

Hecke explaining Ramanujan

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Contents. Hecke operators. Arithmetic properties of the τ function. Poincaré series. Hecke forms.

3.1 The averaging operators

Before defining the main characters, we are going to introduce a widespread notation in the theory of modular forms. Instead of writing $f(z) = j_\gamma^{-k}(z)f(\gamma z)$, the characterizing equation of weakly modular forms, we write $f = f|_\gamma$. The weight k is not shown because it is assumed to be fixed. It is convenient to extend this notation to $\mathrm{GL}_2^+(\mathbb{R})$, the group of real matrices with positive determinant. The fully displayed extended definition is:

$$f|_\gamma(z) = (\det \gamma)^{k/2}(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) \quad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}).$$

In few words, $f|_\gamma(z)$ is $j_\gamma^{-k}(z)f(\gamma z)$ as before except for multiplying by a power of the determinant. The good thing about this notation is that it expresses in a simple way the composition properties derived from the cocycle condition.

Lemma 3.1. *For any $f : \mathbb{H} \rightarrow \mathbb{C}$ and $\gamma_1, \gamma_2 \in \mathrm{GL}_2^+(\mathbb{R})$, we have $(f|_{\gamma_1})|_{\gamma_2} = f|_{\gamma_1\gamma_2}$.*

Note that it clarifies that $f|_T = f|_S = f$ is enough to assure that a function f holomorphic in \mathbb{H} is a weakly modular form.

Proof (Sketch). Write γ_j as $s_j\gamma'_j$ with $s_j = \sqrt{\det \gamma_j}$ and $\gamma'_j \in \mathrm{SL}_2(\mathbb{R})$ and apply the cocycle condition. \square

The map $f \mapsto f|_\gamma$ is commonly called the *slash operator*. This is simply notation. As a motivation for more interesting operators, consider the linear space \mathcal{P} of 1-periodic functions. For $f \in \mathcal{P}$ the function $f(x/N)$ with $N \in \mathbb{Z}_{>1}$ is N -periodic and, in general, it is not in \mathcal{P} , but we obtain a periodic function averaging in this way:

$$f \in \mathcal{P} \mapsto \frac{1}{N} \sum_{n \in \mathbb{Z}/N\mathbb{Z}} f\left(\frac{x+n}{N}\right).$$

It defines an operator $\mathcal{P} \rightarrow \mathcal{P}$, in fact a linear endomorphism. Here, $\mathbb{Z}/N\mathbb{Z}$ can be seen as the quotient between the group generated by the unit translation, preserving \mathcal{P} , and the

group generated by the N translation, preserving the N -periodic functions. The operator is nontrivial. For instance, for $N = 2$ the image of $2 \cos(2\pi x) \sin^2(\pi x)$ is $-\cos^2(\pi x)$.

To adapt the idea to \mathcal{M}_k , let us dream that there exists $G < \mathrm{SL}_2(\mathbb{R})$ such that $\mathrm{SL}_2(\mathbb{Z})$ is a normal subgroup of G of finite index. If $\{\delta_j\}$ is a complete set of representatives of the quotient then $F = \sum_j f|_{\delta_j}$ satisfies $F|_{\gamma} = \sum_j f|_{\delta_j \gamma}$ by Lemma 3.1. The normality assures that if f is weakly modular, $F|_{\gamma} = \sum_j (f|_{\gamma_j})|_{\delta_j} = \sum_j f|_{\delta_j} = F$. In this way, $f \mapsto \sum_j f|_{\delta_j}$ would define an operator acting on weakly modular functions.

The problem is that there is not a clear choice for G . If we revise the argument, the crucial property of δ_j is that for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ there exist $\gamma_j \in \mathrm{SL}_2(\mathbb{Z})$ such that $\{\delta_j \gamma\} = \{\gamma_j \delta_j\}$, assured by the coincidence between left and right cosets in the previous case. We need a kind of commutativity $\mathrm{SL}_2(\mathbb{Z}) G = G \mathrm{SL}_2(\mathbb{Z})$ with a sort of uniqueness. The idea under the operators introduced by Hecke in [2] is to consider G satisfying these properties not being a group.

