Deadline: October 15th

Compute the induced metric (by the usual $\mathbb{R}^3$ metric) on the sphere S^2 when we use the coordinate chart $(S^2 - \{N\}, p_N)$ where p_N the stereographic projection from the north pole N.

Solution. The inverse of the stereographic chart is

$$(x, y) \longmapsto (X, Y, Z) = \Big(\frac{2x}{D}, \frac{2y}{D}, 1 - \frac{2}{D}\Big) \qquad \text{with} \quad D = 1 + x^2 + y^2.$$

Then

$$dX = \frac{2D - 4x^2}{D^2}dx - \frac{4xy}{D^2}dy, \qquad dY = \frac{2D - 4y^2}{D^2}dx - \frac{4xy}{D^2}dy, \qquad dZ = \frac{4x}{D^2}dx + \frac{4y}{D^2}dy$$

and the induced metric is

$$\Big(\frac{2D - 4x^2}{D^2}dx - \frac{4xy}{D^2}dy\Big)^2 + \Big(-\frac{4xy}{D^2}dy + \frac{2D - 4y^2}{D^2}dx\Big)^2 + \Big(\frac{4x}{D^2}dx + \frac{4y}{D^2}dy\Big)^2$$

where we are using the classic notations (the squares really mean tensor products with itself).

The coefficient of $(dx)^2$ is

$$\Big(\frac{2D - 4x^2}{D^2}\Big)^2 + \Big(\frac{4xy}{D^2}\Big)^2 + \Big(\frac{4x}{D^2}\Big)^2 = \frac{4D^2 - 16x^2D + 16x^2(x^2 + y^2 + 1)}{D^4} = \frac{4}{D^2}.$$

By the symmetry $x \leftrightarrow y$ this is also the coefficient of $(dy)^2$. There are not cross terms $dxdy$ because

$$-2(2D - 4x^2)4xy - 8xy(2D - 4y^2) + 32xy = 8xy\big(-2D + 4x^2 - 2D + 4y^2 + 4\big) = 0.$$

Then we obtain finally,

$$4D^{-2}\big((dx)^2 + (dy)^2\big) = 4(1 + x^2 + y^2)^{-2}\big((dx)^2 + (dy)^2\big).$$