roughly like 10^{-8}. Note that if we unwrap the relativistic units, we can interpret post-Newtonian approximations as expansions in the parameter V/c (assumed small) where V is an upper bound for the velocity of the matter (acting as a gravitational source). When V is bounded, the error terms are of the form $O(c^{-n})$. This way of presenting post-Newtonian approximations appears mainly in early papers on the topic [Cha65] but also in some recent works [Kli98].

Now we are going to study how far should we go in the expansion of the metric components to find corrections on the Newtonian theory. We follow the simple approach in [Wil85, §4.1] based on the action corresponding to a single particle of rest mass m

$$S = -m \int \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}.$$

The integral gives the arc length and then it is invariant under re-parametrization. Under our hypothesis of a nearly globally Minkowskian coordinate system, we use t (the Minkowski time) as the parameter. In this way

$$S = -m \int L\, dt \qquad \text{with} \quad L = \sqrt{g_{00} + 2g_{0j}v^j + g_{jk}v^j v^k} \quad \text{where} \quad v^i = \frac{dx^i}{dt}. \tag{3.3}$$

As we have seen, in the Newtonian limit (2.3)

$$g_{00} = 1 - 2\mathcal{U}, \qquad g_{0j} = 0, \qquad g_{ij} = -\delta_{ij} \qquad \text{and then} \quad L = \sqrt{1 - 2\mathcal{U} - v^2}. \tag{3.4}$$

Therefore the Newtonian limit corresponds to terms $O[2]$ in the Lagrangian L and adding $O[3]$ would make a difference. Nevertheless terms depending on odd powers of the velocity are not time reversal invariant, meaning a kind of dissipation of the energy. It suggests that terms of order exactly $O[3]$ usually do not appear in realistic models. The dissipation by gravitational radiation has a much smaller influence under our hypotheses to be include here (see §39.6 and Chapter 36 of [MTW73]). Then we can skip this order and consider the expansion of the Lagrangian L up to $O[4]$. Taking into account (3.3) and the Newtonian limit (3.4), in the post-Newtonian setting

$$g_{00} = 1 - 2\mathcal{U} + O[4], \qquad g_{0j} = O[3], \qquad g_{ij} = -\delta_{ij} + O[2]. \tag{3.5}$$

Note that, as in (2.3), space and time have a different character and *we should rather think that the post-Newtonian spacetime is composed by* 3 *dimensional spaces corresponding to each slice of time*. As it is said in [MTW73, §39.4] "The resultant three-dimensional formalism will look more like Newtonian theory than like general relativity –as, indeed one wishes it to; after all, one's goal is to study small relativistic corrections Newtonian theory!".

Just to illustrate (3.5), consider Schwarzschild's metric [Sch85]

$$\Big(1 - \frac{2GM}{r}\Big)dt^2 - \Big(1 - \frac{2GM}{r}\Big)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \; d\varphi^2 \tag{3.6}$$

that (with different coordinates) was introduced by K. Schwarzschild shortly after the presentation of the field equations. It is an exact solution modeling the external gravitational field of a spherical symmetric and static star of mass M. Changing r by $r(1 + GM/2r)^2$ and using the identity

$$1 - \frac{2GM}{r(1 + GM/2r)^2} = \frac{r - GM + G^2M^2/4r}{r(1 + GM/2r)^2} = \Big(\frac{1 - GM/2r}{1 + GM/2r}\Big)^2$$

we obtain (3.6) in its isotropic form

$$\Big(\frac{1 - GM/2r}{1 + GM/2r}\Big)^2 dt^2 - \Big(1 + \frac{GM}{2r}\Big)^4 \big(dx^2 + dy^2 + dz^2\big) \tag{3.7}$$

with $r^2 = x^2 + y^2 + z^2$, where we have employed that $dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \;\, d\varphi^2$ equals $dx^2 + dy^2 + dz^2$ in spherical coordinates. Recall that the Newtonian potential for this kind of stars (by the shell theorem) is the classic

$$\Phi = -\frac{GM}{r} \quad \text{that we represent with} \quad \mathcal{U} = \frac{GM}{r}.$$

On the other hand, we have the Taylor expansions

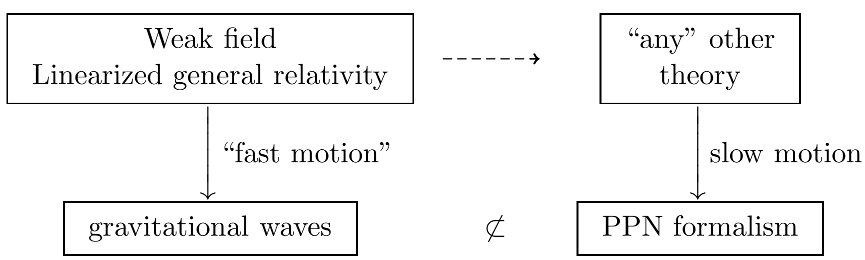
$$\Big(\frac{1 + x/2}{1 - x/2}\Big)^2 = 1 + 2x + 2x^2 + O(x^3) \qquad \text{and} \qquad \big(1 - \frac{x}{2}\big)^4 = 1 - 2x + O(x^2).$$

Using this in (3.7) we have that the post-Newtonian approximation (3.5) in this situation is

$$g_{00} = 1 - 2\mathcal{U} + 2\mathcal{U}^2, \qquad g_{0j} = 0, \qquad g_{ij} = -\delta_{ij}(1 + 2\mathcal{U}). \tag{3.8}$$

Of course, this is only valid under the hypotheses of the Schwarzschild solution. See [Wil85, §5.2] for a full analysis for general relativity.

Post-Newtonian approximations were performed from the very beginning of general relativity because the generic situation is that the field equations are too complicate to be solved (Einstein told around 1936 "If only it weren't so damnably difficult to find exact solutions!" [HT90, p.142]). In this way, the perihelion shift and the deflection of light were initially explained by Einstein [Ein15] [Ein16] using some kind of post-Newtonian approximation. In fact, as we shall see later, his initial wrong prediction about the deflection of light [Ein14] (included in [LEMWed]) can be understood today saying that he used the Newtonian limit instead of the post-Newtonian approximation.



Shortly after Einstein studied linearizations of the field equations to raise the possibility of gravitational waves (as a matter of fact, with ambiguous conclusions by mistake). From the practical point of view, with an eye to astrophysical applications, a basic question is the many-boy motion. Recall that in the Newtonian theory, already the 3-body problem (say the Sun-Earth-Moon system) does not admit a closed solution and there is current research yet even for particular cases [CM00] The problems appear in general for long-time evolution (chaos, stability) or high-accuracy computations because otherwise numerical methods work perfectly at Solar System scale. The situation with general relativity is quite different because its effects are only distinguishable with high precision experiments. Schwarzschild's solution just covers the 1-body problem at not fully because it is not valid for highly rotating stars (one should use Kerr's solution). The challenge of the many-body motion under general relativity was very early taken by W. de Sitter [de 16] and other authors. An important step was given by Einstein and L. Infeld [EI40] preceded by [EIH38] that anticipated some important ideas. Again in connection to the many-body motion, V. Fock [Foc64] and S. Chandrasekhar [Cha65] systematized the post-Newtonian approximations in a way that according to [Wil11] "laid the foundation for modern post-Newtonian theory".

