Two not so well known Taylor expansions

Fernando Chamizo

March 9, 2022

Abstract

We give a proof with almost no prerequisites of the Taylor expansions of $(\arcsin x)^2$ and $(\text{arcsinh } x)^2$ where arcsin and arcsinh are the inverse functions of sin and sinh in a neighborhood of the origin. These expansions are not usually covered in Calculus courses.

1 Some fairly known related series

Consider $f(x) = (1+x)^{-1/2}$. It is clear that its n-th derivative is of the form $f^{(n)}(x) = c_n(1+x)^{-1/2-n}$. Computing $f^{(n+1)}$ one deduces $c_{n+1} = -\left(\frac{1}{2}+n\right)c_n$ with $c_0 = 1$. It is easy to check that $(-1)^n 2^{-2n} n! \binom{2n}{n}$ satisfies this recurrence, giving a closed expression for c_n. Then we have the Taylor expansion

$$(1) \qquad \frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \binom{2n}{n} x^n.$$

Without any consideration about the growth of the central binomial coefficients, we know that this is an actual equality for any x in the open unit disk because $f(z) = (1+z)^{-1/2}$ defines a holomorphic function there. It also proves that we can replace x by x^2 and integrate term by term, getting

$$(2) \qquad \text{arcsinh } x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \binom{2n}{n} \frac{x^{2n+1}}{2n+1} \qquad \text{for} \quad |x| < 1.$$

Recall that arcsinh x is the inverse function of $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ and its derivative is $(1+x^2)^{-1/2}$. As a matter of fact, $\text{arcsinh } x$ equals $\log\left(x + \sqrt{x^2+1}\right)$.

Replacing instead x by $-x^2$ in (1), we get in the same way

$$(3) \qquad \arcsin x = \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \binom{2n}{n} \frac{x^{2n+1}}{2n+1} \qquad \text{for} \quad |x| < 1.$$

In fact, (2) and (3) are equivalent because by the Euler formula, we know $\sinh(ix) = (e^{ix} - e^{-ix})/2 = i \sin x$, which implies $\arcsin t = -i\, \text{arcsinh}(it)$.

2 The expansions

We are going to show the Taylor expansions, valid for $|x| < 1$,

$$(4) \quad (\arcsin x)^2 = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{2n^2 \binom{2n}{n}} \qquad \text{and} \qquad (\text{arcsinh } x)^2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2x)^{2n}}{2n^2 \binom{2n}{n}}.$$

As before, $\arcsin t = -i \operatorname{arcsinh}(it)$ shows that they are equivalent and it is enough to prove the second one.

Consider the integral

$$\int_0^1 \int_0^1 \frac{2\sinh u \cosh u \; dt}{\cosh^2 u - t^2 \sinh^2 u} \, du \, dt$$

and apply Fubini theorem. To perform first the integration in u, we substitute $\cosh^2 x = 1 + \sinh^2 x$ and to reverse the order of integration, we use the partial fraction decomposition

$$\frac{2\sinh u \cosh u}{\cosh^2 u - t^2 \sinh^2 u} = \frac{\sinh u}{\cosh u + t \sinh u} + \frac{\sinh u}{\cosh u - t \sinh u}.$$

Doing both calculations and noting $\cosh u \pm \sinh u = e^{\pm u}$, we have

$$\int_0^1 \frac{\log\big(1 + (1-t^2)\sinh^2 x\big)}{1-t^2} \, dt = \int_0^1 \log \frac{\cosh u + \sinh u}{\cosh u - \sinh u} \, du = x^2.$$

The Taylor expansion $\log(1+x) = \sum (-1)^{n+1} x^n / n$ shows

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} I_{n-1} \sinh^{2n} x = x^2 \qquad \text{with} \quad I_k = \int_0^1 (1-t^2)^k \, dt.$$

Then (4) follows if we prove $n\binom{2n}{n} I_{n-1} = 2^{2n-1}$. This is plain for $n = 1$ and follows easily by induction using $(2n+1)I_n = 2nI_{n-1}$. This relation is well known. A simple proof consists in writing $(1-t^2)^{n-1}$ as $(1-t^2)^{n-1}(1-t^2+t^2)$ to get

$$2nI_{n-1} - 2nI_n = 2n\int_0^1 (1-t^2)^{n-1} t^2 \, dt = 2n \int_0^1 (1-t^2)^{n-1} t^2 \, dt + \int_0^1 d\big(t(1-t^2)^n\big).$$

Expanding $d\big(t(1-t^2)^n\big)$ we cancel the previous integral and we get an extra I_n.

3 A combinatorial reformulation

We can get $x^{-2}(\arcsin x)^2$ in two ways: squaring the expansion (3) and using directly (4). Shifting n by one in the latter formula, we get the shocking relation

$$\left(\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \binom{2n}{n} \frac{x^{2n}}{2n+1} \right)^2 = \sum_{n=0}^{\infty} \frac{2^{2n+1} x^{2n}}{(n+1)^2 \binom{2n+2}{n+1}}.$$

When we open the square, the coefficient of x^{2n} is

$$\sum_{k=0}^{n} \frac{1}{2^{2n}} \cdot \frac{\binom{2k}{k}\binom{2n-2k}{n-k}}{(2k+1)(2n-2k+1)}.$$

Comparing it with the coefficient in the right hand side and cleaning a little the result, we get the cumbersome relation:

$$\frac{(2n+2)!}{(n!)^2} \sum_{k=0}^{n} \frac{\binom{2k}{k}\binom{2n-2k}{n-k}}{(2k+1)(2n-2k+1)} = 2^{4n+1}.$$

It would be nice to have a simple combinatorial interpretation of this formula because it would give an alternative combinatorial proof of the expansions (4).