New groups and new weights

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Contents. Elliptic theta function. Automorphic forms. Eisenstein series for cusps. General valence formula.

1.1 Elliptic functions via theta functions

The theory of modular forms had its origin in the theory of elliptic functions. Jacobi's presentation of the latter involved some functions given by series that are called *theta functions* due to their classic notation. In a nut shell, elliptic functions can be expressed as quotients of theta functions that are not elliptic themselves but the double periodicity is only violated by a factor. In the classic theory there are four theta functions, all of them are minor variations on

$$\theta(z,\tau)=\sum_{n\in\mathbb{Z}} e^{2\pi inz+\pi in^2\tau} \qquad \text{where} \quad z\in\mathbb{C} \quad \text{and} \quad \tau\in\mathbb{H}.$$

It is not difficult to check

$$(1) \qquad \theta(z+1,\tau)=\theta(z,\tau) \qquad \text{and} \qquad \theta(z+\tau,\tau)=e^{-2\pi iz-\pi i\tau}\theta(z,\tau).$$

In this sense, $\theta(z,\tau)$ is elliptic in z with periods 1 and τ except for a factor. It is much harder to get the following result (essentially due to Gauss):

Lemma 1.1 (Jacobi's triple product formula)**.** *We have the identity*

$$\theta(z,\tau)=\prod_{n=1}^{\infty}\big(1-e^{2\pi in\tau}\big)\big(1+e^{2\pi iz+\pi i(2n-1)\tau}\big)\big(1+e^{-2\pi iz+\pi i(2n-1)\tau}\big).$$

There are combinatorial proofs (see [5, §8.3]). We sketch the elegant proof involving elliptic functions.

Proof (sketch). Let $P(z,\tau)$ be the right hand side. For each τ fixed $P(z,\tau)$ has simple zeros exactly at $z=\frac{\tau+1}{2}+m+n\tau$ with $m,n\in\mathbb{Z}$. On the other hand, from (1) and $\theta(z,\tau)=\theta(-z,\tau)$ it is deduced $\theta\big(z+\frac{\tau+1}{2}\big)=-e^{-2\pi iz}\theta\big(\frac{\tau+1}{2}-z\big)$ that implies that $\theta(z,\tau)$ has zeros at the same points. Moreover, by direct manipulations P also satisfies the relations (1). It proves that $\theta(z,\tau)/P(z,\tau)$ is constant in z because it is elliptic and has not zeros.

Proving that this constant K, in principle depending on τ, is actually 1 is rather ingenious. Consider K as a function of $w=e^{i\pi\tau}$. From $K(w)=\theta(\frac{1}{2},\tau)/P(\frac{1}{2},\tau)$ and $K(w)=\theta(\frac{1}{4},\tau)/P(\frac{1}{4},\tau)$ it is deduced with some effort $K(w)=K(w^4)$ and as $K(w)$ is analytic in $|w|<1$, it implies that $K(w)$ is identically 1. $\square$

Now we freeze the variable z putting $z = 0$ and we consider θ as a function of the second argument that we rename as z. In this way we get the *classical theta function*

$$\theta(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z} \qquad \text{with} \quad z \in \mathbb{H}.$$

Proposition 1.2. *The classical theta function θ defines a holomorphic function $\mathbb{H} \longrightarrow \mathbb{C} - \{0\}$ and satisfies $\theta(z+2) = \theta(z)$, $\theta(-1/z) = (-iz)^{1/2}\theta(z)$.*

The branch of the square root in the last part is the standard one: $(-iz)^{1/2}$ is real and positive if $-iz$ is real and positive.

The proof of Proposition 1.2 depends on a result from basic Fourier analysis (see [4]).

Lemma 1.3 (Poisson summation formula)**.** *For $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ a rapidly decreasing function*

$$\sum_{\vec{n}\in\mathbb{Z}^d} f(\vec{n}) = \sum_{\vec{n}\in\mathbb{Z}^d} \widehat{f}(\vec{n})$$

where $\widehat{f}$ is the Fourier transform $\widehat{f}(\vec{\xi}) = \int_{\mathbb{R}^d} f(\vec{x})\, e^{-2\pi i \vec{x}\cdot\vec{\xi}}\, d\vec{x}$.