For each $n \in \mathbb{Z}^+$ and a weight $k \in \mathbb{Z}_{\geq 0}$ even, it is defined the *Hecke operator*

$$T_n f = n^{k/2-1} \sum_{\delta \in \mathrm{SL}_2(\mathbb{Z}) \backslash G_n} f|_{\delta} \quad \text{with} \quad G_n = \{\delta \in \mathcal{M}_{2 \times 2}(\mathbb{Z}) : \det \delta = n\}.$$

Here $n^{k/2-1}$ is a normalizing factor to simplify some statements.

To get a computational friendly alternative to this synthetic definition, it is necessary to explore the structure of $\mathrm{SL}_2(\mathbb{Z}) \backslash G_n$.

Lemma 3.2. For $n \in \mathbb{Z}^+$,

$$G_n = \bigsqcup_{\delta \in \Delta_n} \mathrm{SL}_2(\mathbb{Z}) \delta \quad \text{with} \quad \Delta_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = n, 0 \leq b < d \right\}.$$

where the square union symbol means disjoint union. In particular, for any $\delta \in \Delta_n$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ there exist unique $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$ and $\delta' \in \Delta_n$ with $\delta \gamma = \gamma' \delta'$.

Once we know that Δ_n is a set of representatives of $\mathrm{SL}_2(\mathbb{Z}) \backslash G_n$, an explicit formula for Hecke operators shows up. It is apparent that $T_1 = I$ and the rest are nontrivial.

Corollary 3.3. We have

$$(T_n f)(z) = \frac{1}{n} \sum_{ad=n} a^k \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right).$$

Proof. For $\delta \in \Delta_n$ with $\delta z = (az+b)/d$, the product $(\det \delta)^{k/2} j_{\delta}^{-k}(z)$ is $n^{k/2} d^{-k} = n^{-k/2} a^k$. \square

Proof of Lemma 3.2. Name the rows of an arbitrary element of G_n as $(A, B) \in \mathbb{Z}^2$ and $(C, D) \in \mathbb{Z}^2$. The numbers $a = A/r$ and $d = C/r$ are coprime for $r = \gcd(A, C)$, hence there exist $b, d \in \mathbb{Z}$ with $ad - bc = 1$. A calculation using $(aD - cB)r = n$ shows that for every $t \in \mathbb{Z}$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & b+at \\ c & d+ct \end{pmatrix} \begin{pmatrix} r & dB - bD - nt/r \\ 0 & n/r \end{pmatrix}.$$

For a suitably chosen t , the last matrix is in Δ_n and the last but one is always in $\mathrm{SL}_2(\mathbb{Z})$. It proves $G_n \subset \mathrm{SL}_2(\mathbb{Z})\Delta_n$. The reverse inclusion is trivial.

If $\mathrm{SL}_2(\mathbb{Z})\delta_1$ and $\mathrm{SL}_2(\mathbb{Z})\delta_2$ are not disjoint for some $\delta_1, \delta_2 \in \Delta_n$ then $\delta_2\delta_1^{-1} \in \mathrm{SL}_2(\mathbb{Z})$ which in matrix form reads

$$\begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}^{-1} = \frac{1}{n} \begin{pmatrix} a_2d_1 & b_2a_1 - b_1a_2 \\ 0 & a_1d_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

The only upper triangular matrices in $\mathrm{SL}_2(\mathbb{Z})$ are $\pm T^\ell$, then $a_2d_1 = a_1d_2 = n$ that combined with $a_1d_1 = a_2d_2 = n$ gives $a_1 = a_2$ and $d_1 = d_2$. Hence $(b_2a_1 - b_1a_2)/n = (b_2 - b_1)/d_1$ that implies $b_1 = b_2$ because $|b_1 - b_2| < d_1$. In conclusion, $\delta_1 = \delta_2$. \square

