Problem set 1

Differential Geometry
Deadline: 3/nov/2015

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1) Recall that SU(2) is the group of 2×2 unitary matrices with unit determinant. As any complex number is naturally identified with a point in $\mathbb{R}^2$, SU(2) inherits a natural topology and differentiable structure. Write explicit charts $(\mathcal{U}_i,\varphi_i)$ with $\bigcup\mathcal{U}_i = \mathrm{SU}(2)$. Study if the function

$$F(x^1,x^2,x^3,x^4)=\begin{pmatrix} x^1-ix^4 & -x^3-ix^2\\ x^3-ix^2 & x^1+ix^4\end{pmatrix}\qquad \text{with } i=\sqrt{-1}$$

induces a well-defined diffeomorphism $S^3=\{\vec{x}\in\mathbb{R}^4\ :\ \|\vec{x}\|=1\}\longrightarrow \mathrm{SU}(2)$.

2) Consider a $(0,1)$ tensor field T on a manifold of dimension 2, given by $T=T_1dx^1+T_2dx^2$ when we use the coordinate chart $\big(\mathcal{U},\varphi=(x^1,x^2)\big)$. We associate to T the tensor $\mathrm{rot}(T)$ of type $(0,2)$ given by

$$\mathrm{rot}(T)=\Big(\frac{\partial T_2}{\partial x^1}-\frac{\partial T_1}{\partial x^2}\Big)dx^1\otimes dx^2+\Big(\frac{\partial T_1}{\partial x^2}-\frac{\partial T_2}{\partial x^1}\Big)dx^2\otimes dx^1.$$

Prove that $\mathrm{rot}(T)$ is well-defined, i.e. it only depends on T, not on the choice of the coordinate chart.

3) Consider a (definite positive) scalar product in $\mathbb{R}^3$ given in the standard basis by $\langle V,W\rangle=g_{ij}V^iW^j$ for some fixed constants g_{ij}. How would you compute the volume of a parallelepiped? Explain your answer.

4) Consider the standard stereographic projection $S^2-\{\mathbf{N}\}\longrightarrow\mathbb{R}^2$, with $\mathbf{N}=(0,0,1)$, onto the plane $z=0$. We can identify $\mathbb{R}^2$ and $\mathbb{C}$ in the usual form $(x,y)\leftrightarrow x+iy$ and in this way we define the stereographic projection as a function $\pi: S^2-\{\mathbf{N}\}\longrightarrow\mathbb{C}$. Given $a,b\in\mathbb{C}$ such that $|a|^2+|b|^2=1$, define a fractional linear transformation $f:\mathbb{C}\cup\{\infty\}\longrightarrow\mathbb{C}\cup\{\infty\}$ as $f(z)=(az+b)/(-\overline{b}z+\overline{a})$ where the bar means complex conjugation.

Complete the definition of π with $\pi(\mathbf{N})=\infty$. Prove that for any f as above, the map $\pi^{-1}\circ f\circ\pi$ is a rotation on the sphere.

Comments. **1)** Probably you will solve both parts at the same time. **2)** You are supposed to use just the transformation law for tensors. If you appeal to any other result, you have to prove it. **3)** Here a simple formula is expected. Try, for instance, the parallelepiped generated by the standard basis vectors and $g_{11}=g_{12}=g_{21}=1$, $g_{23}=g_{32}=2$, $g_{22}=g_{33}=5$, $g_{13}=g_{31}=0$. You should be able to get the volume in few seconds. **4)** This problem is long but doable with direct calculations. No tricks are essential but it may help to write dx^2+dy^2 as $dzd\overline{z}$ with $z=x+iy$.