In a different setting of taking into account perturbations or testing alternative theories, simple variations of (3.8) were considered in the 60's by H.P. Robertson, L.I. Schiff,

K. Nordtvedt and other authors (and much earlier by A.S. Eddington) [Rob62] [Sch60] [Nor68]. We shall treat this point later.

4 The potentials of the PPN formalism

Summary. Motivation for the PPN formalism and for the potentials appearing in it.

Each metric theory has its own post-Newtonian approximations and the way of finding the expansion of the metric for the orders indicated in (3.5) can be long and difficult and completely different in each case (see [Wei72, Ch.9], [Wil85, §5.2] for general relativity and [Wil85, Ch.5] for other theories).

Although these potentially big differences, the final expressions keep the same flavor, assuming that we expand in terms of functions involving the matter variables with some basic properties. In this context it appears the *parametrized post-Newtonian formalism*, usually abbreviated as PPN. It is a comprehensive family of post-Newtonian approximations depending on parameters that can be adjusted to represent, in practice, every meaningful metric theory of gravitation. Of course one can always introduce theories, metric or non metric, grounded on purely mathematical terms escaping from any given formalism but the scope of the experiments suggests that the PPN formalism is broad enough to cover any testable theory applicable to the Solar System (and surprisingly, sometimes, far beyond [Wil11]. See [MTW73] for the situation in 1973).

The aims of the PPN can be summarized in the following points:

1. Provide a framework to compare theories.
2. Standardize the way of presenting the experimental results.
3. Compute general relativity corrections in astrophysics.

(In [Wil73] the list a different: To suggest new experiments; To study and classify theories; To use experimental data to eliminate theories).

The first point is the main one. If two theories share the same PPN parameters they can only be distinguished with *post-post-Newtonian* experiments that are commonly out of reach with current methods because the PPN formalism mimics relativistic theories with high accuracy. The second point is closely related to the previous one. Any measurement giving a hint about the metric translates into a property of the parameters that are chosen to represent physical concepts. They are a kind of agreement to compare not only the theories but the particular experiments. The third point very often is not mentioned in the surveys on the topics (an exception is [MTW73, p.1073]) perhaps because the PPN formalism with the choice of the parameters corresponding to general relativity, strictly speaking, should be considered a special post-Newtonian approximation. But on the other hand, the papers [EIH38] [EI40] were important steps in the construction of the formalism although they only deal with general relativity, and the general equations of motion provided by the formalism [Wil85] constitute a way of studying the many-body problem in general relativity.

The history of the PPN formalism, as we have just illustrated, is not independent from the history of the PN approximations, even restricting it to the general relativity case. As the latter was considered in a previous section, we just mention here that arguably the first unified version is due to C. Will and K. Nordtvedt in [WN72]. It contains nine parameters and establishes the modern notation. The general metric is introduced in [WN72, II.c] with the "recipe" to get the parameters that essentially reduces to do the post-Newtonian approximation (this can be quite involved) and to match terms, with some special considerations about preferred frames. Shortly after, Will added a tenth parameter [Wil73]. This is the form included in [Wil85] and still in use.

Once we have fixed the matter variables, in principle there are infinitely many possibilities to construct functions of them appearing in the expansion of the metric. In [Wil85, §4.1d] there are seven restrictions, rather assumptions, on them. Instead of listing them, here we introduce the functions step by step with some brief motivation.

First of all, note that already in the Newtonian limit the potential $\mathcal{U}$ appears as a linear correction of η_{00}. It is a solution of (2.4) and the fundamental solution of this equation in gravitational units ($G = 1$) is $\rho|\vec{x}|^{-1}$, then changing the sign

$$\mathcal{U} = \int \frac{\rho(\vec{y}, t)}{|\vec{x} - \vec{y}|}\, d^3\vec{y}.$$

From here onwards we do not display the dependance of matter variables on time for the sake of brevity. The choice of $\mathcal{U}$ instead of the actual potential $\Phi = -\mathcal{U}$ is just a tradition [MTW73, §39.3].

As we have seen, $\mathcal{U} = O[2]$. In principle, we can use it in g_{ij}. By the separation of space and time (inherited from Newtonian mechanics, see above), we should add to $-\delta_{ij}$ an expression behaving as a 3-dimensional Euclidean tensor. A possibility is $\mathcal{U}\delta_{ij}$ and other of similar nature

$$\mathcal{U}_{ij} = \int \frac{\rho(\vec{y}, t)}{|\vec{x} - \vec{y}|} \frac{x_i - y_i}{|\vec{x} - \vec{y}|} \frac{x_j - y_j}{|\vec{x} - \vec{y}|}\, d^3\vec{y}.$$

Note that $\mathcal{U}$ and $\mathcal{U}_{ij}$ are dimensionless in our units $G = c = 1$ because $[\mathcal{U}] = ML^{-1}$ and $[G] = LM^{-1}(LT^{-1})^2$. We expect the post-Newtonian correction to decay at least as fast as r^{-1} There is not more room for dimensionless symmetric tensors of type $(0, 2)$ with this kind of decay and being $O[2]$. Note that for instance a similar tensor involving $v_i v_j$ gives $O[2] \cdot O[2] = O[4]$. Then these ideas lead to consider

$$g_{ij} = -\delta_{ij} + \mathcal{L}(U\delta_{ij}, \mathcal{U}_{ij}) \tag{4.1}$$

where $\mathcal{L}$ means a linear combination.

For g_{0j} the $O[3]$ size allows to introduce the velocity in $\mathcal{U}$, having in this way $O[2]\cdot O[1] = O[3]$. Note that one must keep the vector behavior. This leads to two possibilities:

$$V_j = \int \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|} v_j(\vec{y})\, d^3\vec{y} \qquad \text{and} \qquad W_j = \int \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}\frac{x_j-y_j}{|\vec{x}-\vec{y}|}\frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|}\cdot\vec{v}(\vec{y})\, d^3\vec{y}.$$

In this case,

$$g_{0j} = \mathcal{L}(V_j, W_j). \tag{4.2}$$

Under the same philosophy, the corrections to g_{00} are pure scalars and the freedom $O[4]$ gives many possibilities. Firstly we can consider $\mathcal{U}^2 = O[4]$. We can also introduce in the formula for $\mathcal{U}$ any factor $O[2]$. Recalling (3.2) the natural possibilities are

$$\Phi_1 = \int \frac{\rho(\vec{y})v^2(\vec{y})}{|\vec{x}-\vec{y}|}d^3\vec{y}, \quad \Phi_2 = \int \frac{\rho(\vec{y})\mathcal{U}(\vec{y})}{|\vec{x}-\vec{y}|}d^3\vec{y}, \quad \Phi_3 = \int \frac{\rho(\vec{y})\Pi(\vec{y})}{|\vec{x}-\vec{y}|}d^3\vec{y}, \quad \Phi_4 = \int \frac{p(\vec{y})}{|\vec{x}-\vec{y}|}d^3\vec{y}.$$

Thinking in contractions of the terms considered for g_{0j}, we have also

$$\mathcal{A} = \int \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}\Big(\frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|}\cdot\vec{v}(\vec{y})\Big)^2\, d^3\vec{y} \qquad \text{and} \qquad \mathcal{B} = \int \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}(\vec{x}-\vec{y})\cdot\frac{d\vec{v}(\vec{y})}{dt}\, d^3\vec{y}$$

(recall that $\vec{v}$ depends also on the coordinate t that is not displayed by brevity). The latter expression seems artificial, and in fact we shall drop it with a gauge choice, but it is a dimensionless scalar $O[4]$, as required.

