L-function elegance

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Contents. Hecke bound. Meromorphic continuation. Functional equation. Euler products.

4.1 The very definition

Associate to $f(z) = \sum_{n=0}^{\infty} a_n\, e^{2\pi i n z} \in \mathcal{M}_k$ the complex variable *L-function*

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

A first question is where it is well-defined. For instance, for $f = E_4$ we have that a_n is proportional to $\sigma_3(n) \ge n^3$ and $L_f(s)$ does not converge for $s \in \mathbb{R}_{\le 4}$ in fact, appealing to the theory of Dirichlet series, it can be proved [4, Cor. 1.2] that it does not converge for any $s \in \mathbb{C}$ with $\Re(s) < 4$. The general problem of convergence is clearly linked to the size of the Fourier coefficients and there is a difference between cusp forms and the rest of modular forms. One could appeal to a profound result due to Deligne giving $|a_p| \le 2p^{(k-1)/2}$ for Hecke forms (this is the *Ramanujan-Petersson conjecture* that was observed by Ramanujan for the τ function). This would be overkill because there is a simple Fourier series argument with similar strength to decide the absolute convergence.

Proposition 4.1. *If* $\sum_{n=1}^{\infty} a_n e^{2\pi i n z} \in \mathcal{S}_k$ *then* $N^{-k} \sum_{n=1}^{N} |a_n|^2$ *remains bounded for* $N \in \mathbb{Z}^+$.

Proof. The function $F(z) = (\Im(z))^{k/2}|f(z)|$ is bounded in the closure of the fundamental domain $\overline{\mathcal{F}}$ because $F(z) \to 0$ when $z \to i\infty$. A calculation shows $F(\gamma z) = F(z)$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Hence F is bounded in $\mathbb{H}$. By Parseval formula applied to $f(x + i/N)$,

$$\sum_{n=1}^{\infty} |a_n|^2 e^{-4\pi n/N} = \int_0^1 \big|f(x+i/N)\big|^2\, dx = N^k \int_0^1 \big|F(x+i/N)\big|^2\, dx$$

and the result follows since the left hand side is greater or equal than $e^{-4\pi} \sum_{n=1}^{N} |a_n|^2$. □

Corollary 4.2. *The L-function* $L_f(s)$ *converges absolutely in the half-plane* $\Re(s) > \frac{k+1}{2}$ *for* $f \in \mathcal{S}_k$ *and defines a holomorphic function there.*

Proof. Using Cauchy's inequality, for $\sigma > \frac{k+1}{2}$,

$$\sum_{n=N}^{\infty} \frac{|a_n|}{n^\sigma} = \sum_{\ell=0}^{\infty} \sum_{n=2^\ell N}^{2^{\ell+1}N-1} \frac{|a_n|}{n^\sigma} \le \sum_{\ell=0}^{\infty} \Big((2^\ell N)^{1-2\sigma} \sum_{n=2^\ell N}^{2^{\ell+1}N-1} |a_n|^2 \Big)^{1/2}.$$

By Corollary 4.1, this is bounded by a multiple of $\sum_{\ell=0}^{\infty}(2^\ell N)^{(1-2\sigma+k)/2}$ and hence tends to 0 when $N\to\infty$.

Once the absolute convergence is settled, L_f is holomorphic by standard theorems in complex analysis [1]. □

For non cuspidal forms the absolute convergence takes place in a smaller half-plane.

Corollary 4.3. *For $f\in\mathcal{M}_k-\mathcal{S}_k$, the L-function $L_f(s)$ converges absolutely in the half-plane $\Re(s)>k$, defines a holomorphic function and $L_f(s)\to\infty$ when $s\to k^+$.*

Proof. Any $f\in\mathcal{M}_k-\mathcal{S}_k$ equals $\lambda_1E_k+\lambda_2 g$ with $\lambda_1\neq 0$ and $g\in\mathcal{S}_k$, so $L_f=\lambda_1L_{E_k}+\lambda_2L_g$. The coefficients of E_k are proportional to $\sigma_{k-1}(n)$ and the inequality

$$\sigma_{k-1}(n)\le n^{k-1}\big(1+2^{1-k}+3^{1-k}+\cdots+n^{1-k}\big)\le\zeta(k-1)n^{k-1}$$

shows that $L_{E_k}(s)$ converges absolutely for $\Re(s)>k$ and $\sigma_{k-1}(n)\ge n^{k-1}$ implies that $L_{E_k}(s)\to\infty$ as $s\to k^+$. By Corollary 4.2, L_g does not affect to the convergence in $\Re(s)>k$. □

4.2 Amazing analytic properties

It turns out that L_f shares with the Riemann zeta function a meromorphic extension and a functional equation. These noticeable properties where established in the celebrated Riemann's memoir [2] about the distribution of prime numbers. The proof for the L-functions of modular forms is quite similar. This is not by chance, in Riemann's original proof a modular-like property was fundamental.

