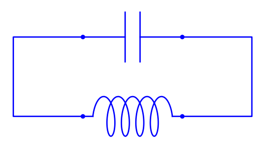
According to our previous study, the resonance happens for $\omega = \omega_0$ with $\omega_0 = (LC)^{-1/2}$ and at large, the current behaves as $I = I_0 \sin(\omega t - \delta)$ for some δ. With our coefficients

$$(1.24) \qquad V_0 = I_0 Z \qquad \text{where} \qquad Z = \sqrt{R^2 + L^2\omega^2(1 - \omega_0^2/\omega^2)^2}.$$

If $\omega = \omega_0$ then $Z = R$ considering only the amplitudes (forgetting the phases) the circuit behaves as if the capacitor and the inductance does not furnish any resistance. On the other hand if ω is not close to ω_0, then Z is much bigger than R. This is the principle to tune a specific radio station or TV channel. In the mathematical context, this primitive machine that allows to select with certain precision specific frequencies is a gateway to an electronic computation of Fourier expansions.

Note that if you omit the source and the resistor in (1.21) the new equation is

$$(1.25) \qquad I'' + (LC)^{-1} I = 0$$



and we have in theory a tireless harmonic oscillator if the capacitor is initially charged. In practice, there is always some resistance in the conductors and, as in a free pendulum, the oscillations fade away quickly. To achieve a real electronic oscillator you have to introduce some kind of amplification. In the early days it was achieved with vacuum tubes (valves) and later with transistors.

Suggested Readings. For a basic mathematically oriented introduction to the Maxwell equations and its relation to modern theoretical physics, I recommend the recent book [Gar15]. The solution of the equations and its meaning is very well explained in the modern classic [FLS64].

1.1.3 Sound waves

The sound consists of changes of pressure that can be detected by the human ear. With some approximations and basic Physics we are going to convince ourselves that it is transmitted as a wave.

If a particle of the air is in a certain position we want to study its displacement u when time evolves and it is disturbed by the sound. We assume that the perturbation acts in the same way at every horizontal line, in other words, $u = u(x,t)$ and we can focus on the X axis. The changes in the pressure p are related to changes in the density ρ. For the sound there are not big variations with respect to the normal pressure and density, say p_0 and ρ_0, then we can write

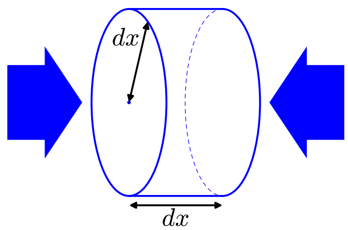
$$(1.26) \qquad p(x,t) = p_0 + p_\epsilon(x,t) \qquad \text{and} \qquad \rho(x,t) = \rho_0 + \rho_\epsilon(x,t)$$

meaning that p_ϵ and ρ_ϵ are much smaller than the constants p_0 and ρ_0. They express some kind of perturbation.

It seems natural than p and ρ have to related in some way by an equation of state[2]. Whatever it is, everything is linear at small scales and then it is not far from the truth to assume $p_\epsilon = \kappa\rho_\epsilon$ for some constant $\kappa > 0$ that measures in some way the elasticity of the air.

We proceed using $F = ma$, as usual in very basic mechanic. The mass of a small cylinder of radius and height (along the X axis) of length dx is approximately $\pi\rho_0(dx)^3$. The forces to each side are $p(x,t)\pi(dx)^2$ and $-p(x+dx,t)\pi(dx)^2$ (recall pressure = force/surface).

(1.27) $F = p(x,t)\text{Area}$



$F = p(x+dx,t)\text{Area}.$

Dividing by $\pi(dx)^3$, the approximation of $F = ma$ becomes when $dx \to 0$

$$-\frac{\partial p_\epsilon}{\partial x} = \rho_0 \frac{\partial^2 u}{\partial t^2}. \tag{1.28}$$

On the other hand, when time evolves, by effect of the displacement from the undisturbed situation, the height of the cylinder pass from dx to $d(x+u)$. The mass must be preserved, then $\rho_0\pi(dx)^3 = \rho d(x+u)\pi(dx)^2$. As $\rho_0/\rho = 1 - \rho_\epsilon/\rho \sim 1 - (\kappa\rho)^{-1}p_\epsilon$ we can approximate

$$-p_\epsilon = \kappa\rho\frac{\partial u}{\partial x} \sim \kappa\rho_0\frac{\partial u}{\partial x} \tag{1.29}$$

and substituting in (1.28) we obtain the wave equation

$$\kappa\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}. \tag{1.30}$$

Probably your mathematical brain is complaining "too many approximations for me". Do not forget that models are models, a sometimes lousy approximation to reality. Anyway it is amazing that we can say something. Gravitational waves have been recently detected and they come from making up a wave equation out of a linear approximations of a scaring nonlinear equation involving the curvature tensor that we have no idea how to solve.

It is not necessary going that far to meet nonlinearity. For instance, if we consider water waves in a thin channel, the waves are transverse and a small oscillation analysis as presented before for sound but taking into account the pressure due to the vertical columns of water to each side of an element of fluid, leads to (1.30) with $\kappa = gh$ where g is the

[2] Just to boast: In cosmology the Universe as a whole is considered to be a fluid with equation of state $p = 0$ since 47000 years after Big Bang (matter dominated era) and $p = \rho/3$ in the early stages of Universe (radiation dominated era). The galaxy separation distance and other astronomical data are explained with the acoustic waves generated in the primordial plasma. If you want to lose your faith, read [Ste16, §16–§19].

gravitational acceleration and h is the deepness of the channel. It implies that the velocity of water waves is $\sqrt{gh}$ according to the model. This is more or less precise for shallow water but it is completely unrealistic in the middle of the ocean where the deepness is of the order of kilometers. The value of $\sqrt{gh}$ does not match with the actual speed and it sounds unnatural that the velocity could be affected in any way by the bottom of the ocean. A revised model taking also into account the vertical acceleration, gives in (1.30) a κ depending on the frequency of the waves. In this way the coefficient of the equation depends on the solution and we have a highly complicated example of non-linearity that is an active field of research for theoreticians.

Suggested Readings. This is classic material that can be found in many books for undergraduates (for instance [AF67]). Talking about classics, perhaps it is worth to have a look to the translations in `http://www.17centurymaths.com/` of early works by Euler, specially E305.