Essentially this is the setting considered in [WN72]. Later it was introduced a new term in [Wil73], a kind of self-correlation of the density

$$\Phi_W = \int \rho(\vec{y})\rho(\vec{z})\frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|^3}\cdot\Big(\frac{\vec{y}-\vec{z}}{|\vec{x}-\vec{z}|} - \frac{\vec{x}-\vec{z}}{|\vec{y}-\vec{z}|}\Big)\, d^3\vec{y}\, d^3\vec{z}.$$

In principle there are other possibilities, for instance involving derivatives of the pressure, but they not appear in current theories. In fact, reading [Wil73] we see that the main reason to introduce Φ_W was to deal with a theory due to A.N. Whitehead, that is nowadays considered obsolete for most of the scholars. In that paper it is claimed "With no exception, every metric theory of gravity analyzed prior this paper fit[sic] into the PPN formalism [...]. The exception was Whitehead's theory, whose post-Newtonian metric contained an additional metric potential Φ_W [...]". It is fair to say that in [Wil73] there is also an interpretation in terms of gauge transformations.

Therefore, we have that the post-Newtonian approximation of the first coefficient of the metric is

$$g_{00} = 1 - 2\mathcal{U} + \mathcal{L}(U^2, \Phi_1, \Phi_2, \Phi_3, \Phi_4, \mathcal{A}, \mathcal{B}, \Phi_W). \tag{4.3}$$

The outcome of all of this analysis is that (4.1), (4.2) and (4.3) allow to express the coefficients of the post-Newtonian metric of any (nowadays) imaginable gravitation theory as a linear form in the twelve functions

$$\mathcal{U}\delta_{ij},\ \mathcal{U}_{ij},\ V_j,\ W_j,\ U^2,\ \Phi_1,\ \Phi_2,\ \Phi_3,\ \Phi_4,\ \mathcal{A},\ \mathcal{B} \text{ and } \Phi_W,$$

called post-Newtonian *potentials*.

5 The PPN metric and its parameters

Summary. A continuation of the previous section: the PPN metric is finally introduced and a brief summary of the meaning of the parameters is given.

In differential geometry, full covariance is itself a target for the sake of generality or beauty. In gravitation, it is very often employed as a tool to be able to choose a coordinate system with special physical properties or symmetries (e.g. Droste coordinates for Schwarzschild's solution).

We are going to use the freedom to change coordinates for specifying a physical context. the formula (4.1) means $g_{ij} = -\delta_{ij} + \lambda\mathcal{U}\delta_{ij} + \mu\mathcal{U}_{ij}$ where $\lambda\mathcal{U}\delta_{ij} + \mu\mathcal{U}_{ij} = O[2]$ is assumed to be a small correction on the Euclidean metric. It is natural to consider a frame in which this corrected metric does not show preferred directions (isotropic) and angles are not modified (conformal). In this natural frame the 3-dimensional metric g_{ij} should be diagonal (and, of course, close to $-\delta_{ij}$). We have to check that this is actually possible.

Consider the *superpotential* whose partial derivatives give the potential for g_{ij}. Namely,

$$\chi(\vec{x}) = -\int \rho(\vec{y})|\vec{x}-\vec{y}|\, d^3\vec{y} \quad \text{that verifies} \quad \frac{\partial^2\chi}{\partial x_i \partial x_j} = -\mathcal{U}\delta_{ij} + \mathcal{U}_{ij}.$$

Let us consider the change of coordinates

$$x^i = \widetilde{x}^i - \kappa g^{ij}\frac{\partial\chi}{\partial x^j}, \quad \text{then its jacobian matrix is} \quad \frac{\partial\widetilde{x}}{\partial x} \approx \mathrm{Id} - G^{-1}H$$

where G is the matrix of the metric (g_{ij}) and H is the Hessian matrix of $\kappa\chi$ with κ a constant, $H = \kappa\big(\partial^2\chi/\partial x^i\partial x^j\big)$. In the approximation, essentially we have not differentiated the metric because it behaves as a constant (in some sense we are considering derivatives as covariant derivatives. Recall Ricci's lemma). The effect on the metric is

$$\widetilde{G} = (\mathrm{Id} - G^{-1}H)^t G(\mathrm{Id} - G^{-1}H) = G - 2H + HG^{-1}H.$$

The last term is $O[4]$. Then, omitting negligible terms,

$$\widetilde{g}_{ij} \approx gh_{ij} - 2\kappa\frac{\partial^2\chi}{\partial x^i\partial x^j} = g_{ij} + 2\kappa\big(\mathcal{U}\delta_{ij} - \mathcal{U}_{ij}\big).$$

In this way, when $g_{ij} = -\delta_{ij} + \lambda\mathcal{U}\delta_{ij} + \mu\mathcal{U}_{ij}$ we can kill the term $\mu\mathcal{U}_{ij}$ with the previous change of coordinates choosing $\kappa = \mu/2$.

As noted before, there is a not explicitly indicated dependance on t. With this change, it affects to g_{00} and g_{0j} modifying the specific linear combinations but the general structures (4.3) and (4.2) are preserved.

It is also possible to perform a change only in time

$$t = \widetilde{t} - \kappa\Big(g^{00}\frac{\partial\chi}{\partial t} + g^{0j}\frac{\partial\chi}{\partial x^j}\Big)$$

where κ is again a constant to be adjusted. This change, of course, does not affect to g_{ij}. It preserves the structure of (4.2) and with some calculations [Wil85, p.97] one gets

$$\widetilde{g}_{00} \approx g_{00} - 2\kappa\big(\mathcal{A} + \mathcal{B} - \Phi_1\big).$$

Then with a suitable choice of κ we can drop any linear dependance on $\mathcal{A}$, $\mathcal{B}$ or Φ_1 in (4.3). As $\mathcal{B}$ is more involved, we prefer to cancel it.

Summarizing, with a judicious choice of the coordinates we can always disregard $\mathcal{U}_{ij}$ and $\mathcal{B}$. This is called the *standard post-Newtonian gauge*.

There remain 10 potentials that enter linearly in the post-Newtonian metric. In principle we can call their coefficients $\lambda_1, \lambda_2, \dots, \lambda_{10}$ and write (4.3), (4.2) and (4.1) in the standard gauge as

$$(5.1) \qquad \begin{cases} g_{00} = 1 - 2\mathcal{U} + \lambda_1\mathcal{U}^2 + \lambda_2\Phi_W + \lambda_3\Phi_1 + \lambda_4\Phi_2 + \lambda_5\Phi_3 + \lambda_6\Phi_4 + \lambda_7\mathcal{A} \\ g_{0j} = \lambda_8 V_j + \lambda_9 W_j \\ g_{ij} = -\delta_{ij} + \lambda_{10}\mathcal{U}\delta_{ij} \end{cases}$$

but this notation is not in use. The standard choice is a set of parameters called

$$\gamma,\ \beta,\ \xi,\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \zeta_1,\ \zeta_2,\ \zeta_3,\ \text{and}\ \zeta_4.$$

They are preferred because they have some physical significance (see Box 39.5 in [MTW73] for other sets of parameters and their translation). The linear relation between these physical parameters and the previous ones is

$$(5.2) \qquad \begin{cases} \lambda_1 = 2\beta \\ \lambda_2 = 2\xi \\ \lambda_3 = -2\gamma - 2 - \alpha_3 - \zeta_1 + 2\xi \\ \lambda_4 = -2(3\gamma - 2\beta + 1 + \zeta_2 + \xi) \\ \lambda_5 = -2(1 + \zeta_3) \end{cases} \qquad \begin{cases} \lambda_6 = -2(3\gamma + 3\zeta_4 - 2\xi) \\ \lambda_7 = \zeta_1 - 2\xi \\ \lambda_8 = \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi) \\ \lambda_9 = \frac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi) \\ \lambda_{10} = -2\gamma \end{cases}$$

The full covariance of general relativity allows to choose any coordinate frame but in some theories there is a kind of *universal rest frame*. Note that it contradicts Newton's first law but the absolute space introduced by him and its famous rotating bucket argument supports this view for acceleration[4].