Proof of Proposition 1.2. It is holomorphic by the general convergence results [1] thanks to the exponential decay of $e^{\pi i n^2 z}$. It does not vanish by Lemma 1.1 with $z = 0$. The periodicity $\theta(z+2) = \theta(z)$ is obvious.

It is well-known that $e^{-\pi x^2}$ coincides with its Fourier transform. With a change of variables it is deduced that $f(x) = e^{-\pi t x^2}$ with $t > 0$ satisfies $\widehat{f}(\xi) = t^{-1/2}\, e^{-\pi\xi^2/t}$. Then Poisson summation formula implies $\theta(it) = t^{-1/2}\theta(i/t)$. By analytic continuation, we can take $t = -iz$ with $z \in \mathbb{H}$. □

Corollary 1.4. *If $k \in \mathbb{Z}^+$ is divisible by four, $\theta^{2k}|_\gamma = \theta^{2k}$ for every $\gamma \in \Gamma_\theta$.*

Proof. This follows from Proposition 1.2 using that $\Gamma_\theta = \langle S, T^2\rangle$. □

Note that opening the $2k$ power, even if the divisibility condition is not fullfilled,

$$\theta^{2k}(z) = \sum_{n=0}^{\infty} r_{2k}(n) e^{\pi i n z} \qquad \text{with} \quad r_{2k}(n) = \#\big\{(n_1,\dots,n_{2k}) \in \mathbb{Z}^{2k} \;:\; n_1^2 + \cdots + n_{2k}^2 = n\big\}.$$

In this way, the powers of θ are linked to the number of representations as a sum of squares.

1.2 A general definition

Having in mind Corollary 1.4, a first step to generalize modular forms is to define a *weakly modular form* of weight $k \in \mathbb{Z}_{\ge 0}$ for a congruence subgroup Γ as a holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ satisfying $f|_\gamma = f$ for every $\gamma \in \Gamma$. Note that if $-I \notin \Gamma$ the weight can be odd without forcing $f = 0$.

So, Corollary 1.4 claims that θ^{2k} is a weakly modular form of weight k for a congruence subgroup Γ_θ for $4 \mid k$. An infinite family of examples for each even weight is $E_k(Nz)$ which is a weakly modular form of weight k for the congruence subgroup $\Gamma_0(N)$ thanks to the following result.

Lemma 1.5. *If f is a weakly modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ then g defined as $g(z) = f(Nz)$ is a weakly modular form of weight k for $\Gamma_0(N)$.*

Proof. Given

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{we consider} \qquad \gamma' = \begin{pmatrix} a & bN \\ c/N & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Then $g(\gamma z) = f(N\gamma z) = f\big(\gamma'(Nz)\big)$. This shows $g(\gamma z) = j^k_{\gamma'}(Nz) f(Nz) = j^k_\gamma(z) g(z)$. □

To pass from "weakly modular" to "modular" it is necessary to generalize the Fourier expansion. This is not trivial. Note, for instance, that the weakly modular functions for $\Gamma(N)$ are not in general 1-periodic because $T \notin \Gamma(N)$. A more important problem is that $\Gamma\backslash\mathbb{H}$ in general has several cusps, points at infinity, and it is unclear how to define Fourier series at them when they are different of $i\infty$.

A handful solution is to introduce *scaling matrices*. Recall that for a congruence subgroup the fundamental domain can be obtained gluing copies of $\mathcal{F}$ and the cusp are the points of the closure in the boundary of $\mathbb{H} \cup \{i\infty\}$, i.e., they are real (in fact rational) or $i\infty$. Consider a cusp $\mathfrak{a}$ not having another equivalent cusp in the fundamental domain (we can always assume so changing the domain). The aspect of a neighborhood of $\mathfrak{a}$ is like

$\mathfrak{a}$ real

or

$\mathfrak{a}$ infinity

where the different tones are given by images of $\mathcal{F}$ under coset representatives. In fact the first figure is the lower part of a fundamental domain for $\Gamma_0(5)$ with $\mathfrak{a} = 0$ and the second figure is the upper part of a fundamental domain for $\Gamma(3)$ with $\mathfrak{a} = i\infty$.

