Gravitational lensing: The number of images

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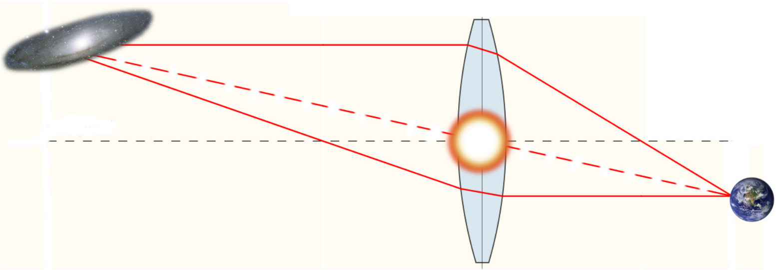
Msc Theoretical Physics

May 31, 2016

Gravitational lensing

A massive object deflects the light and it behaves as a lens.



The observer sees distorted images of the background objects.

Einstein was rather skeptical in 1939 about the possibility of measuring gravitational lensing:

Of course, there is no hope of observing this phenomenon directly. First, we shall scarcely ever approach closely enough to such a central line. Second, the angle β will defy the resolving power of our instruments. For, α_0 being of the order of magnitude

deviating star B is seen, is much smaller. Therefore, the light coming from the luminous circle can not be distinguished by an observer as geometrically different from that coming from the star B, but simply will manifest itself as increased apparent brightness of B.

Even in the most favorable cases the length l is only a few light-seconds, and x must be small compared with this, if an appreciable increase of the apparent brightness of A is to be produced by the lens-like action of B.

Therefore, there is no great chance of observing this phenomenon, even if dazzling by the light of the much nearer star B is disregarded. This apparent

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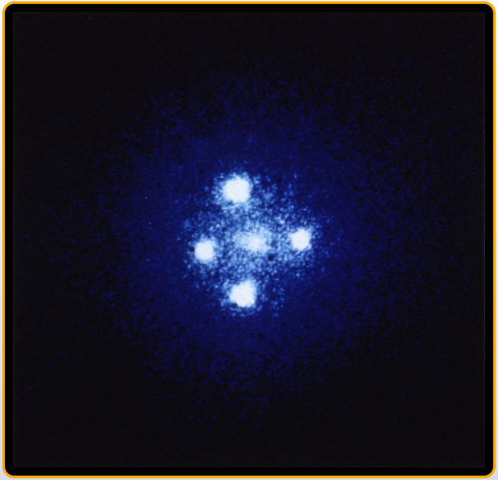
Fortunately, he was too skeptical and since its astronomical discovery in 1979, we have amazing images like...

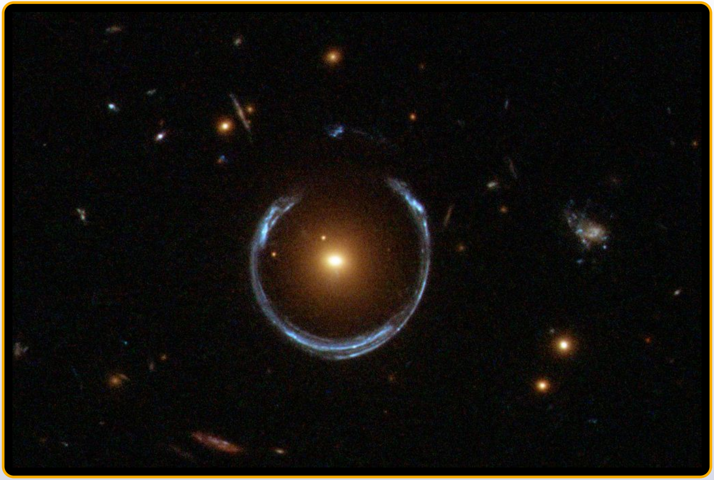
a cross



Credit: NASA, ESA, & STScI.

Source: `http://hubblesite.org/newscenter/archive/releases/1990/20/image/a/`

a fried egg



Credit: ESA/Hubble & NASA.

Source: http://apod.nasa.gov/apod/image/1112/lensshoe_hubble_3235.jpg

and even a smiley!

Credit: NASA & ESA.

Source: `http://www.spacetelescope.org/images/potw1506a/`

A naive goal

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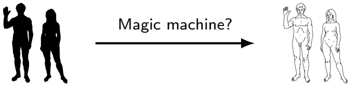
Can we say something about the structure of the lens from the image? This is not a well-posed problem. It is like trying to reconstruct a human body from just one radiograph.



(from the Pioneer plaque)

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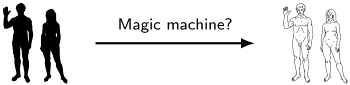

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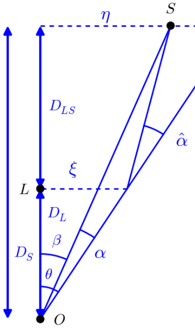
Nevertheless, we can study if our educated guess matches the properties of the observed lensing. The property considered here is the number of images.