The possible existence of a universal rest frame implies that PPN formalism is not invariant under Lorentz transformations. It is not obvious what is the concrete meaning of this assertion because we have taken some decisions about the gauge and the form of the PPN metric. The idea is to perform a change of coordinates mimicking the effect of a Lorentz boost

$$\tau = \frac{t - wx}{\sqrt{1-w^2}}, \qquad \xi = \frac{x - wt}{\sqrt{1-w^2}}, \tag{5.3}$$

followed by a gauge adjusting change [WN72] [MTW73, §39.9]. The change of of coordinates is [WN72, (13)]

$$\begin{cases} \tau = t\big(1 + \frac{1}{2}w^2 + \frac{3}{8}w^4\big) - \vec{x}\cdot\vec{w}\big(1 + \frac{1}{2}w^2\big) + tO[5] \\ \vec{\xi} = \vec{x} - \big(1 + \frac{1}{2}w^2\big)\vec{w}t + \frac{1}{2}(\vec{x}\cdot\vec{w})\vec{w} + xO[4] \end{cases}$$

where we assume $w = O[1]$. Note that

$$\frac{1}{\sqrt{1-w^2}} = 1 + \frac{1}{2}w^2 + \frac{3}{8}w^4 + \dots$$

then (for transversal velocities) it is an approximation of (5.3). Moreover

$$\frac{d\vec{\xi}}{d\tau} = \frac{d\vec{x}}{dt} - \vec{w} + O[3]$$

that keeps the interpretation of $\vec{w}$ as the relative velocity between both frames.

It is impossible to preserve the form of the metric (5.1) and we are forced to introduce again the potentials U_{ij} in g_{00} and g_{0j} to deal with this relative velocity. For instance, we have

$$g_{00}^{(w)} = g_{00} + (\alpha_1 - \alpha_2 - \alpha_3)w^2\mathcal{U} + \alpha_2 w^i w^j \mathcal{U}_{ij} + (\alpha_1 - 2\alpha_3)w^i V_i$$

where g_{00} is like in (5.1) and $g_{00}^{(w)}$ is the metric for velocity $\vec{w}$ with respect to the universal rest frame. We have independence of $\vec{w}$, and consequently no universal rest frame, if $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Then these parameters measure in some way the existence and influence of the rest frame.

The parameters ζ_1, ζ_2, ζ_3 and ζ_4 (and in part α_3) are linked to global conservation laws. It is difficult to give details without entering into cumbersome calculations (see [Wil85, §4.4]). We

[4] A different interpretation was given by E. Mach who claimed that the inertia here is generated by the far stars. Naturally, Einstein praised Mach and his amazing idea (although he was perhaps the only outstanding physicist doubting the existence of the atoms in the 20th century).

just mention some basic general ideas. Noether's theorem implies conservation of currents in the form $\partial_\mu J^\mu = 0$. In the Newton setting they are translated into integral (global) conservation laws. For instance, for the electromagnetic field we have

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{J} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}\int_V \rho = \int_V \frac{\partial \rho}{\partial t} = -\int_{\partial V} \vec{J}$$

and if $\vec{J}$ vanishes at infinity, we have that the total charge $\int \rho$ remains constant. This is "global" because it does not depend on a property measured at a local frame. In a geometric theory the conservation law for the energy-momentum tensor $T^{\mu\nu}_{;\nu} = 0$ cannot be translated in general into a global conservation law because the presence of the Christoffel symbols prevent from separating a component as we did with the charge density above. Nevertheless with the PPN formalism there is a chance to consider a wise variation to be conserved after the approximation. This is promising because in the Newtonian theory we have conservation laws. It turns out that a kind of conservation of the total momentum can be stated if $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$. This is the situation in general relativity and in conservative theories. These parameters then measure a kind of deviation from the conservation law for momentum.

As we have said before, the parameter ξ was introduced to cover a specific gravitational theory. But it can be interpreted in terms of the motion with respect to the universal rest frame [Wil73] and in Table 4.3 of [Wil85] it is said, without further explanations, that it measures "preferred-location effects". This parameter is also null in general relativity.

When all the considered parameters vanish, only β and γ remain and the metric (5.1) becomes

$$\begin{cases} g_{00} = 1 - 2(\mathcal{U} + \Phi_3) + 2\beta\mathcal{U}^2 - (2\gamma+2)\Phi_1 - 2(3\gamma - 2\beta + 1)\Phi_2 - 6\gamma\Phi_4 \\ g_{0j} = \big(2\gamma + \frac{3}{2}\big)V_j + \frac{1}{2}W_j \\ g_{ij} = -\delta_{ij}(1 + 2\gamma\mathcal{U}) \end{cases}$$

In the case $p = \Pi = 0$ (what occurs with point masses) and static (or very slow motion), Φ_j, V_j and W_j are negligible and we arrive to the metric considered by Robertson and Schiff (see [Rob62] [Sch60])

$$(5.4) \qquad \big(1 - 2\mathcal{U} + 2\beta\mathcal{U}^2\big)dt^2 - (1 + 2\gamma\mathcal{U})\big(dx^2 + dy^2 + dz^2\big).$$

Here γ measures the amount of space curvature induced by the gravitational field. The length is modified with respect to the usual Euclidean metric in a proportion given by

$$\frac{1}{\sqrt{1 + 2\gamma\mathcal{U}}} - 1 \sim -\gamma\mathcal{U}.$$

On the other hand, β allows to control how nonlinear the theory is. The constant factors 2 multiplying β and γ are motivated by (3.8).

The equations of motion under the post-Newtonian (5.1) are quite complicate specially for massive bodies if one wants to keep the goal of covering all reasonable theories because the motion of self-gravitating bodies can depend on its internal structure (this is called the Nordtvedt effect). Then the center of the mass of the Earth could not to follow a geodesic. The problem of motion already called the attention of the authors of the early paper [EIH38]. In the first lines it is made the strong claim "Previous attacks on this problem have been based upon gravitational equations in which some specific energy-momentum tensor for matter has been assumed. [The] entry [of these tensors] into the equations makes it impossible to determine how far the results obtained are independent of the particular assumption made concerning the constitution of matter".