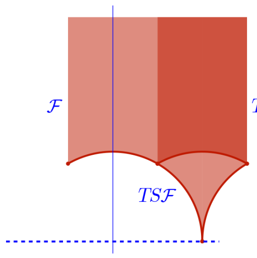
Geometrically, a scaling matrix $\sigma_\mathfrak{a}$ is an element of $\mathrm{SL}_2(\mathbb{R})$ such that $\sigma_\mathfrak{a}\infty = \mathfrak{a}$ and applies this neighborhood is the upper part of the $\mathcal{F}$ i.e., of $|\Re(z)| < 1/2$. In fact any vertical band of width 1 makes the job because we can always rearrange $\mathcal{F}$ to get it.

A less illustrative but more formal definition is to consider the scaling matrix as a matrix $\sigma_\mathfrak{a} \in \mathrm{SL}_2(\mathbb{R})$ such that $\sigma_\mathfrak{a}\infty = \mathfrak{a}$ and $\sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = \langle \pm T\rangle$ where $\Gamma_\mathfrak{a}$ is the stability group of the cusp, the elements of Γ leaving $\mathfrak{a}$ invariant. The $\pm$ becomes a $+$ if $-I \notin \Gamma$ or if working in $\mathrm{PSL}_2(\mathbb{Z})$. It is just to manage the ambiguity $\gamma z = (-\gamma)z$. This algebraic definition shows that a scaling matrix post-multiplied by any translation in $\mathrm{SL}_2(\mathbb{R})$ is still a scaling matrix. This reflects the fact that we can choose any band of width 1 in the geometric interpretation.

For instance, if in the first figure above we consider $\mathfrak{a} = 0$ then $\sigma_0 z = -1/(5z)$ while in the second figure $\sigma_\infty z = 3z$ and we have

$$\sigma_0 = \begin{pmatrix} 0 & -1/\sqrt{5} \\ \sqrt{5} & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma_\infty = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1/\sqrt{5} \end{pmatrix}.$$

If we recall the following fundamental domain for Γ_θ, we see the two inequivalent cusps $i\infty$ and 1 and observe that the scaling matrices compatible with this fundamental domain must act like $\sigma_\infty z = 2z + 1/2$ and $\sigma_1 z = TSz = 1 - 1/z$. It is also valid $\sigma_\infty z = 2z$ although it does not match the standard fundamental domain.



$$\mathfrak{a} = \infty,\ 1$$

$$\sigma_\infty = \begin{pmatrix} \sqrt{2} & \sqrt{2}/4 \\ 0 & 1/\sqrt{2} \end{pmatrix} \quad \text{or} \quad \sigma_\infty = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

After this definition, a *modular form* of weight k for Γ (a congruence subgroup) is a holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ satisfying $f|_\gamma = f$ for every $\gamma \in \Gamma$ and such that $f|_{\sigma_\mathfrak{a}}$ has a (convergent) Fourier expansion of the form $\sum_{n=0}^\infty a_n e^{2\pi i n z}$ for every cusp $\mathfrak{a}$.

The Fourier coefficients a_n are, of course, cusp dependent. The scaling matrix forces $f|_{\sigma_\mathfrak{a}}$ to be periodic, as the following result shows, then the actual requirement is that the Fourier series has not negative frequencies. Thinking in terms of Riemann surfaces, it assures that f is holomorphic at $\mathfrak{a}$, it does not blow out at that point of infinity.

Lemma 1.6. *If f is a weakly modular form then $f|_{\sigma_\mathfrak{a}}$ is 1-periodic for every cusp $\mathfrak{a}$. In particular it admits a Fourier series $\sum_{n\in\mathbb{Z}} a_n e^{2\pi i n z}$.*

