Deadline: December 1st

Let ω_1 and ω_2 be linearly independent (at each point) 1-forms on a manifold of dimension 2. Let A_1 and A_2 be functions such that $d\omega_1 = A_1\, \omega_1 \wedge \omega_2$ and $d\omega_2 = A_2\, \omega_1 \wedge \omega_2$, and define the 1-form $\theta = -A_1\omega_1 - A_2\omega_2$. Given a function f, consider now the same construction starting with $\widetilde{\omega}_1 = \cos f\, \omega_1 - \sin f\, \omega_2$ and $\widetilde{\omega}_2 = \sin f\, \omega_1 + \cos f\, \omega_2$ to get $\widetilde{\theta}$. Prove that $\widetilde{\theta} = \theta + df$.

Solution. Most of you have done long calculations. This short solution does not include new ideas, it just organizes the notation in a more efficient way.

We have (if you are not familiar with differential forms think about it, it is easy)

$$d\widetilde{\omega}_1 = d(\cos f) \wedge \omega_1 + \cos f\, d\omega_1 - d(\sin f) \wedge \omega_2 - \sin f\, d\omega_2.$$

Let B_1, B_2 be such that $df = B_1\omega_1 + B_2\omega_2$. Expanding $d(\cos f)$ and $d(\sin f)$ and substituting $d\omega_1$ and $d\omega_2$, we get

$$d\widetilde{\omega}_1 = \Big(\cos f\, (A_1 - B_1) - \sin f\, (A_2 - B_2)\Big)\omega_1 \wedge \omega_2$$

and a similar formula holds for $d\widetilde{\omega}_2$ formally replacing f by $f - \pi/2$, i.e.

$$d\widetilde{\omega}_2 = \Big(\sin f\, (A_1 - B_1) + \cos f\, (A_2 - B_2)\Big)\omega_1 \wedge \omega_2.$$

Note that the trigonometric coefficients appearing in the big parentheses correspond to the matrix G_f of a rotation of angle f and they are the same coefficients relating $\widetilde{\omega}_i$ and ω_i. Then we can write

$$\widetilde{\theta} = -\widetilde{A}_1\widetilde{\omega}_1 - \widetilde{A}_2\widetilde{\omega}_2 = -G_f \begin{pmatrix} A_1 - B_1 \\ A_2 - B_2 \end{pmatrix} \cdot G_f \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

where $\cdot$ indicates the usual scalar product. Since it is preserved by rotations, we have that $\widetilde{\theta} = -(A_1 - B_1)\omega_1 - (A_2 - B_2)\omega_2 = \theta + df$ as expected.