We are going to consider the post-Newtonian equations of motion for photons, that are specially simple in comparison with those of massive bodies. In a metric theory, photons follow null geodesics, i.e.

$$g_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}=0 \qquad \text{and} \qquad \frac{d^2x^\lambda}{dp^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}=0. \tag{5.5}$$

Here p is a non-physical parameter. To enter into post-Newtonian approximations, we would like to separate space and time expressing in the trajectories the space coordinates as a function of the time. In the first equation, replacing p by t is harmless just multiplying by dp/dt twice. For the second equation, note that by the chain rule

$$\frac{d^2x^\lambda}{dt^2}=\frac{d}{dt}\Big(\frac{dx^\lambda}{dp}\frac{dp}{dt}\Big)=\frac{d^2x^\lambda}{dp^2}\Big(\frac{dp}{dt}\Big)^2+\frac{dx^\lambda}{dp}\frac{d^2p}{dt^2}.$$

Then multiplying the second equation of (5.5) by $(dp/dt)^2$, we have

$$\frac{d^2x^\lambda}{dt^2}-\frac{dx^\lambda}{dp}\frac{d^2p}{dt^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=0.$$

Taking $\lambda=0$ ($x^0=t$) and multiplying by dx^λ/dt we have

$$-\frac{dx^\lambda}{dt}\frac{d^2p}{dt^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\lambda}{dt}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=0.$$

Subtracting both equations, we have finally that (5.5) turns into

$$g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=0 \qquad \text{and} \qquad \frac{d^2x^j}{dt^2}+\Big(\Gamma^j_{\mu\nu}-\Gamma^0_{\mu\nu}\frac{dx^j}{dt}\Big)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=0.$$

With the PPN metric (5.1), (5.2), the Christoffel symbols are quite complicate (see Table 6.1 in [Wil85]) but we only need post-Newtonian precision: First corrections on straight lines. The resulting equations are

$$
(5.6) \qquad \begin{cases} 1 - 2\mathcal{U} - |\frac{d\vec{x}}{dt}|^2(1 + 2\gamma\mathcal{U}) = 0 \\ \frac{d^2x^j}{dt^2} = \frac{\partial\mathcal{U}}{\partial x^j}\Big(1 + \gamma|\frac{d\vec{x}}{dt}|^2\Big) - 2\frac{dx^j}{dt}\Big(\frac{d\vec{x}}{dt} \cdot \nabla\mathcal{U}\Big)(1 + \gamma) \end{cases}
$$

Note that in first approximation we have the Newtonian equation $d^2\vec{x}/dt^2 = \partial\mathcal{U}/\partial x^j$.

The only PPN parameter appearing in these equations of motion is γ. This is relevant from the experimental point of view because it allows to separate the measurement of γ and to design simpler experiments. In fact, as we shall see, the deflection of light was the first prediction of general relativity being checked.

6 The classical tests and the equivalence principle

Summary. Statement of the classical tests and a critic view of the relation of two of them with the equivalence principle.

Usually the following facts are considered the classical tests of general relativity

(i) The gravitational red shift

(ii) The deflection of light

(iii) The perihelion shift of Mercury

Really only (iii) was experimentally known when the general relativity was born. In fact it should not be considered a completely independent experiment because the theory was guided to explain this phenomenon (and an early non covariant wrong theory by Einstein also explained it).

In [Wil85], (i) is substituted by the time delay of light because it is claimed that "[it] is really not a test of general relativity, rather it is a test of the Einstein equivalence Principle". Schiff made a stronger claim in 1960 [Sch60]: "Only the planetary orbit precession provides a real test of general relativity". It is difficult to share this opinion today because years later W. Rindler [Rin68] proved that [Sch60] was flawed (we shall see it in an indirect way in the next section), but the point here is to distinguish specific general relativity effects from the common effects of many reasonable metric theories. For instance, very often it is said that most of the relativistic corrections letting GPS works come from general relativity but, rigorously speaking, it comes from the equivalence principle that would be considered a principle underlying any acceptable theory. According to [Sch60] "it will be extremely difficult to design a terrestrial or satellite experiment that really tests general relativity, and does not merely supply corroborative evidence for the equivalence principle and special relativity".

In the next lines we follow mainly [Sch60] to deduce (i) from the equivalence principle and we also review briefly some points in its history. That of the other tests is treated in subsequent sections. We also include the deduction of (ii) from the equivalence principle in [Sch60] because, although it is not valid, it leads to the correct result and has some interest.

Imagine two clocks A and B placed at rest over the surface of the Earth in the same radial line at distances r_A and r_B from the center of the Earth. From the equivalence principle (and Newton's law) this is equivalent to assume no gravitational field and upward accelerations GM/r_A and GM/r_B where M is the mass of the Earth. If we compare with an inertial clock C in the trajectories of A and B in this fictitious system, by Lorentz contraction, we have

$$T_A = \frac{T_C}{\sqrt{1-v_A^2}} \qquad \text{and} \qquad T_B = \frac{T_C}{\sqrt{1-v_B^2}} \tag{6.1}$$

where T_C is the period of C and T_A and T_B are the periods of A and B observed by C.

Basic classical mechanics gives $v_B^2 - v_A^2 = 2GM/r_B - 2GM/r_A$ and for non-relativistic velocities

$$\frac{T_B}{T_A} \sim \sqrt{1+v_B^2-v_A^2} = \sqrt{1+2GM(r_B^{-1}-r_A^{-1})} \sim \sqrt{\frac{1-2GM/r_A}{1-2GM/r_B}}.$$

This implies that oscillatory phenomena under a gravitational field seems to be slower when observed by a distant observer [Wei72]. In particular, spectral lines from stars should appear red shifted by effect of the gravity (at long scale the cosmological red shift is largely dominant). In fact using the Sun to perform the experiment was suggested by Einstein and even in [Ein14] he refers shyly in a footnote to available experimental data that could support the result but he doubts if they are due to the influence of pressure and temperature.

There was another dubious experiment in 1925. The definitive verification is nowadays attributed to the *Pound-Rebka experiment*. It was done in a tower 22.5 meters high in Harvard using gamma rays and atoms of iron to detect frequencies. Since then, the gravitational red shift has been measured in many experiments (see Table 2.3 in [Wil85] and the updates in [Wil01]). It seems that the highest precision was reached in 1976 comparing a maser clock on Earth and another to an altitude of about 10000km.

Let us now analyze the deflection of light with similar, but more involved, arguments. Consider a ray of light following the horizontal line $y = R_0 \neq 0$ and we want to study the deflection induced by a point mass located at the origin of the XY plane.

Let us say that the measurements of time and radial space are (t', r') for an observer far from the mass (essentially not affected by gravitation). We also consider another observer close to the mass and to the trajectory of the ray, measuring (t, r). By (6.1) and a similar argument involving length contraction instead of time dilation[5], we have

$$dt' = \frac{dt}{\sqrt{1 - 2GM/r}} \qquad \text{and} \qquad dr' = dr\sqrt{1 - 2GM/r}.$$

The lengths orthogonal to radial directions are not affected by gravitation, then we have by Pythagorean theorem $dx^2 - dr^2 = (dx')^2 - (dr')^2$. Substituting dr'

$$dx^2 = (dx')^2 + \frac{2GM}{r} dr^2 \qquad \text{that implies} \qquad (dx)^2 \Big(1 - \frac{2GM}{r}\frac{x^2}{r^2}\Big) = (dx')^2$$

because for y =constant, $dr/dx = x/r$. The velocity of the ray of light measured by the close observer is $1 = dx/dt$ (because the system ray-observer can be considered an accelerated system under no gravitational forces by the equivalence principle) and by the the far observer is

$$\frac{dx'}{dt'} \sim \frac{dx}{dt}\sqrt{1 - \frac{2GM}{r}\frac{x^2}{r^2}}\sqrt{1 - \frac{2GM}{r}} \sim 1 - \frac{2GM}{r}\Big(1 + \frac{x^2}{r^2}\Big). \tag{6.2}$$