Proof. By the definitions of the slash operator and the scaling matrix,

$$f|_{\sigma_\mathfrak{a}}(Tz) = j_{\sigma_\mathfrak{a}}^{-k}(Tz) f(\sigma_\mathfrak{a} T z) = j_{\sigma_\mathfrak{a}}^{-k}(Tz) f(\gamma \sigma_\mathfrak{a} z)$$

for $\gamma = \sigma_\mathfrak{a} T \sigma_\mathfrak{a}^{-1} \in \Gamma_\mathfrak{a}$ (in fact a generator except for the sign). If f is weakly modular, $f(\gamma\sigma_\mathfrak{a} z) = j_\gamma^k(\sigma_\mathfrak{a} z) f(\sigma_\mathfrak{a} z) = j_\gamma^k(\sigma_\mathfrak{a} z) j_{\sigma_\mathfrak{a}}^k(z) f|_{\sigma_\mathfrak{a}}(z)$. Using the cocycle condition, $j_{\sigma_\mathfrak{a}}^{-1}(Tz) j_\gamma(\sigma_\mathfrak{a} z) j_{\sigma_\mathfrak{a}}(z) = j_T(z) = 1$, and it is deduced $f|_{\sigma_\mathfrak{a}}(Tz) = f|_{\sigma_\mathfrak{a}}(z)$.

The function $f|_{\sigma_\mathfrak{a}}(z)$ is a C^∞ 1-periodic function as a function of $\Re(z)$, hence for each fixed imaginary part admits a Fourier expansion. By analytic continuation, it is a function of z. □

In the literature a more common approach (see [3], [2]) is to use Fourier expansions of the form $\sum a_n e^{2\pi i n z/L}$ for $f|\gamma$ where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\gamma\infty$ gives the selected cusp. Here L is the *width of the cusp*, a divisor of the level of the subgroup, the scaling matrix essentially is this γ combined with a dilation.

Strictly speaking, the possibility of composing with a translation causes certain dependence of the Fourier expansion on the scaling matrix. It is not a big deal, it is like saying that $f(z) = \sum a_n e^{2\pi i n z}$ is equivalent to $f(z + r) = \sum b_n e^{2\pi i n z}$ with $b_n = a_n e^{2\pi i r n}$ for $r \in \mathbb{R}$.

We use $\mathcal{M}_k(\Gamma)$ to denote the linear space of modular forms of weight k for Γ and $\mathcal{S}_k(\Gamma)$ to denote the subspace of *cusp forms* defined as the modular forms having $a_0 = 0$ for every cusp. It may happen that $a_0 = 0$ for the cusp $\mathfrak{a}$ but not necessarily for other cusps. It is said that the modular form is cuspidal at $\mathfrak{a}$.

Some authors reserve the name modular form to the case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and use *automorphic form* in general. The requirement of Γ to be a congruence subgroup is not essential, automorphic forms can be defined for Fuchsian groups of the first kind. We have restricted ourselves to weight in $\mathbb{Z}_{\geq 0}$, otherwise we would find problems in the definition of the slash operator because a determination of the argument should be specified. It is possible to extend the theory to nonintegral weights and we will do it later in connection with θ.

Promoting θ^{2k} from weakly modular to modular is not difficult appealing to Poisson summation formula.

Proposition 1.7. *For $k \in \mathbb{Z}^+$ divisible by four $\theta^{2k} \in \mathcal{M}_k(\Gamma_\theta)$ and it is cuspidal at* 1.

Proof. After Corollary 1.4 we have to check that $\theta^{2k}|_{\sigma_\mathfrak{a}}$ has valid Fourier series (with $n \geq 0$) for the two inequivalent cusps $\mathfrak{a} = \infty$ and $\mathfrak{b} = 1$. For $\mathfrak{a}$ it is rather trivial because choosing the scaling matrix $\sigma_\mathfrak{a} z = 2z + 1$

$$\theta^{2k}|_{\sigma_\mathfrak{a}}(z) = 2^{-k/2}\theta^{2k}(2z+1) = 2^{-k/2}\Big(\sum_{n\in\mathbb{Z}}(-1)^n e^{2\pi i n^2 z}\Big)^{2k} = 2^{-k/2}\sum_{n=0}^{\infty}(-1)^n r_{2k}(n) e^{2\pi i n^2 z}$$

with r_{2k} the number of representation as a sum of $2k$ squares, which gives a valid Fourier series.

