Una Solución

Enero
Álgebra
24/25

(1) Verdadero

A,B,C conjuntos $\Longrightarrow A\setminus(B\cup C) = (A\setminus B)\cap(A\setminus C)$

Dem:

$$A\setminus(B\cup C) = \{a\in A \mid a\notin B \wedge a\notin C\} =$$

$$\left.\begin{aligned} A\setminus B &= \{a\in A \mid a\notin B\} \\ A\setminus C &= \{a\in A \; a\notin C\}\end{aligned}\right\} (A\setminus B)\cap(A\setminus C) = \left\{a\in A \;\middle|\; \begin{aligned} a&\notin B \\ \wedge\; a&\notin C\end{aligned}\right\}$$

(2) En $\mathbb{N}\times\mathbb{N}$:

$$(a,b)R(n,m) \Longleftrightarrow 2a+3b = 2n+3m$$

(a) Veamos es una relación de equivalencia:

(i) Reflexiva: $(a,b)R(a,b)$ ya que $2a+3b=2a+3b$

(ii) Simétrica: $\underbrace{(a,b)R(n,m)} \Longrightarrow \underbrace{(n,m)R(a,b)}$

$$2a+3b=2n+3m \Longleftrightarrow 2n+3m=2a+3b$$

(iii) Transitiva:

$$\underbrace{(a,b)R(n,m)} \wedge (n,m)R(u,v) \Longrightarrow (a,b)R(u,v)$$

$$2a+3b=2n+3m \wedge 2n+3m=2u+3v \Rightarrow 2a+3b=2u+3v$$

(b) $[(3,4)] = \{ (n,m) \in \mathbb{N} \times \mathbb{N} \mid \underbrace{2\cdot3+3\cdot4}_{18} = 2n+3m \}$

$2n+3m = 18 \iff 2n = 18-3m = 3(6-m) \iff 2 \mid 6-m$

con $\underline{6-m>0}$
$m<6$

$m=1 \Rightarrow 2 \nmid 6-1=5$

$m=2 \Rightarrow 2 \mid 6-2=4 \rightarrow n = 3\cdot\frac{4}{2} = 6 \rightarrow (n,m) = (6,2)$

$m=3 \Rightarrow 2 \nmid 6-3=3$

$m=4 \Rightarrow 2 \mid 6-4=2 \rightarrow n = 3\cdot\frac{2}{2} = 3 \rightarrow (n,m) = (3,4)$

$m=5 \Rightarrow 2 \nmid 6-5$

Conclusión: $[(3,4)] = \{ (3,4), (6,2) \}$

(3) $7^{668} + 17^8 \cdot 19^6 \pmod{18}$

$\left.\begin{array}{l} \varphi(18) = \varphi(2)\varphi(3^2) = 1\cdot6 = 6 \\ 668 = 6\cdot111+2 \end{array}\right\} \Rightarrow 7^{668} = (\underbrace{7^6}_{\substack{\equiv \\ 1 \pmod{18}}})^{111} \cdot 7^2 \equiv 7^2 \pmod{18}$

Teorema Euler-Fermat

$17 \equiv -1 \pmod{18}$

$19 \equiv 1 \pmod{18}$

$7 \equiv 7 \pmod{18}$

Entonces:

$7^{668} + 17^8 \cdot 19^6 \equiv 7^2 + (-1)^8 \cdot 1^6 \equiv 49+1 \equiv 13+1 \equiv 14 \pmod{18}$

$49 = 18\cdot2+13$

Conclusión: El resto de dividir $7^{668}+17^8\cdot19^6$ entre 18 es:

$\boxed{14}$

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$$U_a = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$A_a = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & a \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 1 & a \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & a-1 \end{pmatrix}$$

$$\dim U_a = 4 - \text{rango}(A_a) = \begin{cases} 3 & \text{si } a \neq 1 \\ 2 & \text{si } a = 1 \end{cases}$$