The rest of the argument is mainly geometric. By elementary differential geometry, the integral of the curvature is the variation of the angle of the tangent vector. The ansatz is that we have a small deflection, then approximately the angle of deflection is

$$\alpha = \int_{-\infty}^{\infty} \kappa \, dx \quad \text{with } \kappa \text{ the curvature of the trajectory of the ray.}$$

By Huygen's principle (or variations on the Snell's law and Fermat's principle) the curvature κ at a point (x, y) is approximated by the derivative of dx'/dt' with respect to y at $y = R_0$. This is not obvious, although is mentioned in [Sch60] and [Ein16] without any detail. See [Ein14] for a more careful explanation. An idea is that the secondary circular waves become osculating circles. A calculation, using $r^2 = x^2 + y^2$, proves

$$\frac{\partial}{\partial y}\Big(1 - \frac{2GM}{r}(1 + \frac{x^2}{r^2})\Big) = -\frac{\partial}{\partial y}\Big(\frac{2GM}{r} - \frac{GMy^2}{r^3}\Big) = 4\frac{GMy}{r^3} - 3\frac{GMy^3}{r^5}.$$

The formula

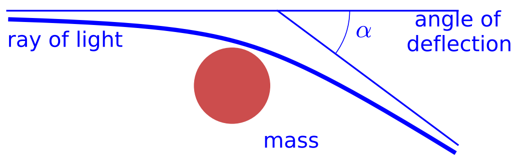
$$\int_{-\infty}^{\infty} (1 + x^2)^{-\beta} = \frac{\pi\Gamma(2\beta - 1)}{2^{2\beta-2}\Gamma^2(\beta)} \quad \text{implies} \quad \int_{y=R_0} \frac{dx}{r^3} = \frac{2}{R_0^2} \quad \text{and} \quad \frac{dx}{r^5} = \frac{4}{3R_0^3}. \tag{6.3}$$

Then the angle of deflection is

$$\alpha = \frac{4GM}{R_0}. \tag{6.4}$$

[5]Rindler [Rin68] criticized this claim and showed that in a parallel gravitational field the length contraction is not the inverse of the time dilation, as claimed. With this purpose, he introduced the Rindler's metric that represents a uniformly accelerating frame.

This angle is the excess over the angle π. If the ray comes from the left $x = -\infty$ and $y = R_0 > 0$, it will be bent to negative values values of y eventually after passing the mass ($x = 0$). In the Newtonian setting it is like thinking that photons are massive and hence affected by gravitational forces.



von Soldner 1804	$\to \alpha \sim 2GM/R_0$
Einstein 1911	$\to \alpha \sim 2GM/R_0$
Einstein 1916	$\to \alpha \sim 4GM/R_0$

7 The deflection of light

Summary. After a historical digression, a post-Newtonian study of the deflection of light and extra considerations about the effect of a negligible highly spinning mass.

Among the classical tests, probably the deflection of light has the most involved history.

In 1911 Einstein predicted a gravitational deflection of the rays of light [Ein14] and he suggested to several astronomer to carry on the experiment. With the modern view, he was using the Newtonian limit (2.3) without the post-Newtonian effects (3.5). In fact, he was unaware that more that 100 years before, G.J. von Soldner had stated a similar prediction using basic Newtonian theory. The best situation considered to observe the effect was to compare pictures of stars visually close to the Sun during a total solar eclipse. These stars during the eclipse should appear displaced in comparison with their position when the Sun is not visually close. WWI and bad weather prevent the experiment from being done. In 1916 [Ein16], with general relativity created, Einstein changed completely the prediction to the double of the previous value.

In 1919 a favorable solar eclipse took place especially visible at the island of Principe and an expedition went there (and to other station in Brazil) to do the measurements. Very often in the literature this is presented as the definitive corroboration of general relativity but the truth is that the precision of the experiment was controversial from the very beginning. This is still evident in old books written many years later like [Ber42]. Even in the bestselling modern book [Haw88] S.W. Hawking claims that experiment was not valid. In the conclusions of the original paper [DED20], it calls the attention of the suspicious reader that two measures, not matching the prediction, are rejected, and for the remaining one the expected value is barely covered by the estimated interval. Probably the origin of part of the controversy is due to the fact that Eddington seemed biased to accept Einstein (second) prediction.

On the other hand, the author of [Ken09] points out that the reanalysis of the data done in 1979 proves that the experiment was valid. In the final part of this paper, the sometimes alleged prejudices of Eddington are balanced saying "I argue that they had reasonable grounds for making their central claim that their results were not compatible with Einstein's".

This controversy has become just historical because the deflection of the light has been verified with completely reliable experiments (see [Wil01]). One of the most evident astronomical realizations is the gravitational lensing.

We are going to consider now the deflection of the light from the post-Newtonian point of view. Recall that we had the equations of motion for photons (5.6). We consider the potential corresponding to the solar eclipse experiment, $\mathcal{U} = GM/r$ with M the mass of the Sun. Let us say, as in the previous section, that the ray comes from the left along the horizontal line $y = R_0 > 0$ with the Sun located at the origin and R_0 very close to its radius. By the symmetry we can assume $z = 0$. In the post-Newtonian setting we assume a small correction on straight lines. Then we write

$$\vec{x}(t) = \big(t - M + \epsilon_1(t), R_0 + \epsilon_2(t), 0\big) \qquad \text{with } \dot{\epsilon}_1, \dot{\epsilon}_2 \text{ small.}$$

By the first equation in (5.6),

$$(1+\dot{\epsilon}_1)^2 + \dot{\epsilon}_2^2 = \frac{1 - 2GM/r}{1 + 2\gamma GM/r} \sim \Big(1 - \frac{2GM}{r}\Big)\Big(1 - 2\gamma\frac{GM}{r}\Big) \sim 1 - 2(1+\gamma)\frac{GM}{r}.$$

The first term is approximated by $1 + 2\dot{\epsilon}_1$. Then we conclude

$$\frac{dx}{dt} = 1 - (1+\gamma)\frac{GM}{r}.$$

The arguments leading to (6.4) from (6.2) can be copied mutatis mutandi to obtain the angle of deflection

$$\alpha \sim \int_{y=R_0} \frac{\partial}{\partial y}\Big(1 - (1+\gamma)\frac{GM}{r}\Big)dx = (1+\gamma)GMR_0 \int_{y=R_0} r^{-3} dx$$

and (6.3) gives finally

$$\alpha \sim 2(1+\gamma)\frac{GM}{R_0}$$

In the Newtonian limit $\gamma = 0$, compare (2.3) and (5.1) or (5.4). Substituting M and R_0 this gives for the Sun the deflection of $0.83''$, the value wrongly predicted in [Ein14]. For general relativity, $\gamma = 1$ and one gets the double of this value as claimed correctly in [Ein16]. This formula implies that metric theories with different values of γ give different deflections and it proves indirectly that the deflection of light cannot be explained using only the equivalence principle. In particular the arguments by Schiff reproduced in the previous section cannot be based just in the equivalence principle. As noted in [Rin68] the flaw is in the treatment of the space coordinates that would give an inconsistent result for a parallel gravitational field.

The value of γ has been tested experimentally many times (see [Wil01]). The sharpest results come from the tracking of Cassini space probe (a probe launched on 1997 mainly to study Saturn) and suggest $\gamma = 1 \pm 2.3 \cdot 10^{-5}$.