Before stating the result, recall that the Γ function is a meromorphic function on $\mathbb{C}$ that for $\Re(s)>0$ is given by

$$\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}\,dx. \tag{1}$$

Partial integration shows $\Gamma(s+1)=s\Gamma(s)$ and this allows to extend it as a meromorphic function having single poles at $\mathbb{Z}_{\le 0}$. Clearly $\Gamma(1)=1$ and hence $\Gamma(n)=(n-1)!$ for $n\in\mathbb{Z}^+$. A deeper property of the Γ function is that it does not vanish. For a more detailed discussion about this special function, see [1].

Theorem 4.4. *If $f\in\mathcal{S}_k$ then L_f extends to an entire function and if $f\in\mathcal{M}_k-\mathcal{S}_k$ it extends to a meromorphic function on $\mathbb{C}$ with a single pole at $s=k$ which is simple and with residue $\frac{(2\pi i)^k}{(k-1)!}a_0$. In both cases, L_f satisfies the functional equation*

$$\Lambda_f(s)=(-1)^{k/2}\Lambda_f(k-s)\qquad\textit{with}\quad\Lambda_f(s)=(2\pi)^{-s}\Gamma(s)L_f(s).$$

Proof. Assume $\Re(s)>k$ and $f(z)=\sum_{n=0}^\infty a_ne^{2\pi inz}\in\mathcal{M}_k$. Using the integral representation (1) with the change of variables $x=2\pi ny$, it is deduced

$$\Lambda_f(s)=\int_0^\infty y^{s-1}\big(f(iy)-a_0\big)\,dy.$$

We have $f(iy) = i^k y^{-k} f(i/y)$ because $f \in \mathcal{M}_k$. Applying this relation in the part $[0,1]$ of the integral and performing in it a subsequent change of variables $y \mapsto y^{-1}$,

$$\Lambda_f(s) = \int_1^\infty y^{-s-1}\big(i^k y^k f(iy) - a_0\big)\,dy + \int_1^\infty y^{s-1}\big(f(iy) - a_0\big)\,dy.$$

This can be arranged as

$$\Lambda_f(s) = \int_1^\infty \big(i^k y^{k-s} + y^s\big) y^{-1}\big(f(iy) - a_0\big)\,dy + a_0 \int_1^\infty \big(i^k y^{k-s-1} - y^{-s-1}\big)\,dy.$$

The first integral defines an entire function because $f(iy)$ tends exponentially to a_0 when $y \to +\infty$. The second integral is elementary, giving

$$\Lambda_f(s) = \frac{i^k a_0}{s-k} - \frac{a_0}{s} + \int_1^\infty \big(i^k y^{k-s} + y^s\big) y^{-1}\big(f(iy) - a_0\big)\,dy.$$

If $f \in \mathcal{S}_k$ then $a_0 = 0$ and $\Lambda_f(s)$ defines an entire function and it implies that L_f extends to an entire function because $\Gamma(s)$ does not vanish. If $f \in \mathcal{M}_k - \mathcal{S}_k$, $\Lambda_f(s)$ defines a meromorphic function on $\mathbb{C}$ with single poles at $s = 0$ and $s = k$.

Irrespective of whether $f \in \mathcal{M}_k$ is a cusp form or not, changing s by $k - s$ we deduce $\Lambda_f(s) = (-1)^{k/2}\Lambda_f(k-s)$. Note that $i^k = i^{-k} = (-1)^{k/2}$ because k is even.

Using the definition, the entire function $(s-k)s\Lambda_f(s)$ equals $(2\pi)^{-s}\Gamma(s+1)(s-k)L_f(s)$. It implies that $(s-k)L_f(s)$ is entire. We also have

$$\lim_{s\to k}(s-k)L_f(s) = \lim_{s\to k}\frac{(2\pi)^s}{\Gamma(s)}(s-k)\Lambda_f(s) = \frac{(2\pi)^k}{\Gamma(k)} i^k a_0$$

as expected. □

Hecke and Weil proved *converse theorems* showing that, under some technical conditions, functional equations imply modularity. In the version of Weil, they are related to a possible approach to get the modularity needed to prove Fermat last theorem.