Finally, here it is the result we were looking for:

Theorem 3.4. *For fixed weight k and $n \in \mathbb{Z}^+$, the Hecke operator T_n defines linear maps $\mathcal{M}_k \rightarrow \mathcal{M}_k$ and $\mathcal{S}_k \rightarrow \mathcal{S}_k$.*

Proof. The property of the slash operator (Lemma 3.1) and Lemma 3.2 show for $f \in \mathcal{M}_k$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

$$(T_n f)|_\gamma = n^{k/2-1} \sum_{\delta \in \Delta_n} f|_{\delta\gamma} = n^{k/2-1} \sum_{\delta' \in \Delta_n} f|_{\gamma'\delta'} = T_n f.$$

Then $T_n f$ is weakly modular. Indeed $T_n f \in \mathcal{M}_k$ because $\sum_{b=0}^{d-1} e^{2\pi im(az+b)/d}$ vanishes if $d \nmid m$ and a finite sum cannot ruin the absolute convergence of the series expansion.

Clearly T_n is linear. Noting that $\lim_{z \rightarrow i\infty} \delta z = i\infty$, it follows that \mathcal{S}_k is an invariant subspace. \square

3.2 A bunch of similar properties and τ

The general idea is that Hecke operators act on the Fourier coefficients in a reasonably simple way. Theorem 3.4 shows that they preserve the modular form spaces and it opens the door to find surprising relations between Fourier coefficients of modular forms like those experimentally found by Ramanujan for the discriminant function [6].

Here we only deal with the algebra generated by the Hecke operators. On the other hand, there is a big algebraic theory establishing a far deeper connection between them, Galois theory and number theory that was crucial to prove *Fermat Last Theorem* [3]. Hecke never suspected that his operators after the years would be understood as an action on homology groups.

The aforementioned simple action on the Fourier coefficients is summarized in the following result.

Theorem 3.5. *For any absolutely convergent Fourier series*

$$T_n \left(\sum_{m=0}^{\infty} a_m e^{2\pi imz} \right) = \sum_{m=0}^{\infty} b_m e^{2\pi imz} \quad \text{with} \quad b_m = \sum_{d|\mathrm{gcd}(m,n)} d^{k-1} a_{mn/d^2}.$$

For $m = 0$ the formula gives $b_0 = \sigma_{k-1}(n)a_0$ which sounds rather trivial because $T_n 1 = \sigma_{k-1}(n)$ by Lemma 3.3.

Proof. Using $\sum_{b=0}^{d-1} e^{2\pi imb/d} = 0$ for $d \nmid m$ in Lemma 3.3,

$$T_n \left(\sum_{m=0}^{\infty} a_m e^{2\pi imz} \right) = \frac{1}{n} \sum_{m=0}^{\infty} a_m \sum_{\substack{ad=n \\ d|m}} a^k d e^{2\pi iamz/d}.$$

Note that $\frac{1}{n} a^k d = a^{k-1}$. Applying a change of variables $m = dd'$ and putting later $m' = ad'$, we have

$$\sum_{ad=n} \sum_{d'=0}^{\infty} a_{dd'} a^{k-1} e^{2\pi i ad'z} = \sum_{m'=0}^{\infty} \sum_{\substack{ad=n \\ ad'=m'}} a^{k-1} a_{dd'} e^{2\pi im'z}.$$

Note that a in the limits of the double sum divides $\gcd(m, n)$ and $mn/a^2 = dd'$. So, renaming the variables the proof is finished. \square

This result is enough to deduce the arithmetic properties observed by Ramanujan in [6] for the coefficients $\tau(n)$ of the discriminant function. They reduce the computation of $\tau(n)$ to the case n prime.

