Deadline: October 30

Exercises

1) Compute all the components of the tensor $T = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \otimes dy$ defined on $\mathbb{R}^2$ when we use the polar coordinates chart $\phi = (r, \theta)$.

2) Consider SO(3), the group of orthogonal matrices 3×3 with determinant 1. Each element SO(3) has 9 entries, in this way, it inherits the topological structure of $\mathbb{R}^9$ (and also the differential structure by restriction). Write an explicit chart of SO(3).

3) Let F be the function defined as $F(x,y) = \big(\Re f(x+iy), \Im f(x+iy)\big)$ where $\Re$ and $\Im$ denote the real and the imaginary part and $f(z) = (az+b)/(cz+d)$ with $a, b, c, d \in \mathbb{R}$ fixed such that $ad - bc = 1$. Check that $F : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R} \times \mathbb{R}^+$ is a well-defined diffeomorphism and prove that it leaves the tensor $y^{-2}(dx \otimes dy - dy \otimes dx)$ invariant.

4) Consider a particle of mass m constrained to move on the unit sphere. Find the differential equations of motion in the presence of gravitation ($V = mgz$) employing the usual spherical coordinates. Solve this equations for a particle starting at the north pole when we are in the International Space Station ($g \to 0$).

Notes and hints

1) This should be just simple automatic computations using the chain rule.

2) Only a chart is required, do not worry about a full atlas. Recall that SO(3) is exactly the group of rotations in $\mathbb{R}^3$. Depending on your previous knowledge, this problem could be harder than it seems.

3) The grade depends on how elegant the proof is. Of course, we are assuming that $\mathbb{R} \times \mathbb{R}^+$ is endowed with the usual differential structure (the identity defines a chart).

4) Really the ISS is only 420 km over the Earth then the value of g in principle is only an 11 % less than on the Earth but due to the rotation the gravitational force is compensated (by the centrifugal force). The fact that the gravitation can be simulated or compensated in an accelerated frame of reference is related to the *equivalence principle*, that played a role in the development of general relativity.