Making the lens equation complex

[Straumann, 1997]



$$\boxed{\beta = \theta - \frac{D_{LS}}{D_S}\hat{\alpha}} \text{ (lens equation)}$$

$D_S\beta = \vec{\eta},\ D_L\theta = \vec{\xi}$ (ang. diam. dist.)

$$\hat{\alpha} = 4G\int \frac{\vec{\xi}-\vec{\zeta}}{|\vec{\xi}-\vec{\zeta}|^2}\Sigma(\vec{\zeta})\, d^2\zeta$$

(width of the lens $\ll$ dist(O, L), dist(S, L))

$$\left.\begin{matrix}\text{Lens plane}\\ \text{Source plane}\end{matrix}\right\} \leftrightarrow \text{copies of } \mathbb{C}$$

Renaming, with some normalization, $\vec{\eta}, \vec{\xi}, \vec{\zeta}$ $(\in \mathbb{R}^2) \to w, z, \zeta$ $(\in \mathbb{C})$ and using $\frac{\xi-\zeta}{|\xi-\zeta|^2} = \frac{1}{z^*-\zeta^*}$

Complex form of the lens equation

$$w = z - \int_L \frac{d\sigma(\zeta)}{z^* - \zeta^*}$$

where $d\sigma/d\mathrm{Area}$ is essentially the surface mass density.

Chang–Refsdal lens $\to$ Lensing of stars (or point-masses) in a background galaxy (originally a quasar).

More complex form of the lens equation

$$w = z - \int_L \frac{d\sigma(\zeta)}{z^* - \zeta^*} - \gamma z^*$$

where γ is the background shear.

The number of images

For point masses $d\sigma =$ sum of Dirac deltas and we have

$$w = z - \sum_n \frac{\mathcal{M}_n}{z^* - z_n^*} - \gamma z^*$$

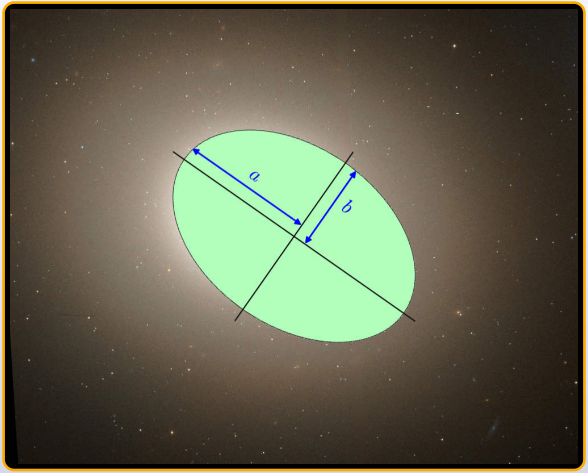
where z_n are the positions and $\mathcal{M}_n$ are related to the masses.

In any case

 # Images = # solutions (in z) for w fixed

Perhaps, we could use the number of images not as a signature but as a hint for some kind of structures.

Idealized elliptical galaxies



[Fassnacht, Keeton, Khavinson, 2009]

$d\sigma = K dx\, dy$ (evenly distrib.), lens $= E = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}$.

The integral can be computed explicitly!

The complex lens equation becomes ($c =$ focal distance)

$$\begin{cases} w = z + \dfrac{2ab}{c^2}\left(z - \sqrt{z^2 - c^2}\right)^* - \gamma z^* & \text{if } z \notin E \\[2ex] w = \dfrac{a^2 + b^2 - 2ab}{c^2} z^* - \gamma z^* & \text{if } z \in E \end{cases}$$

The second equation is linear in x and y ($z = x + iy$). Then for each w in the source there is at most one solution $z \in E$.

The first equation is more complicated

$$w = z + \frac{2ab}{c^2}\left(z - \sqrt{z^2 - c^2}\right)^* - \gamma z^*$$

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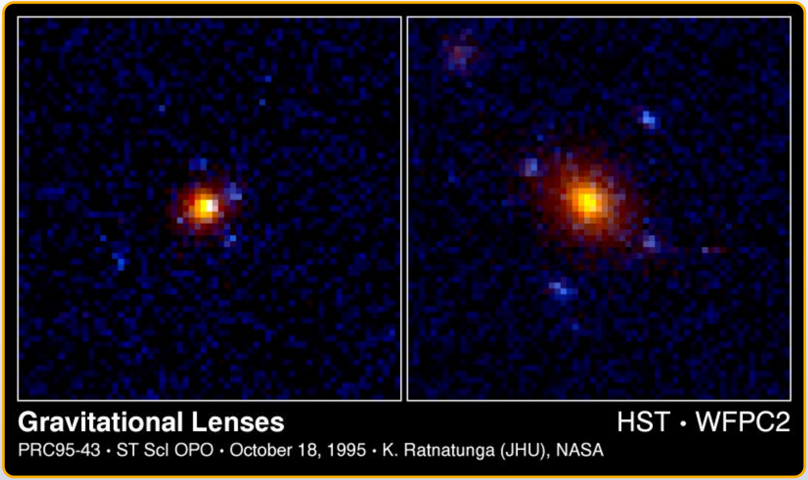
$$\begin{cases} \text{First equation} & \to \text{at most 4 solutions} \\ \text{Second equation} & \to \text{at most 1 solutions} \end{cases}$$

In this idealized lensing with elliptical lenses

At most 4 clear images $+$ 1 probably invisible image

A real photo for the ideal model

Unfortunate low quality. Blame Hubble telescope!