Por lo tanto:

$$\boxed{\dim U_a = 2 \iff a = 1}$$

$a=1$

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \longleftrightarrow \begin{cases} x_1 = x_2 \\ x_3 = -x_4 \end{cases}$$

$$U_1 = \{ \underbrace{(x_2, x_2, -x_4, x_4)}_{\overset{\shortparallel}{x_2(1,1,0,0) + x_4(0,0,-1,1)}} \mid x_2, x_4 \in \mathbb{R} \}$$

$$U_1 = \langle \underbrace{(1,1,0,0), (0,0,-1,1)}_{L.I.} \rangle$$

$\Rightarrow \mathcal{B} = \{(1,1,0,0), (0,0,-1,1)\}$ son base de U_1

Escaneado con CamScanner

⑥ $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^4$

$$f(x,y) = (3x+y, -x+y, -2x+y, 2x+y)$$

ⓐ $B = \{(1,0),(0,1)\}$, $B' = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$

$$M_{BB'}(f) = \begin{array}{c} \begin{array}{cc} f(1,0) & f(0,1) \end{array} \\ \begin{pmatrix} 3 & 1 \\ -1 & 1 \\ -2 & 1 \\ 2 & 1 \end{pmatrix} \end{array}$$

$f(1,0) = (3,-1,-2,2) = (3,-1,-2,2)_{B'}$

$f(0,1) = (1,1,1,1) = (1,1,1,1)_{B'}$

ⓑ $U+V = \langle (1,1,0,0),(1,-1,-1,1), f(1,0), f(0,1) \rangle$

$V = Im(f) = \langle f(1,0), f(0,1) \rangle$

Buscamos base de $U+V$:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ 3 & -1 & -2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 \\ 0 & -4 & -2 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \{(1,0,0,1),(0,1,0,-1),(0,0,1,1)\}$ base de $U+V$.

$(x_1, x_2, x_3, x_4) \in U+V$ ssi:

$$\text{rango} \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \\ 1 & -1 & 1 & x_4 \end{pmatrix} = 3 : \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & \underset{\overset{\parallel}{0}}{\boxed{x_4 - x_1 + x_2 - x_3}} \end{pmatrix}$$

$$\boxed{\begin{array}{c} \text{Ecuación de } U+V: \\ x_1 - x_2 + x_3 - x_4 = 0 \end{array}}$$

7 $\quad A_a = \begin{pmatrix} 2-a & -2+2a \\ 1-a & -1+2a \end{pmatrix}$

Vamos a intentar diagonalizar A:

$$P_{A_a}(x) = |A_a - xI_2| = \begin{vmatrix} 2-a-x & -2+2a \\ 1-a & -1+2a-x \end{vmatrix} =$$

$$= x^2 - ax - x + a = (x-1)(x-a)$$

$\boxed{a \neq 1} \Rightarrow A_a$ diagonalizable

Autovalores:

$\lambda=1$ $\quad Ker(A_a - I_2) = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} 1-a & -2+2a \\ 1-a & -2+2a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$x(1-a) = 2(1-a)y \Leftrightarrow x = 2y \Rightarrow u_1 = (2,1)$ autovector de autovalor 1

$\lambda=a$: $\quad Ker(A_a - aI_2) =$
$\left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} 2-2a & -2+2a \\ 1-a & -1+a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$(1-a)x = (1-a)y \Leftrightarrow x = y \Rightarrow u_2 = (1,1)$ autovector de autovalor a

$$P_a = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad D_a = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad P_a^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\Rightarrow A_a = P_a D_a P_a^{-1}$$

$$\Rightarrow A_a^{200} = P_a D_a^{200} P_a^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & a^{200} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

$$\boxed{A_a^{200} = \begin{pmatrix} 2-a^{200} & -2+2a^{200} \\ 1-a^{200} & -1+2a^{200} \end{pmatrix}}$$

$\boxed{a=1} \Rightarrow A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A^{200} = I_2$

$$\boxed{A_a^n = A_{a^n}}$$