We finish this section with a slightly original approach related to Problem 20.4 of [LPPT75]. In this problem it is considered an approximation to the deflection of the light when one takes into account the angular momentum. An "exact" approach would require the Kerr solution [HT90] but here the point is to find approximation in the lines of the post-Newtonian setting. In [LPPT75] the effect of the static mass is separated from the angular moment and consequently the mass is set to 0 to study the pure angular moment effect.

The metric is in this case approximated by [MTW73, (19.13)]

$$dt^2 + 4\epsilon_{jki} S^k \frac{x^l}{r^3}\, dt dx^j - dx^2 - dy^2 - dz^2$$

where $x = x^1$, $y = x^2$, $z = x^3$, as usual. In [LPPT75] the problem is reduced to the case with light rays included in a plane orthogonal to the angular momentum because in the other coordinate planes there is no deflection (we include below comments on this and on the reduction $M = 0$). The use of spherical coordinates and conservation laws solves the problem but at the cost of long calculations. In particular the non-displayed calculations to arrive to [LPPT75, (6)] require some effort. This a nonlinear equation whose solution is approximated by a perturbation method. In our approach we reduce the amount of computations (all of them are displayed in detail), we do not rely on non-obvious conservation laws and we get a differential equation that can be solved exactly.

By the rotational symmetry, we can assume that the angular momentum is of the form $(S^1, S^2, S^3) = (0, 0, J)$

$$dt^2 + 4Jr^{-3} dt(xdy - ydx) - dx^2 - dy^2 - dz^2 \tag{7.1}$$

In cylindrical coordinates the associated Lagrangian is

$$L = \dot{t}^2 + \frac{4Jr^2}{(r^2+z^2)^{3/2}} \dot{\alpha}\dot{t} - \dot{r}^2 - r^2\dot{\alpha}^2 - \dot{z}^2.$$

Recall that the ray of light is contained in an the orthogonal plane to $(0, 0, J)$, with a translation $z \mapsto z + k$ we can take $z = 0$. The variables t and α are cyclic giving

$$\begin{cases} \dot{t} + \dfrac{2J}{r}\dot{\alpha} = K_1 \\ -\dfrac{2J}{r}\dot{t} + r^2\dot{\alpha} = K_2 \end{cases} \quad \text{that, disregarding terms in } J^2\text{, implies} \quad \begin{cases} \dot{\alpha} = \dfrac{K_2}{r^2} + \dfrac{2J}{r^3}K_1 \\ \dot{t} = K_1 - \dfrac{2J}{r^3}K_2 \end{cases}$$

We have a null geodesic, then

$$\dot{t}^2 + \frac{4J}{r}\dot{\alpha}\dot{t} - \dot{r}^2 - r^2\dot{\alpha}^2 = 0 \quad \text{that implies} \quad \dot{r}^2 - K_1^2 + \frac{K_2^2}{r^2} + \frac{4J}{r^3}K_1K_2 = 0$$

where, again, we omit the terms in J^2. We approximate

$$\dot{r}^2 = \big(\frac{dr}{d\alpha}\big)^2\Big(\frac{K_2}{r^2} + \frac{2J}{r^3}K_1\Big)^2 \sim \big(\frac{dr}{d\alpha}\big)^2\frac{K_2^2}{r^4}\Big(1 - \frac{4JK_1}{K_2 r}\Big)^{-1}$$

that when substituted in the last equation leads to

$$\big(\frac{dr}{d\alpha}\big)^2 = \frac{K_1^2}{K_2^2}r^4 - \frac{4JK_1^3}{K_2^3}r^3 - r^2.$$

As the middle coefficient is assumed to be small, the derivative es zero when $r = K_2/K_1$, hence $b = K_2/K_1$. This ordinary differential equation can be solved explicitly. The quickest way is the change $r = u^{-1}$ and completing squares with $v = u + 2J/b$

$$\big(\frac{du}{d\alpha}\big)^2 = b^{-2} - \frac{4J}{b^3}u - u^2 \qquad \text{and} \qquad \big(\frac{dv}{d\alpha}\big)^2 = b^{-2} - v^2.$$

that implies $v = b^{-1}\cos(\alpha + \alpha_0)$. We can rotate the system and assume $\alpha_0 = 0$ (closest point at $\alpha = 0$) then

$$u(\alpha) = \frac{\cos\alpha}{b} - \frac{2J}{b^3} \qquad \text{and} \qquad u = 0 \quad \text{implies} \quad \alpha \approx \pm\frac{2J}{b^2}.$$

We have asymptotes at this angles, then the deflection amounts $-4J/b^2$. The sign comes from the minus sign in the formula for u.

8 The perihelion shift

Summary. The classical perihelion shift analysis with no restrictions about the eccentricity and some considerations about a possible Newtonian explanation.

In the mid-19th century, U. Le Verrier noticed a tiny anomaly in the motion of Mercury. In an idealized form, its elliptic orbit suffers a rotation of $43''$ per century, known as perihelion shift. This quantity is, of course, very small and it seems even less relevant when we know that in the actual measurements the influence of the rest of the planets contributes with a shift of around $530''$ per century[6]. In fact, it was conjectured a hypothetical small planet, provisionally

[6]The precession of the equinoxes (the continuous change in the orientation of the axis of the Earth) induces an almost ten times bigger apparent contribution.

named Vulcan explaining the excess in the shift. Curiously enough, it was even "discovered" and "observed" several times (of course we now consider that they were experimental errors).

General relativity offers an explanation predicting a variation of an angle

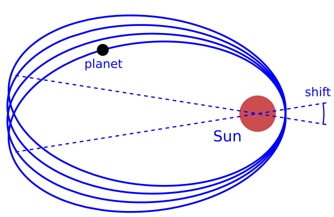
$$\alpha \sim 3\pi GM\Big(\frac{1}{r_a}+\frac{1}{r_p}\Big) \tag{8.1}$$

per each orbital period, where r_a and r_p are the (distances to) the aphelion and perihelion. This was firstly stated in [Ein15]. Substituting the orbital data of Mercury we have a quite good agreement with the experimental data.

In the PPN formalism, difficult and long computations with the equations of motion (not completely reproduced in [Wil85, §7.3]) show that in this general context the shift is given by

$$\alpha \sim \pi GM\big(\frac{1}{r_a}+\frac{1}{r_p}\big)\Big(2+2\gamma-\beta+(2\alpha_1-\alpha_2+\alpha_3+2\zeta_2)\frac{\mu}{2M}\Big)+\frac{3\pi}{4}J_2R^2\big(\frac{1}{r_a}+\frac{1}{r_p}\big)^2$$

where μ is the reduced mass (of the system Sun-planet), J_2 is the solar quadrupole moment and R is the mean radius of the Sun. For general relativity $\beta=\gamma=1$, $\alpha_1=\alpha_2=\alpha_3=\zeta_2=0$ and one recovers (8.1) when $J_2=0$. A problem appeared in the late 60's and early 70's when a measurement of J_2 gave a correction on (8.1) not matching astronomical data, raising in this way some doubts on general relativity. But more recent estimates (see the updates in §14 of [Wil85]) suggest that the measured value of J_2 was wrong. With the present available data [Wil01], we can say that general relativity has surmounted so far any experimental test at Solar System scale.



