Deadline: January 20

Exercises

1) Answer briefly the following questions:

1. In Riemannian manifolds of dimension 2 with metric $ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{11}dy^2$, the area of a region R is defined as $\int_R \sqrt{g_{11}g_{22} - (g_{12})^2}\ dxdy$. Why is this a natural definition?

2. Imagine a space-time $\{(t,x) \in \mathbb{R} \times (-1,1)\}$ endowed with the metric $(1-x^2)^{-1}dt^2 - dx^2$. Compute $x(\tau)$ for the geodesic $c(\tau) = \big(t(\tau), x(\tau)\big)$ with $G(\dot{c},\dot{c}) = 1$, $x(0) = x_0$, $\dot{x}(0) = t(0) = 0$. Why the inhabitants of this space-time would think that there is a Sun in the middle of $(-1,1)$ attracting the material particles?

3. Consider in $\mathbb{R} \times \mathbb{R}^+$ the metric $y^{-2}dx^2 + y^{-2}dy^2$. If $\nabla_2\big(a(y)\frac{\partial}{\partial x}\big) = 0$ and $a(1) = 1$, what is the formula for $a = a(y)$?

4. When we consider in $\mathbb{R}^2$ the metric $\big(3 + \sin^4(2\pi x)\big)dx^2 + \sin^4(x+y)\ dxdy + (x^2+1)\ dy^2$, its scalar curvature at $(0,0)$ is $-2/3$. What is the scalar curvature at $(1,-1)$ when we consider the metric $\big(y^2 + 2y + 2\big)dx^2 + \sin^4(x+y)\ dxdy + \big(3 + \sin^4(2\pi y)\big)dy^2$?

Notes and hints

1) For your convenience, I strongly insist on brevity. Please, do not lose your time trying to repeat the format of the answers in previous sheets. This is shorter and easier.

1. Think that a metric is, in some sense, a way of specifying lengths and angles (a scalar product). Then if you want to give a fully geometrical answer you should wonder how could you compute areas using lengths and angles.

2. I assume that you recall that in general relativity the geodesics of the space-time with $G(\dot{c}, \dot{c}) = 1$ correspond to the trajectories of the material particles. As a hint, I suggest to employ $G(\dot{c}, \dot{c}) = 1$ only to substitute the initial conditions.

3. Recall that, by definition, $\nabla_2 V = \nabla_{\partial_2} V = V^i_{;2}\partial_i$.

4. If your answer takes more than three lines, you are missing the simple solution.