Corollary 3.6. *If m and n are coprime $\tau(mn) = \tau(m)\tau(n)$ and if p is prime $\tau(p^{\ell+1}) = \tau(p)\tau(p^{\ell}) - p^{11}\tau(p^{\ell-1})$ for $\ell \in \mathbb{Z}^+$.*

Proof. We know that Δ generates \mathcal{S}_{12} and $T_n(\mathcal{S}_{12}) \subset \mathcal{S}_{12}$. Then $T_n = \lambda\Delta$ for some $\lambda \in \mathbb{C}$ and Theorem 3.5 gives

$$\lambda\tau(m) = \sum_{d|\gcd(m,n)} d^{11}\tau(mn/d^2).$$

Taking $m = 1$ it is deduced $\lambda = \tau(n)$. If m and n are coprime, $d = 1$ and the equality reads $\tau(m)\tau(n) = \tau(mn)$. If $n = p$ and $m = p^{\ell}$, $d \in \{1, p\}$ and it reads $\tau(p)\tau(p^{\ell}) = \tau(p^{\ell+1}) + p^{11}\tau(p^{\ell-1})$. \square

For instance, to get the values $\tau(2) = -24$ and $\tau(3) = 252$ is rather quick from the product formula for Δ , but computing $\tau(72)$ would be absolutely not affordable with this method. Using the second part of Corollary 3.6 it is deduced

$$\begin{cases} \tau(8) = \tau(2^3) = \tau(2)\tau(2^2) - 2^{11}\tau(2) = \tau(2)(\tau(2)^2 - 2^{11}) - 2^{11}\tau(2) = 84480, \\ \tau(9) = \tau(3^2) = \tau(3)^2 - 3^{11} = -113643 \end{cases}$$

and the first part gives the impressive value

$$\tau(72) = \tau(8)\tau(9) = -9\,600\,560\,640.$$

The function τ at prime numbers is quite chaotic. The *Sato-Tate conjecture* claims that $\arccos(\frac{1}{2}p^{-11/2}\tau(p))$ behaves as a random variable with density function $\frac{2}{\pi}\sin^2 x$ supported on $[0, \pi]$. This conjecture and its generalization to other modular forms was proved in 2011. There are still some mysteries to unveil. Probably the oldest is *Lehmer conjecture* claiming that $\tau(n) \neq 0$ for all n . It is known that a counterexample would be greater than $8 \cdot 10^{23}$.

The multiplicative structure of the Fourier coefficients in the theory of modular forms in full generality derives, as we will see, from the structure of the *Hecke algebra* (the algebra generated by Hecke operators). Note the similarity between the next result and Theorem 3.5 and between its corollary and the properties of the τ function.

Theorem 3.7. *The Hecke operators satisfy*

$$T_m T_n = \sum_{d|\gcd(m,n)} d^{k-1} T_{mn/d^2}.$$

In particular, they commute.

Corollary 3.8. *If m and n are coprime $T_m T_n = T_{mn}$ and $T_{p^{\ell+1}} = T_p T_{p^\ell} - p^{k-1} T_{p^{\ell-1}}$ for p prime and $\ell \in \mathbb{Z}^+$.*

Proof. Using the explicit formula in Lemma 3.3,

$$mnT_m(T_n f) = \sum_{a_1 d_1 = m} \sum_{a_2 d_2 = n} (a_1 a_2)^k \sum_{b_1=0}^{d_1-1} \sum_{b_2=0}^{d_2-1} f\left(\frac{a_1 a_2 z + a_1 b_2 + b_1 d_2}{d_1 d_2}\right).$$