Source: `http://hubblesite.org/newscenter/archive/releases/1995/43/image/a/`

Less idealized elliptical galaxies

(Galaxy M60) Credit: NASA, & STScI.

[Keeton, Mao, Witt, 2000]

A uniform distribution of the mass in a galaxy is far from being realistic. Very often for gravitational lensing it is assumed isothermal or isothermal elliptic density $\propto r^{-2}$

Some authors have studied the problem in a more general setting allowing the galaxies to be an ellipsoid.

In this general situation the caustics admit explicit formulas. But too long to be displayed here.

For (planar) elliptic galaxies with isothermal density. The complex lens equation leads to study the number of preimages of the function

$$F(z) = z - K \int_0^c \left((z^*)^2 - u^2\right)^{-1/2} du - \gamma z^*.$$

Perhaps this is not so hard!

There is a result (the so-called odd number theorem) that implies that when there is no shear, the number of images is odd. It possible to give a simple proof in the planar case, essentially an application of the argument principle

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} = \#\text{zeros} - \#\text{poles}.$$

Radial symmetry

[An, Evans, 2006]

Astronomically a star ($\neq$ Sun) is a point mass and the complex lens equation becomes

$$w = z - \frac{K}{z^*} - \gamma z^* \qquad \text{(Chang–Refsdal lens)}$$

(take $K = 1$ for simplicity).

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(take $K = 1$ for simplicity).

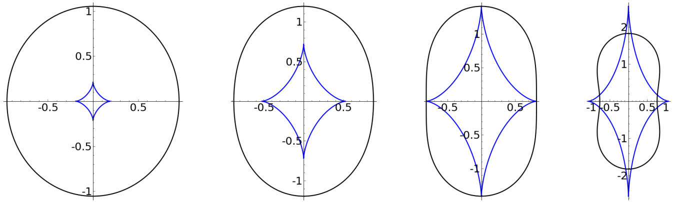
For $\gamma = 0$, one gets readily the Einstein ring: $w = 0 \Rightarrow |z|^2 = 1$.

In general, the Jacobian determinant is

$$\frac{\partial w}{\partial z}\frac{\partial w^*}{\partial z^*} - \frac{\partial w}{\partial z^*}\frac{\partial w^*}{\partial z} = 1 - \left|\gamma - z^{-2}\right|.$$

Then the critical curve is $z^2 = \left(\gamma + e^{i\theta}\right)^{-1}$ and we get also the caustics.

Caustics and **critical curves** for several values of the shear:



γ=0.1 γ=0.3 γ=0.5 γ=0.7

One can count the number of images substituting the equation into itself:

$$w = z - \frac{1}{z^*} - \gamma z^* \qquad \Longrightarrow \qquad z^* = w + \frac{1}{z} + \gamma z.$$

Then

$$\frac{\sum_{n=0}^{4} a_n(w,\gamma) z^n}{z(1 + w^* z + \gamma z^2)} = 0 \qquad \Longrightarrow \qquad \text{4 images generically.}$$

Surprisingly this analysis generalizes to any $d\sigma$ with radial symmetry. The key point is by Cauchy's integral formula

$$\int_{|\zeta|=r} \frac{1}{z-\zeta}\frac{d\zeta}{\zeta} = \int_0^{2\pi} \frac{ir}{z - re^{i\theta}} = \begin{cases} -2\pi i/z & \text{if } |z| < r \\ 0 & \text{if } |z| > r \end{cases}$$

Then for $d\sigma = \rho(r)\, dA$,

$$\int_{\mathbb{C}} \frac{d\sigma(\zeta)}{z-\zeta} = \int_0^\infty \int_0^{2\pi} \frac{\rho(r) r\, dr d\theta}{z - re^{i\theta}} = -\frac{2\pi}{z} \int_0^{|z|} \rho(r) r\, dr.$$

If ρ is supported in $r < R$ then the integral is a constant for $|z| > R$ (outside of the lens) and we recover the Chang-Refsdal lens equation (with a different constant).

The counting of the images reduces to study how many zeros of the polynomial $\sum_{n=0}^{4} a_n(w,\gamma) z^n$ lie in the lens (the support of ρ).

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