Let $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $n > 2$, a self-adjoint linear map with a double eigenvalue $\lambda_1 = \lambda_2$ (we assume that the rest are simple), let V be the corresponding 2-dimensional eigenspace and $W = V^\perp$. We saw in class that $L\big|_V = \lambda_1 \mathrm{Id}$ (note that $V \cong \mathbb{R}^2$). As L, and hence $L\big|_W$, is self-adjoint, we have an orthonormal basis[1] of eigenvectors $\{\vec{v}_i\}_{i=3}^n \in W$. Given such vectors, $L\big|_W$ is determined by their images $\{\lambda_i\vec{v}_i\}_{i=3}^n$. Reciprocally, $L\big|_W$ determines the eigenvalues $\{\lambda_i\}_{i=3}^n$ and the orthonormal eigenvectors $\{\lambda_i\vec{v}_i\}_{i=3}^n$, up to the sign and the ordering.

Summing up, to define L we have to choose the eigenvalues and the orthonormal vectors to generate W (note that V is determined by W).

Data	**degrees of freedom**
$n-1$ distinct eigenvalues $\lambda_1, \lambda_3, \lambda_4, \dots, \lambda_n \in \mathbb{R}$	$n-1$
$n-2$ orthonormal vectors $\vec{v}_3, \vec{v}_4, \dots, \vec{v}_n \in \mathbb{R}^n$	$(n-1)+(n-2)+\cdots+2$

The explanation of the last cell is that $\vec{v}_3$ is in S^{n-1} (dimension $n-1$), $\vec{v}_4$ is in the intersection of S^{n-1} with the hyperplane $\vec{v}_3 \cdot \vec{x} = \vec{0}$, that is the same as (diffeomorphic to) S^{n-2} and successively each new vector has to lie on a new hyperplane, decreasing the dimension by 1.

Adding these values, we get

$$\dim M = (n-1) + (n-1) + (n-2) + \cdots + 2 = \frac{n(n+1)}{2} - 2.$$

Note: Some of you sent some references and asked me some questions. I am composing a document with complementary information and with some references to basic algebraic geometry. I shall post it on my website.

[1] Just in case you do not remember linear algebra, note that $\langle L\vec{v}_i, \vec{v}_j\rangle = \lambda_i\langle \vec{v}_i, \vec{v}_j\rangle$, $\langle \vec{v}_i, L\vec{v}_j\rangle = \lambda_j\langle \vec{v}_i, \vec{v}_j\rangle$ implies $\langle \vec{v}_i, \vec{v}_j\rangle = 0$ for $i \neq j$ if $\langle L\vec{v}_i, \vec{v}_j\rangle = \langle \vec{v}_i, L\vec{v}_j\rangle$ because $\lambda_i \neq \lambda_j$.