Applying Lemma 1.3 to $f(x) = e^{-\pi(x+1/2)^2 t}$ with $t > 0$ as in the proof of Proposition 1.2 we get a similar result with an extra $e^{\pi i n} = e^{\pi i n^2}$ factor. Then

$$\sum_{n\in\mathbb{Z}} e^{-\pi(n+1/2)^2 t} = t^{-1/2}\sum_{n\in\mathbb{Z}} e^{-\pi n^2/t + \pi i n^2}.$$

Raising to the power $2k$ and putting $t = -iz$ (by analytic continuation) it is deduced

$$\Big(\sum_{n\in\mathbb{Z}} e^{\pi i (n+1/2)^2 z}\Big)^{2k} = j_{\sigma_1}^{-k}(z)\theta^{2k}(\sigma_1 z).$$

Finally note that $(n+1/2)^2 - 1/4 \in 2\mathbb{Z}^+$. Then when opening the power in the left hand side the exponents are nonnegative even integer multiples of $\pi i z$, giving a valid Fourier series. □

1.3 Creating and counting modular forms for subgroups

There is a generalization of the Eisenstein series to congruence subgroups. The novelty is that there is one for each cusp and they do not vanish for odd weight if $-I$ is not in the group.

Given a cusp $\mathfrak{a}$ of a congruence subgroup Γ and a corresponding scaling matrix $\sigma_\mathfrak{a}$ we define the *Eisenstein series*

$$E_k^\mathfrak{a}(z) = \sum_{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma} j^{-k}_{\sigma_\mathfrak{a}^{-1}\gamma}(z).$$

As before, $\Gamma_\mathfrak{a}$ is the group of parabolic elements in Γ leaving $\mathfrak{a}$ invariant. We impose $k \in \mathbb{Z}_{\geq 3}$ to assure the absolute convergence to a holomorphic function.

Proposition 1.8. *The Eisenstein series $E_k^\mathfrak{a}$ is a modular form of weight k for Γ. Moreover, it is cuspidal at every cusp not equivalent to $\mathfrak{a}$ and the Fourier expansion at $\mathfrak{a}$ has $a_0 = 1$.*

Proof. If $\delta \in \Gamma$

$$E_k^{\mathfrak{a}}|_\delta(z) = j_\delta^{-k}(z) E_k^{\mathfrak{a}}(\delta z) = j_\delta^{-k}(z) \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} j_{\sigma_{\mathfrak{a}}^{-1}\gamma}^{-k}(\delta z) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} j_{\sigma_{\mathfrak{a}}^{-1}\gamma\delta}^{-k}(z)$$

by the cocycle condition. Changing $\gamma \mapsto \gamma\delta^{-1}$ we get $E_k^{\mathfrak{a}}|_\delta = E_k^{\mathfrak{a}}$, hence it is weakly modular.

It is possible to write a general Fourier expansion of $E_k^{\mathfrak{a}}$ at every cusp [6, (3.15)] but it is involved and it would require to introduce new notation. We instead appeal to Lemma 1.6 and prove $E_k^{\mathfrak{a}}|_{\sigma_{\mathfrak{a}}}(i\infty) = 1$ and $E_k^{\mathfrak{a}}|_{\sigma_{\mathfrak{b}}}(i\infty) = 0$ if $\mathfrak{a} \neq \gamma\mathfrak{b}$ for every $\gamma \in \Gamma$.

A calculation with the cocycle condition as before, shows

$$E_k^{\mathfrak{a}}|_{\sigma_{\mathfrak{b}}}(z) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} j_{\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}}^{-k}(z) \qquad \text{and} \qquad E_k^{\mathfrak{a}}|_{\sigma_{\mathfrak{a}}}(z) = \sum_{\gamma' \in \langle \pm T\rangle \backslash \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}} j_{\gamma'}^{-k}(z).$$

The only chance to avoid $j_{\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}}^{-k}(i\infty) = 0$ is $\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}\infty = \infty$ (an upper triangular matrix). Applying $\sigma_{\mathfrak{a}}$ in both sides, it gives $\mathfrak{a} = \gamma\mathfrak{b}$ showing that, $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent. Hence $E_k^{\mathfrak{a}}$ is cuspidal at every cusp not equivalent to $\mathfrak{a}$.

