Deadline: October 15th

Consider to parallel circles of radius 2, say $x^2 + y^2 = 4$, $z = 0$ and $x^2 + y^2 = 4$, $z = h$. If h is small there is a connected surface of revolution of minimal area having the two circles as boundary. Our intuition suggests that when h grows, the surface collapses in some way. Give a numerical estimation of the critical separation h_c.

Solution. (If you can read more about this, check [1] and [2, §2.6]).

There are two possible solutions. One can think that the collapse occurs when the area equals the area of the boundary circles because for a larger separation they give a solution of the problem of minimizing the area. One can also think that a collapse occurs when we cannot solve the Euler-Lagrange equations under the stated conditions.

Which of these scenarios corresponds to the actual physical situation of the collapse of a soap film? In my opinion, it is the second one because the physical laws acts "locally". Anyway, solutions based on any of these setting are considered valid answers for the stated problem.

Let us translate the figure by $(0, 0, -h/2)$. As we saw in class, the solution obtained using calculus of variations (Euler-Lagrange equations) is given by

$$r(z) = K \cosh \frac{z}{K}$$

where r means the radial cylindrical coordinate $r = \sqrt{x^2 + y^2}$. The boundary conditions give the following implicit equation relating h and K

$$2 = K \cosh \frac{h}{2K}.$$

First approach We compute the area of solution

$$A = 2\pi \int_{-h/2}^{h/2} r(z) \sqrt{1 + \big(r'(z)\big)^2}\, dz = \pi K h + \pi K^2 \sinh \frac{h}{K}.$$

The integral is very simple, just recall $\cosh^2 x - \sinh^2 x = 1$. According to the previous considerations the critical h is given by the solution of the system of equations

$$\begin{cases} 2 = K \cosh \frac{h}{2K} \\ \pi K h + \pi K^2 \sinh \frac{h}{K} = 8\pi \end{cases}$$

Note that the area of the circles is $2 \cdot \pi 2^2 = 8\pi$.

Using the formula $\sinh(2x) = 2\sinh x \cosh x$ and the first equation we can write $\pi K^2 \sinh \frac{h}{K}$ as $4\pi\sqrt{4-K^2}$ and we obtain the single variable equation

$$2\pi K^2 \operatorname{arccosh} \frac{2}{K} + 4\pi\sqrt{4-K^2} - 8\pi = 0.$$

Of course there are many iterative methods to approximate a solution. We outline a simple one. By inspection, we have that the solution is in $[1.6, 1.7]$ and the derivative there is close to -24, then the simple scheme $x_{n+1} = x_n + f(x_n)/24$ should converge quickly (because the derivative of $x - f(x)/(-24)$ is small). Starting with $x_0 = 1.6$ with 3 iterations we obtain the 6 significant digits approximation $K = 1.65103$ that leads to $h = 2K \operatorname{arccosh}(2/K) = 2.110789\ldots$

Second approach Naming $x = h/2K$ and $\alpha = 4/h$ the equation imposing the boundary conditions can be written as $\cosh x = \alpha x$ and we have to determine the smallest value of α such that this equation has a solution. In the critical value we have a double root (the straight line $y = \alpha x$ is tangent to $y = \cosh x$). Then

$$\begin{cases} \cosh x - \alpha x = 0 \\ (\cosh x - \alpha x)' = \sinh x - \alpha = 0 \end{cases}$$

This implies $x \tanh x - 1 = 0$ that it is easy to solve with iterative methods. We proceed as below with a simple one. By inspection, the solution belongs to $[1.1, 1.2]$, in fact it is close to 1.2 and the derivative there is not very far from 1, then the simple scheme $x_{n+1} = x_n - f(x_n)$ should converge quickly (because the derivative of $x - f(x)$ is small). Starting with $x_0 = 1.2$ with 3 iterations we obtain the 5 significant digits approximation $x = 1.19967$ that leads to $h = 4/\alpha = 4/\sinh x = 2.65097\ldots$

References

[1] http://mathworld.wolfram.com/MinimalSurfaceofRevolution.html

[2] H. Sagan, *Introduction to the calculus of variations*, Dover Publications, Inc., New York, 1992.