Bonnet's recursion from Rodrigues

Introduction. Legendre polynomials are very important in mathematics and physics. They can be defined in several ways. One of them is the *Rodrigues formula*

$$P_n = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

An algorithmic friendly alternative is Bonnet's recursion formula

$$(n+1)P_{n+1} = (2n+1)x\,P_n - n\,P_{n-1} \qquad \text{with} \quad P_0 = 1, \quad P_1 = x.$$

Many people have probably proven the latter from the former, but I was unable to find such proof on the internet. I finally gave up my search and wrote my own proof on the back of an envelope. It is convenient to abbreviate $X_n = 2^n n! P_n$. With this notation the recursion becomes

$$X_{n+1} = (4n+2)xX_n - 4n^2 X_{n-1}.$$

By the Leibniz product rule

$$(A_n) \qquad X_{n+1} = \frac{d^{n+1}}{dx^{n+1}}\big((x^2-1)(x^2-1)^n\big) = (x^2-1)X_n' + \binom{n+1}{1}2xX_n + \binom{n+1}{2}2\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n.$$

On the other hand, computing a derivative before the application of the Leibniz product rule

$$(B_n) \qquad X_{n+1} = \frac{d^n}{dx^n}\big(2(n+1)x(x^2-1)^n\big) = 2(n+1)xX_n + 2(n+1)\binom{n}{1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n.$$

With $2(A_n)-(B_n)$ the nasty last term is eliminated to get

$$(C_n) \qquad X_{n+1} = 2(n+1)xX_n + 2(x^2-1)X_n'.$$

Finally, $(C_n) - 2nx(C_{n-1}) + 2(x^2-1)(B_{n-1})'$ is

$$X_{n+1} - 2nxX_n + 2(x^2-1)X_n' = \\ 2(n+1)xX_n - 4n^2x^2X_{n-1} + 2(x^2-1)(X_n' - 2nxX_{n-1}') + 2(x^2-1)(2n^2X_{n-1} + 2nxX_{n-1}').$$

The terms with X_n' and X_{n-1}' cancel out and the expected recursion comes out.

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