The values taken by $D = \gcd(a_1, d_2)$ are the common divisors of m and n . Replacing a_1 and d_2 by $a'_1 = a_1/D$ and $d'_2 = d_2/D$ has no effect in the argument of f and $a'_1 b_2 + b_1 d'_2$, for a'_1 and d'_2 fixed, determines b_2 modulo d'_2 . Therefore

$$mnT_m(T_n f) = \sum_{D|\gcd(m,n)} \sum_{a'_1 d_1 = m/D} \sum_{a_2 d'_2 = n/D} D^k (a'_1 a_2)^k D \sum_{b_1=0}^{d_1-1} \sum_{b_2=0}^{d'_2-1} f\left(\frac{a'_1 a_2 z + a'_1 b_2 + b_1 d'_2}{d_1 d'_2}\right).$$

The extra D factor comes from passing from $0 \leq b_2 < d_2$ to $0 \leq b_2 < d'_2$. Note that $b = a'_1 b_2 + b_1 d'_2$ covers all the residue classes modulo $d_1 d'_2$ when a'_1 and d_2 are fixed and b_1 and b_2 are in their ranges of summation. Write $a = a'_1 a_2$ and $d = d_1 d'_2$. Then $ad = mn/D^2$ and this relation determines a'_1 and d'_2 for each a and d , in fact $a'_1 = m/\gcd(m, dD)$ and $d'_2 = dD/\gcd(m, dD)$. Hence

$$mnT_m(T_n f) = \sum_{D|\gcd(m,n)} D^{k+1} \sum_{ad=mn/D} a^k \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right)$$

and the inner sum is mn/D^2 times T_{mn/D^2} . □

A natural question is to find a formula for T_{mn} in terms of Hecke operator with smaller index. It is a direct corollary of Theorem 3.7 when one knows Möbius inversion [7].

Corollary 3.9. *The Hecke operators satisfy $T_{mn} = \sum_{d|\gcd(m,n)} \mu(d) d^{k-1} T_{m/d} T_{n/d}$.*

To fully understand the relation between Hecke operators and Fourier coefficients in the next subsection, we need to enlarge the family of modular forms introducing a new member, a cuspidal version of the Eisenstein series.

For $k \in \mathbb{Z}_{>2}$ even and $m \in \mathbb{Z}^+$ the *Poincaré series* are defined as

$$P_m(z) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} e^{2\pi imz} |_\gamma = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} j_\gamma^{-k}(z) e^{2\pi m\gamma z}$$

where $\Gamma_\infty = \{\pm T^n\}_{n \in \mathbb{Z}}$, the group of parabolic elements fixing $i\infty$. It is an exercise to check that each class in $\Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})$ is composed by the matrices with a fixed lower row modulo a sign. In other words, the matrices with the same $j_\gamma^2(z)$.

If we put $m = 0$ in the definition, it is obtained E_k . In this sense Poincaré series generalize Eisenstein series.

Following the scheme of the proof of Proposition 1.3 it can be proved that P_m has a valid Fourier expansion with real Fourier coefficients and vanishing 0-coefficient [4, §3.2]. The difference is that a difficult integral appears leading to Bessel functions and some finite exponential sums in the expansion. We take here the existence of this Fourier expansion for granted and only prove the weakly modular part of the following result.

Lemma 3.10. *With the previous notation, $P_m \in \mathcal{S}_k$.*

Proof (only weakly modular). For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, by the multiplicative property of the slash operator (Lemma 3.1) $P_m|_\gamma(z) = \sum_{\gamma' \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} e^{2\pi imz} |_{\gamma'\gamma}$. Post-multiplying $\Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})$ by $\mathrm{SL}_2(\mathbb{Z})$ leaves it obviously invariant. Then we can sum on $\gamma'\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})$ and renaming $\gamma'\gamma$ as γ it is proved the equality $P_m|_\gamma = P_m$. \square

Theorem 3.11. *The action of Hecke operators on Poincaré series satisfies*

$$T_n P_m = n^{k-1} \sum_{d|(m,n)} d^{1-k} P_{mn/d^2}.$$

Proof. By the definition of the Hecke operators

$$T_n P_m = n^{k/2-1} \sum_{\delta \in \Delta_n} P_m|_\delta = n^{k/2-1} \sum_{\delta \in \Delta_n} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} e^{2\pi imz} |_{\gamma\delta}$$