4.3 Arithmetic in a complex function

In the title of [3] we grasp that there are *Euler products* associated to modular forms. An Euler product is a formula resembling the following relation found by Euler between the Riemann zeta function and the prime numbers:

$$\sum_{n=1}^\infty \frac{1}{n^s} = \prod_p \big(1 - p^{-s}\big)^{-1} \qquad \text{for} \quad \Re(s) > 1.$$

Roughly speaking the modern study of the distribution of prime number is based on eliminating in this formula the primes in terms of the ζ function (the left hand side).

There is a close relation between Euler products and multiplicative functions [4] and the Fourier coefficients of Hecke forms are an example of them.

Theorem 4.5. *If $f \in \mathcal{S}_k$ is a Hecke form then L_f has the following Euler product*

$$L_f(s) = \prod_p \big(1 - a_p p^{-s} + p^{k-1-2s}\big)^{-1} \qquad \text{for} \quad \Re(s) > \frac{k+1}{2}$$

where a_p is the p-th Fourier coefficient of f.

Proof. We proceed formally without worrying about convergence issues. They can be approached considering partial products [4, Th. 1.9]. In this formal way, using the fundamental theorem of arithmetic and $a_{nm} = a_n a_m$ for m and n coprime,

$$L_f(s) = \prod_p \Big(1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{2s}} + \frac{a_{p^3}}{p^{3s}} + \dots\Big).$$

The expected formula is a consequence of taking $x = p^{-s}$ in

$$\big(1 - a_p p^{-s} + p^{k-1-2s}\big) \sum_{\ell=0}^{\infty} a_{p^\ell} x^\ell = 1.$$

To derive this power series identity, we open the parenthesis to get in the left hand side

$$\sum_{\ell=0}^{\infty} a_{p^\ell} x^\ell - \sum_{\ell=0}^{\infty} a_p a_{p^\ell} x^{\ell+1} + p^{k-1} \sum_{\ell=0}^{\infty} a_{p^\ell} x^{\ell+2}.$$

Shifting ℓ by 1 in the last sum and by -1 in the last, it equals

$$1 + a_p x + \sum_{\ell=1}^{\infty} a_{p^{\ell+1}} x^{\ell+1} - a_p x + \sum_{\ell=1}^{\infty} a_p a_{p^\ell} x^{\ell+1} + p^{k-1} \sum_{\ell=1}^{\infty} a_{p^{\ell-1}} x^{\ell+1}.$$

Finally, Corollary 3.17 shows that the sums cancel. □

We have just proved hat the arithmetic properties of the a_n implies the Euler product. It is not difficult to observe that it also works the other way around. The Euler product implies the arithmetic properties of the Fourier coefficients.

Exercises

Exercise 1. Prove that for $f = -\frac{B_k}{2k} E_k$ we have $L_f(s) = \zeta(s)\zeta(s-k+1)$.

Exercise 2. If $f \in \mathcal{S}_k$ show that $L_f(-n) = 0$ for every $n \in \mathbb{Z}_{\ge 0}$.

Exercise 3. If $f \in \mathcal{M}_k$ give a formula for the value of L_f'/L_f at $s = k/2$ in terms of Γ'/Γ.

Exercise 4. Performing in $\int_0^\infty \int_0^\infty x^{w-1}(1+x)^{-s} y^{s-1} e^{-y}\, dxdy$ the change of variables $x = u/v$, $y = u+v$ deduce $\Gamma(s) \int_0^\infty x^{w-1}(1+x)^{-s}\, dx = \Gamma(s-w)\Gamma(w)$ for $\Re(w) > 0$ and $\Re(w-s) < 0$.

Exercise 5. Deduce from the formula of the previous exercise that Γ does not vanish on $\mathbb{C}$.

References

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[2] H. M. Edwards. *Riemann's zeta function.* Academic Press, New York-London, 1974. Pure and Applied Mathematics, Vol. 58.

[3] E. Hecke. Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung. I. *Math. Ann.*, 114(1):1–28, 1937.

[4] H. L. Montgomery and R. C. Vaughan. *Multiplicative number theory. I. Classical theory*, volume 97 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2007.