Differential Geometry
Deadline: 19/jan/2016

Historical comments. This year we celebrate the 100th anniversary of general relativity. It is just a number counting from the publication of the right field equations but Einstein was learning differential (Riemannian) geometry during years with the help of M. Grossmann, a mathematician and former classmate, and producing partial approaches to the metric theory of gravitation. In a letter[1] addressed to A. Sommerfeld in 1912 he wrote:

> *I am working exclusively on the problem of gravitation now, and, with the help of a mathematician friend here, I believe I will master all difficulties. But one thing is certain: I have never toiled this much in all my life, and I have been instilled with great reverence for mathematics, the subtler parts of which I naively used to regard as luxury! Compared with this problem, the original theory of relativity was child's play.*

We infer that differential geometry gave a bad time to Einstein. It is curious that this topic considered by a genius like him as one of the "subtler parts" of Mathematics, is nowadays standard in graduate and undergraduate studies and not specially abstruse[2]. At that time Riemannian geometry was unknown by physicists (and it was a specialized topic for mathematicians) then a first task for Einstein was introducing it to his colleagues. After the publication of the field equations in a short note of 3 pages, he wrote a very long (more than 50 pages in the original) fundamental paper: "The Foundation of the General Theory of Relativity". About one half of it is devoted to explain basic concepts in Riemannian geometry (metrics, geodesics, tensors, covariant derivatives, curvature).

Considerations about the paper. We are going to work with the English translation[3] provided by the site "The collected papers of Albert Einstein" `http://einsteinpapers.press.princeton.edu/vol6-trans`. It can be more convenient for most of you to use a PDF version, for instance the one marked as *A. Einstein (1916)* in `http://hermes.ffn.ub.es/luisnavarro/clasicos_2.htm` (check also Moodle).

You are not supposed to read it word by word because the questions below focus on some specific paragraphs and a (small) part of the paper requires something beyond a basic level on physics. An overview of the general notation and the contents may help anyway:

Notation. The notation in the paper is surprisingly modern with few exceptions: The coordinate functions are denoted by x_μ instead of x^μ. The "covariant derivative" is called

[1] You can see a facsimile of the original letter (in German) here: `http://einstein-virtuell.mpiwg-berlin.mpg.de/VEA/SC95647728_MOD129969944_SEQ-1877920857_SL-1688271241_de.html`.

[2] Hilbert exaggerated this point when said: "Every boy in the streets of Göttingen understands more about four-dimensional geometry than Einstein. Yet, in spite of that, Einstein did the work and not the mathematicians".

[3] Check `http://einsteinpapers.press.princeton.edu/vol6-doc/311?ajax` if you prefer the German original.

the "extension" of a vector field or of a tensor in general. The Christoffel symbols Γ^c_{ab} are denoted by $\{ab, c\}$, unfortunately at certain point it is defined $\Gamma^c_{ab} = -\{ab, c\}$. The no so common nowadays notation $[\mu\nu, \sigma]$ means $\frac{1}{2}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})$. The usual names for the components of the Riemann and Ricci tensors, $R^\alpha_{\beta\gamma\delta}$ and $R_{\alpha\beta}$, are not employed here, they are called, with certain normalization, $B^\alpha_{\beta\gamma\delta}$ and $G_{\alpha\beta}$. The coordinates are labeled as x_μ, $\mu = 1, 2, 3, 4$ with x_4 the time coordinate while in modern expositions the labeling is usually $\mu = 0, 1, 2, 3$ with x^0 the time coordinate.

Contents. The paper is divided into five parts:

A) §1-§4. There are general considerations (with few formulas) about the fundamental physical ideas. Check the lines in italic type in p.149 and 153 that one can rephrase saying that the laws of physics should be invariant under changes of coordinate charts.

B) §4-§12. This is the mathematical framework. The explanations are clear, simple and practical to some extent. Recall that Einstein is trying to introduce a part of differential geometry to his colleagues and in general they do not have a previous background on it.

C) §13-§18. The field equations are introduced here. The motivation for the vacuum case ($R_{\alpha\beta} = 0$ in modern notation) is very weak. Essentially it is claimed that as the vanishing of the Riemann tensor gives a wrong result, that of the Ricci tensor may do the job. The last paragraph of §14 suggests that Einstein is aware of the weakness of the argument and provides some extra support for the theory. The sections §15-§18 require some extra knowledge on physics.

D) §19-§20. It seems that the motivation here is the "open question" mentioned in the introduction leading to a possible unification of gravitation and electromagnetism. Again some extra knowledge on physics is needed here. If you have this knowledge, note that the electromagnetic tensor is antisymmetric and the energy-momentum tensor in general relativity is symmetric then something as obvious as adding both tensors cannot work.

E) §21-§22. Here it is shown that the Newtonian theory is recovered in first approximation and new effects are predicted.

Questions. The number of pages and lines refer to the translations in the previous links. Most of the questions are easy and admit a brief answer. Conciseness is largely appreciated. Please, do not answer questions that are not asked.

1) In §6 (lines 11-13) it is claimed essentially that the $(2, 0)$ tensors can be represented as sums of products of vectors. Can you make this statement rigorous?

2) In §10 there is a long explanation to conclude that the natural definition of covariant derivative (named here "extension") of a 1-form of components A_μ is (in modern notation) the tensor with components $A_{\mu;\nu}$ given by $\partial_\nu A_\mu - \Gamma^\tau_{\mu\nu} A_\tau$. Give a shorter explanation using that you know the formula for $V^\mu_{;\nu}$ (the covariant derivative of an arbitrary vector field) and it is natural to assume $\partial_\nu(A_\mu V^\mu) = A_{\mu;\nu} V^\mu + A_\mu V^\mu_{;\nu}$.

3) In §11 a long argument culminates with the formula (35) for the divergence Φ of a vector field in Riemannian geometry (replace $-g$ by $|g|$ in general). Can you give a non-trivial smooth vector field on S^2 (with the induced usual metric) having zero divergence? "Nontrivial" means that the identically 0 vector field is not admitted as a valid answer.

4) In the first line of (44) it is defined $G_{\mu\nu}$ as $B^{\rho}_{\mu\nu\rho}$. How is it related to the components of the Ricci tensor $R_{\mu\nu}$ defined in class? Hint: Note that Einstein calls $B^{\rho}_{\mu\sigma\tau}$ to the components $R^{\rho}_{\mu\sigma\tau}$ of the Riemann tensor.

5) [non mandatory] Where does (70) come from? Hint: Note that Einstein assumes $\det(g_{\alpha\beta}) = -1$ and the terms with decay beyond r^{-1}, like r^{-2}, are neglected.

6) After the comments that precede it, the first formula in p. 199 means that the bending angle of an almost straight plane curve with respect to the straight line is approximated by the integral of its curvature. Could you write some exact mathematical formula explaining it? Hint: Recall Frenet-Serret formulas.

7) [non mandatory] In p. 199 (line 10) it is claimed that (74) implies that a ray of light dodging Jupiter would suffer a deflection of about $0.02''$. Assume that (74) is exact (it is not by the approximations and by the influence of other planets), look up the radius and mass of Jupiter and get a finer value in seconds of arc with two significant digits instead of one.

Non mandatory questions. Questions **5)** and **7)** are non mandatory exercises and you can get the full grade in this problem set if you skip them. They are included here because they refer to Einstein's paper.

To my taste the hardest question is the fifth. Good short and clear explanations in the answer to this question will be graded with 0.5 to be added to the non mandatory exercises. Question seventh is very easy but a little off-topic (there have been recent experiments to verify corrected predictions). It counts 0.1 as a non mandatory exercise.