By Lemma 3.2, $\gamma\delta$ are representatives of $\Gamma_\infty \backslash G_n$. It turns out that they $\delta\gamma$ are also representatives of these classes¹. Therefore

$$T_n P_m(z) = n^{k-1} \sum_{\delta \in \Delta_n} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} j_{\delta\gamma}^{-k}(z) e^{2\pi im\delta\gamma z}.$$

Writing the explicit form of δ

$$T_n P_m(z) = n^{k-1} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \sum_{ad=n} \sum_{b=0}^{d-1} d^{-k} j_\gamma^{-k}(z) e^{2\pi imad^{-1}\gamma z} e^{2\pi imab/d}.$$

¹[I must confess that I have taken this claim from [4], where it is mentioned without any explanation, but it does not convince to me.]

The sum in b vanishes except if $d \mid m$ in which case it is d . Hence

$$T_n P_m(z) = n^{k-1} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \sum_{\substack{ad=n \\ d \mid m}} d^{1-k} j_\gamma^{-k}(z) e^{2\pi i m a d^{-1} \gamma z}$$

that gives the result substituting $a = n/d$ in the exponential. \square

This result implies a noticeable symmetry.

Corollary 3.12. *For any $n, m \in \mathbb{Z}^+$, $n^{1-k} T_n P_m = m^{1-k} T_m P_n$.*

3.3 All modular forms are arithmetic

We introduce the *Petersson inner product* in \mathcal{S}_k

$$\langle f, g \rangle = \int_{\mathcal{F}} f(z) \overline{g(z)} \Im(z)^k d\mu(z) \quad \text{where} \quad d\mu(x+iy) = \frac{dx dy}{y^2}.$$

This definition also works when $f \in \mathcal{S}_k$ and $g \in \mathcal{M}_k$ or vice versa.

The measure $d\mu(z)$ is invariant under $\mathrm{SL}_2(\mathbb{Z})$ because it is the volume element associated to the Poincaré metric. Easy calculations involving the relation between $j_\gamma(z)$ and $\Im(\gamma z)$, shows that the other part is also invariant, namely,

$$\langle f, g \rangle = \int_{\gamma \mathcal{F}} f(z) \overline{g(z)} \Im(z)^k d\mu(z) \quad \text{for any } \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

In other words, Petersson inner product is independent of the choice of the fundamental domain.

The most important property of the Poincaré series is that they give the Fourier coefficients of any modular form via this inner product.

Theorem 3.13. *If $f(z) = \sum_{m=0}^{\infty} a_m e^{2\pi i m z} \in \mathcal{M}_k$ then $\langle f, P_m \rangle = \frac{(k-2)!}{(4\pi m)^{k-1}} a_m$ for any $m \geq 1$.*

Proof. Unwrapping the definition of P_m , using $\Im(z) \overline{j_\gamma(z)} j_\gamma(z) = \Im(\gamma z)$ and the modular relation $f(z) = j_\gamma^{-k}(z) f(\gamma z)$, we have

$$\langle f, P_m \rangle = \int_{\mathcal{F}} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} (\Im(\gamma z))^k f(\gamma z) e^{-2\pi i m \gamma \bar{z}} d\mu(z).$$

Now we use the invariance under $\mathrm{SL}_2(\mathbb{Z})$ of the integral and that $\bigcup_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \gamma \overline{\mathcal{F}}$ gives the vertical band $\mathbb{H} \cap \{|\Re(z)| \leq \frac{1}{2}\}$ for some choice of the representatives (exercise) to get

$$\langle f, P_m \rangle = \int_0^\infty \int_{-1/2}^{1/2} y^k f(x+iy) e^{-2\pi i m(x-iy)} d\mu(x+iy) = \int_0^\infty y^k (a_n e^{-2\pi m y}) e^{-2\pi m y} y^{-2} dy.$$

The well-known integral $(k-2)! = \int_0^\infty t^{k-2} e^{-t} dt$ implies the result. \square

The reason to state the previous result is to obtain the following property of the Hecke operators that could be difficult to obtain with direct calculations [1].

