Problem set 2

Differential Geometry
Deadline: 10/dec/2015

1) Let g be the metric induced on S^2 from the Euclidean one when we employ the coordinate chart given by the north stereographic projection

a) Find the Christoffel symbols, deduce the differential equations of the geodesics and compute explicitly two solutions with $\big(x(0), y(0)\big) = (3/5, 4/5)$ parametrized by arc length.

b) Let p_N and p_S be the north and south stereographic projections, respectively. Study what happens with the components of the tensor g when we change variables with $(x, y) = p_S \circ p_N^{-1}(u, v)$. Try to find a simple geometric explanation of this fact.

2) Assume that with certain metric in $\mathbb{R}^2$ using a coordinate chart $\phi = (x, y)$ we have that the only non zero Christoffel symbols are $\Gamma^1_{12} = \Gamma^1_{21} = y$, $\Gamma^2_{11} = -ye^{y^2}$. Compute Γ^1_{22} when we use the "polar chart" $\psi = (r, \theta)$ related with the previous one by $x = r\cos\theta$, $y = r\sin\theta$.

3) In relativity, the metric $a^2x^2dt^2 - dx^2 - dy^2 - dz^2$ with $x > 0$ corresponds to the "natural" one employed by an accelerated observer (with acceleration proportional to a). Find the explicit equations of motion parametrized by proper time of a massive particle under the initial conditions $\dot{t}(0) = a^{-1}$, $x(0) = 1$ and assuming that the rest of the initial values of the coordinates and their derivatives are zero.

4) Some basic electrodynamic laws imply that if a magnetic monopole is placed in the origin, the motion of a charged particle is described by a curve $c(t) = \big(x(t), y(t), z(t)\big)$ that solves the differential equation $\ddot{c} = \|c\|^{-3}(c \times \dot{c})$ where $\times$ is the usual cross-product. Prove that if the particle is initially at $(1, 0, 1)$ with velocity $(0, 1/\sqrt{2}, 0)$, then $c(t)$ is a geodesic of the cone $z = \sqrt{x^2 + y^2}$.

Comments. **1)** You can look up and copy the explicit form of g without further explanations. It was worked out in a non mandatory exercise. For the last part of a) and for b), try to think geometrically to reduce/avoid calculations. **2)** Just assume (wrongly or not) that these Christoffel symbols corresponds to a metric, you do not need to know $g_{\alpha\beta}$. In case you are wondering, Γ^i_{jk} are not the components of a tensor. **3)** This problem is neither long nor difficult. If you have problems to solve the ODEs probably you are missing something. **4)** Of course we assume that the metric is the induced by the usual one in $\mathbb{R}^3$. This problem is more difficult than the rest to my taste.