If $\mathfrak{a} = \mathfrak{b}$ then $\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}}\infty = \infty$ implies $\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}} \in \langle \pm T\rangle$ concluding that the only γ' in the range of the second sum with $j_{\gamma'}^{-k}(i\infty) \neq 0$ belongs to the class of I and $E_k^{\mathfrak{a}}|_{\sigma_{\mathfrak{a}}}(i\infty) = 1$. □

Corollary 1.9. *Every $f \in \mathcal{M}_k(\Gamma)$ can be written as $f = g + \sum_{\mathfrak{a}} a_0^{\mathfrak{a}} E_k^{\mathfrak{a}}(z)$ where $a_0^{\mathfrak{a}}$ is the 0-coefficient of f at the cusp $\mathfrak{a}$, $g \in \mathcal{S}_k(\Gamma)$ and the sum is over a complete set of inequivalent cusps.*

Once we have constructed a family of modular forms, the natural question is how many there are for each weight. There exist general formulas for the dimension of $\mathcal{M}_k(\Gamma)$ for any congruence subgroup [3, 3.5–6] but they are quite messy to state and not so easy to prove. Roughly speaking they depend on the number of cusps and on the number of points fixed by elliptic elements in the closure of a fundamental domain. Actually, the cusps must be divided into two classes in some situations and the order of the fixed points also plays a role.

We only explore here a generalization of the valence formula that gives a bound. The idea is imposing the symmetries of $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ to get a modular form for $\mathrm{SL}_2(\mathbb{Z})$ and apply the original valence formula.

There is a little technical issue caused by the ambiguity $\gamma z = (-\gamma)z$. To avoid it, it is convenient to work in $\mathrm{PSL}_2(\mathbb{Z})$ and consider $\overline{\Gamma} = \Gamma/\{\pm I\}$ if $-I \in \Gamma$ and $\overline{\Gamma} = \Gamma$ otherwise.

Theorem 1.10 (general valence formula)**.** *If $f \in \mathcal{M}_k(\Gamma) - \{0\}$*

$$\sum_{z \in \overline{\Gamma}\backslash\mathbb{H}} \frac{n(f,z)}{|\overline{\Gamma}_z|} + \sum_{\mathfrak{a}} n(f,\mathfrak{a}) = \frac{k}{12}[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}]$$

where $\overline{\Gamma}_z$ is the stability group of z, $\mathfrak{a}$ runs over a complete set of inequivalent cusps and $n(f,\mathfrak{a})$ is the order at the cusp $\mathfrak{a}$ *given by the index of the first nonvanishing Fourier coefficient.*

Proof. With this notation, the original valence formula reads

$$\sum_{z\in \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}} \frac{n(g,z)}{|\mathrm{PSL}_2(\mathbb{Z})_z|} + n(g,\infty) = \frac{K}{12} \qquad \text{for} \quad g\in\mathcal{M}_K-\{0\}.$$

Consider $\{\gamma_j\}_{j=1}^m$ representatives of $\overline{\Gamma}\backslash\mathrm{PSL}_2(\mathbb{Z})$ with $m=[\mathrm{PSL}_2(\mathbb{Z}):\overline{\Gamma}]$. Then $g=\prod_{j=1}^m f|_{\gamma_j}$ belongs to $\mathcal{M}_{km}$ and we can take this g with $K=km$ in the original valence formula. Note that

$$n(g,z)=\sum_{j=1}^m n(f,\gamma_j z) = \sum_{z'\in\{\gamma_j z\}_{j=1}^m} \#\{j\,:\,\gamma_j z=z'\}n(f,z') = \sum_{z'\in\{\gamma_j z\}_{j=1}^m} [\mathrm{PSL}_2(\mathbb{Z})_{z'}:\overline{\Gamma}_{z'}]n(f,z').$$

