Deadline: December 11

Exercises

1) Let f be the map than includes $\mathbb{R}^3$ into the quaternions as $f(a,b,c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Consider $F(\vec{x}) = f^{-1}\big(q^{-1}f(\vec{x})q\big)$ with $q = \cos t + f(\vec{n})\sin t$ where $t \in \mathbb{R}$ and $\vec{n}$ is a unit vector. Prove that $F \in \mathrm{SO}(3)$, in fact it is a rotation in $\mathbb{R}^3$ around $\vec{n}$. Compute its angle in terms of t.

2) Let $\omega = a\ dx + b\ dy + c\ dz$ be a 1-form in $\mathbb{R}^3$ with a non-vanishing and consider the distribution $\Delta = \{X\ :\ \omega(X) = 0\}$. Prove that Δ is completely integrable if and only if

$$a\Big(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}\Big) + b\Big(\frac{\partial a}{\partial z} - \frac{\partial c}{\partial x}\Big) + c\Big(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\Big) = 0.$$

3) Compute the cohomology groups of the (surface of the) torus with two points removed.

4) Consider the differential form in $\mathbb{R}^3 - \{\vec{0}\}$

$$\omega = \|\vec{x}\|^{-3}\big(x\ dy \wedge dz - y\ dx \wedge dz + z\ dx \wedge dy\big)$$

where $\vec{x} = (x,y,z)$. Compute $\int_{S^2} i^*\omega$ where $i : S^2 \longrightarrow \mathbb{R}^3$ is the inclusion. Compute the same integral when S^2 is replaced by any sphere of radius R and center $\vec{c}$ such that $\|\vec{c}\| < R$.

Notes and hints

1) Believe or not, this problem is relevant in video game programming (in 3D animation, in general). In case you are wondering, I agree, this problem has nothing to do with this part of the course except for some resemblances with our study of the angular momentum, but it is still geometry. Try to find an elegant and short proof.

I assume that you know the quaternions. If not, you should read the following crash course in barely four lines: The quaternions are the $\mathbb{R}$-vector space generated by $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ equipped with the associative non-commutative vector multiplication induced by $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. For instance, $\mathbf{ij} = -\mathbf{ijk}^2 = -(\mathbf{ijk})\mathbf{k} = \mathbf{k}$ and $\mathbf{jk} = \mathbf{i}$, $\mathbf{ki} = \mathbf{j}$ in the same way. Note that if $q = \lambda_0 + \lambda_1\mathbf{i} + \lambda_2\mathbf{j} + \lambda_3\mathbf{k} \neq 0$, then $q^{-1} = \overline{q}/\|q\|^2$ with $\overline{q} = \lambda_0 - \lambda_1\mathbf{i} - \lambda_2\mathbf{j} - \lambda_3\mathbf{k}$ and $\|q\|^2 = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2$.

2) Convince yourself that Δ is really a distribution and get a local basis. Recall that 1-forms act on vectors. For instance $dx(V)$ is the first coordinate of V. This problem is closer to the original statement given by Frobenius for his theorem. A local basis with many constant coordinates eases the verification of the condition in the theorem.

3) We have seen the groups for a simple punctured torus. Try to relate to it and to the usual torus.

4) This form (of course without the modern notation) was introduced by Gauss to define the *linking number*. An integer with topological meaning expressing how two closed curves are entangled in $\mathbb{R}^3$.