General relativity predictions

Mercury	→	42.98"/century
Venus	→	8.63"/century
Earth	→	3.84"/century
Mars	→	1.35"/century

In many books of general relativity (e.g. [Sch85], [LPPT75]) it is assumed a nearly circular orbit to get (8.1). This is not very convincing by two reasons: The main interest is to apply it to Mercury and its orbit is clearly eccentric. On the other hand, in a nearly circular orbit the perihelion is not well-localized, meaning that there is a wide band of points more or less to the same distance of the focus and any loss in a possible approximation method could change substantially the estimated position of the perihelion.

We introduce a mathematical device (that is implicit in [Ein15]) to treat the case of non negligible eccentricity avoiding these drawbacks.

Consider a cubic polynomial $P(x) = \epsilon x^3 - bx^2 + cx - d$ with $\epsilon, b > 0$, and assume that it has three positive real roots $x_1 < x_2 < x_3$. Then we are going to prove that

$$\int_{x_1}^{x_2} \frac{dx}{\sqrt{P(x)}} \sim \frac{\pi}{\sqrt{b}} + \frac{3\pi}{4b^{3/2}}(x_1 + x_2)\epsilon \qquad \text{when } \frac{\epsilon x_2}{b} \text{ is small.} \tag{8.2}$$

By Vieta's formulas, we know that $x_3 = b\epsilon^{-1} - x_1 - x_2$, and $P(x) = \epsilon(x - x_1)(x_2 - x)(x_3 - x)$ implies

$$P(x) = b(x - x_1)(x_2 - x)\big(1 - \epsilon b^{-1}(x + x_1 + x_2)\big) \sim \frac{b(x - x_1)(x_2 - x)}{\big(1 + \epsilon(x + x_1 + x_2)/2b\big)^2}$$

because $1-h \sim (1+h/2)^{-2}$. Changing variables $x = y-(x_1+x_2)/2$ and writing $l = (x_2-x_1)/2$, we have that the considered approximated integral is

$$\int_{x_1}^{x_2} \frac{1 + \epsilon(x + x_1 + x_2)/2b}{\sqrt{b(x - x_1)(x_2 - x)}} dx = \frac{1}{\sqrt{b}} \int_{-l}^{l} \frac{1 + 3\epsilon(x_1 + x_2)/4b}{\sqrt{l^2 - y^2}} dy$$

and this elementary integral (arcsin) gives (8.2).

To deduce (8.1) we employ the Lagrangian $L = (1 - 2GM/r)\dot{t}^2 - (1 - 2GM/r)^{-1}\dot{r}^2 - r^2\dot{\varphi}^2$ corresponding to the planar motion ($\theta = \pi/2$) under Schwarzschild's metric (3.6). The quantities $(1 - 2GM/r)\dot{t}$ and $r^2\dot{\varphi}$ are constants of motion that we call $1/\sqrt{E}$ and $J/\sqrt{E}$ (we follow [MTW73] although renaming E^{-1} as $E + 1$ seems more appropriate, since the next equation). Using the definition of the proper time ($L = 1$), we obtain the energy equation for massive particles

$$\Big(\frac{dr}{d\tau}\Big)^2 + V_{\text{eff}} = \frac{1}{E} \qquad \text{with} \qquad V_{\text{eff}} = \Big(1 - \frac{2GM}{r}\Big)\Big(1 + \frac{J^2}{Er^2}\Big). \tag{8.3}$$

Multiplying by $(d\tau/d\varphi)^2 = r^4EJ^{-2}$, we have that there exists a cubic polynomial Q with $Q(0) = 2GM$ and linear coefficient -1 such that

$$\big(\frac{dr}{d\phi}\big)^2 = rQ(r) \qquad \text{and changing } r = u^{-1}, \quad \big(\frac{du}{d\phi}\big)^2 = P(u) \quad \text{where } P(u) = u^3Q(1/u).$$

Note that P is also a cubic polynomial and its leading coefficient is $2GM$ and its quadratic coefficient -1. The first two smallest roots u_a and u_p of P correspond to the inverse of the aphelion and perihelion (V_{eff} reaches firstly a maximum and later a minimum).

Starting from the aphelion when passing to the next perihelion u increases, reciprocally, when passing form the previous perihelion to the current aphelion, u decreases. Then we have the following formulas for the angle variation:

$$\phi_{\text{next per.}} - \phi_{\text{aph.}} = \int_{u_a}^{u_p} \frac{du}{\sqrt{P(u)}}, \qquad \phi_{\text{aph.}} - \phi_{\text{prev. per.}} = \int_{u_p}^{u_a} \frac{du}{-\sqrt{P(u)}}.$$

Adding both formulas and using (8.2) with $\epsilon = 2GM$ and $b = 1$, we get that the perihelion advances as stated in (8.1)

$$2\int_{u_a}^{u_p} \frac{du}{\sqrt{P(u)}} - 2\pi \sim 2\Big(\frac{\pi}{1} + \frac{3\pi}{4}(u_a + u_p)2GM\Big) - 2\pi = 3\pi GM\Big(\frac{1}{r_a} + \frac{1}{r_p}\Big).$$

Note that $\epsilon u_p/b \approx M/r$ is small for any planet in the Solar System.

As we have mentioned before, a proposal to explain the perihelion shift of Mercury using Newtonian mechanics was the existence of a hypothetical new little planet. A more reasonable proposal is the influence of the nonzero quadrupole moment of the Sun (this calculation is the content of [LPPT75, 15.7]), as it was later considered in the PPN approach to general relativity. The gravitational potential Φ becomes $-\Phi = \mathcal{U} = GM/r + AGM/r^3$ with A a constant related to the quadrupole moment. As it corresponds to a central force, the orbits are planar (assume $\theta = \pi/2$) and the angular moment (per unit of mass) $J = r^2 d\phi/dt$ is a constant of motion. The conservation of the energy reads

$$\frac{1}{2}\Big(\big(\frac{dr}{dt}\big)^2 + r^2\big(\frac{d\phi}{dt}\big)^2\Big) - \frac{GM}{r} - A\frac{GM}{r^3} = E$$

or equivalently, we have the following Newtonian analog of (8.3)

$$\big(\frac{dr}{dt}\big)^2 = 2E + \frac{2GM}{r} - \frac{J^2}{r^2} + 2\frac{AGM}{r^3}.$$

Dividing by $(d\phi/dt)^2 = J^2/r^4$, as before,

$$\big(\frac{dr}{d\phi}\big)^2 = rQ(r) \qquad \text{and changing } r = u^{-1}, \quad \big(\frac{du}{d\phi}\big)^2 = P(u) \quad \text{where } P(u) = u^3 Q(1/u).$$

with Q a cubic polynomial such that its constant term is $2AGM/J^2$ and its linear coefficient is -1. Consequently P is also a cubic polynomial and $2AGM/J^2$ and -1 are, respectively, the leading and quadratic coefficients. The approximation formula (8.2) gives as before that the perihelion advances

$$2\int_{u_a}^{u_p} \frac{du}{\sqrt{P(u)}} - 2\pi \sim 2\Big(\frac{\pi}{1} + \frac{3\pi}{4}(u_a + u_p)\frac{2AGM}{J^2}\Big) - 2\pi = 3\pi\frac{AGM}{J^2}\big(\frac{1}{r_a} + \frac{1}{r_b}\big).$$

This means that we can mimic the relativistic perihelion shift (8.1) if the quadrupole moment of the Sun and the angular moment of the planet satisfy $A = J^2$. There are two experimental ways to rule out this possibility. The first is the admitted measurement of the quadrupole moment. The second is that even with an ad hoc value of A, it is impossible to match simultaneously the perihelion shift of Mercury and Venus.

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