The stability groups $\mathrm{PSL}_2(\mathbb{Z})_z$ and $\mathrm{PSL}_2(\mathbb{Z})_{z'}$ are conjugate because z and z' are related by elements of $\mathrm{PSL}_2(\mathbb{Z})$. In particular, $[\mathrm{PSL}_2(\mathbb{Z})_{z'}:\overline{\Gamma}_{z'}]=|\mathrm{PSL}_2(\mathbb{Z})_z|/|\overline{\Gamma}_{z'}|$ and

$$\sum_{z\in \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}} \frac{n(g,z)}{|\mathrm{PSL}_2(\mathbb{Z})_z|} = \sum_{z\in \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}} \sum_{z'\in\{\gamma_j z\}_{j=1}^m} \frac{n(f,z')}{|\overline{\Gamma}_{z'}|} = \sum_{z\in\overline{\Gamma}\backslash\mathbb{H}} \frac{n(f,z')}{|\overline{\Gamma}_{z'}|}.$$

It remains to deal with the cusps. The point here is that the γ_j are scaling matrices except for multiplying by a dilation to adjust the width. Namely, for any γ_j with $\gamma_j\infty=\mathfrak{a}$ we have $\sigma_{\mathfrak{a}}=\lambda\gamma_j\tau$ with $\lambda,\tau\in\mathrm{PSL}_2(\mathbb{R})$ corresponding respectively to multiply by $N=\#\{j\,:\,\gamma_j\infty=\mathfrak{a}\}$ and to a translation. Hence $f|_{\gamma_j}(z)=\sum_{n=0}^\infty a_n e^{2\pi i n(z+\alpha)/N}$ where a_n are the Foruier coefficients at the cusp $\mathfrak{a}$. This justifies

$$n(g,\infty)=\sum_{\mathfrak{a}} n\Big(\prod_{\gamma_j\infty=\mathfrak{a}} f|_{\gamma_j},\infty\Big) = \sum_{\mathfrak{a}} n(f,\mathfrak{a})$$

and the proof is finished. □

Corollary 1.11. *We have*

$$\dim\mathcal{M}_k(\Gamma)\le 1+\frac{k}{12}[\mathrm{PSL}_2(\mathbb{Z}):\overline{\Gamma}].$$

In particular the only modular forms of weight 0 *are the constants.*

Proof. With a non trivial linear combination of m linear independent modular forms we can vanish the first $m-1$ coefficients Fourier coefficients at one of the cusps $\mathfrak{a}$, getting $n(f,\mathfrak{a})=m-1$ and the result follows from the valence formula. □

Exercises

EXERCISE 1. Prove that if $\gamma\sigma_{\mathfrak{a}}=\sigma_{\mathfrak{a}}T$ then $j_{\sigma_{\mathfrak{a}}}^{-1}(Tz)j_\gamma(\sigma_{\mathfrak{a}}z)j_{\sigma_{\mathfrak{a}}}(z)=1$. This fact was employed to show that $f|_{\sigma_{\mathfrak{a}}}(z)$ is 1-periodic.

Exercise 2. For $A \in \mathrm{SL}_d(\mathbb{R})$ with $8 \mid d$ consider

$$f_A(z) = \sum_{\vec{n} \in \mathbb{Z}^d} e^{\pi i z \|A\vec{n}\|^2} \qquad \text{with} \quad z \in \mathbb{H}.$$

Let B be the cofactor matrix of A. Check $B \in \mathrm{SL}_d(\mathbb{R})$ and show the identity $f_A(z) = z^{-d/2} f_B(-1/z)$.

Exercise 3. If $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent cusps, explain how to get $\sigma_{\mathfrak{b}}$ out of $\sigma_{\mathfrak{a}}$.

Exercise 4. Show that for p prime $\Gamma_0(p)$ has only two inequivalent cusps and give scaling matrices for them.

Exercise 5. Show that $\Gamma_0(4)$ has three inequivalent cusps.

Exercise 6. Show that $\left(\frac{1}{2}\sum_{n=-15}^{15} e^{-n^2/4}\right)^2$ differs form π in less than 10^{-15}.

Exercise 7. Formally we could assign to $\theta(2z)$ the L-function $\zeta(2s) = \sum_{n=1}^{\infty} n^{-2s}$. Repeating the reasoning done with the L-functions associated to modular forms, show the functional equation for the Riemann zeta function: $\Lambda(s) = \Lambda(1-s)$ for $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

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