Deadline: November 19th

Prove that $r^4\dot\theta^2 + r^4\dot\varphi^2\sin^2\theta$ remains constant along any geodesic of the Schwarzschild metric.

Solution. (Most of you have done more or less long calculations. I include here a short proof for illustration not involving explicit second derivatives of the variables. The notation p (generalized momentum) is taken from Physics.

The Lagrangian corresponding to the Schwarzschild metric is

$$\mathcal{L} = \big(1-\frac{2GM}{r}\big)\dot t^2 - \big(1-\frac{2GM}{r}\big)^{-1}\dot r^2 - r^2\dot\theta^2 - r^2\sin^2\theta\,\dot\varphi^2.$$

We have

$$p_\theta := \frac{\partial\mathcal{L}}{\partial\dot\theta} = -2r^2\dot\theta \qquad \text{and} \qquad p_\varphi := \frac{\partial\mathcal{L}}{\partial\dot\varphi} = -2r^2\sin^2\theta\dot\varphi.$$

The Euler-Lagrange equations prove

$$\dot{p_\theta} = \frac{\partial\mathcal{L}}{\partial\theta} = -2r^2\sin\theta\cos\theta\,\dot\varphi^2 \qquad \text{and} \qquad \dot{p_\varphi} = \frac{\partial\mathcal{L}}{\partial\varphi} = 0.$$

The key observation is that $4r^4\dot\theta^2 + 4r^4\dot\varphi^2\sin^2\theta = p_\theta^2 + p_\varphi^2/\sin^2\theta$. Then

$$4\frac{d}{d\lambda}\big(r^4\dot\theta^2 + r^4\dot\varphi^2\sin^2\theta\big) = 2p_\theta\dot p_\theta + 2\frac{p_\varphi\dot p_\varphi}{\sin^2\theta} - 2p_\varphi^2\frac{\dot\theta}{\sin^3\theta} = 0,$$

where the last equality follows easily substituting the values of p_θ, $\dot p_\theta$, p_φ and $\dot p_\varphi$.