Theorem 3.14. *The Hecke operators are self-adjoint operators $T_n : \mathcal{S}_k \rightarrow \mathcal{S}_k$ when \mathcal{S}_k is endowed with the Petersson inner product.*

Proof. Consider the linear space V generated by $\{P_m\}_{m=1}^\infty$. By Lemma 3.10 $V \subset \mathcal{S}_k$, in particular it is finite dimensional. On the other hand, Theorem 3.13 shows that $\mathcal{S}_k \cap V^\perp = \{0\}$, hence $V = \mathcal{S}_k$ and it is enough to prove

$$\langle T_n P_\ell, P_m \rangle = \langle T_n P_m, P_\ell \rangle \quad \text{for every } \ell, m \in \mathbb{Z}^+.$$

The order is irrelevant because these inner products are real thanks to Theorem 3.11 and Theorem 3.13 using that Poincaré series have real coefficients. Let us indicate with $\text{coeff}_m(f)$ the m -th Fourier coefficient of f . By Corollary 3.12 and Theorem 3.13,

$$n^{1-k} \langle T_n P_\ell, P_m \rangle = \ell^{1-k} \langle T_\ell P_n, P_m \rangle = \frac{(k-2)!}{(4\pi m)^{k-1}} \ell^{1-k} \text{coeff}_m(T_\ell P_n).$$

By Theorem 3.5 this latter coefficient equals $\text{coeff}_\ell(T_m P_n)$ and, again by the symmetry and the formula for the Fourier coefficients,

$$\text{coeff}_\ell(T_m P_n) = m^{k-1} n^{1-k} \text{coeff}_\ell(T_n P_m) = m^{k-1} n^{1-k} \frac{(4\pi \ell)^{k-1}}{(k-2)!} \langle T_n P_m, P_\ell \rangle.$$

Substituting, the expected equality follows. \square

The following linear algebra result is central in some areas of physics and it is less known in basic linear algebra courses for mathematicians. See [5] for a proof.

Proposition 3.15. *Let V be an inner product space with $\dim V < \infty$ and $\{T_\alpha\}$ is a set of self-adjoint commuting operators then there exists an orthogonal basis in which all the operator diagonalize simultaneously.*

A consequence of Theorem 3.14 and Proposition 3.15 is our goal:

Corollary 3.16. *There exists a basis \mathcal{B} of \mathcal{S}_k , called a Hecke basis, such that $T_n f = a_n f$ for any $f(z) = \sum_{n=1}^\infty a_n e^{2\pi i n z} \in \mathcal{B}$ and any $n \in \mathbb{Z}^+$. In particular, $a_1 = 1$.*

Proof. Let f be an element of the basis assured by Proposition 3.15 for the Hecke operators. Then $T_n f = \lambda_n f$ implies $a_n = \lambda_n a_1$ by Theorem 3.5 with $m = 1$. Replacing f by $a_1^{-1} f$ the proof is finished. \square

The elements of a Hecke basis are called *Hecke forms*. The following result shows that the properties observed by Ramanujan hold for Hecke forms. Hence they are a general phenomenon in the sense that any modular form can be written as a linear combination of modular forms satisfying them. Note that $\{\sigma_{k-1}(n)\}_{n=1}^\infty$ are the Fourier coefficients of $-\frac{B_k}{2k} E_k$ and they also satisfy the relations.

Corollary 3.17. *For any Hecke form its Fourier coefficients $\{a_n\}_{n=1}^\infty$ satisfy $a_n a_m = a_{nm}$ for $\gcd(m, n) = 1$ and $a_p a_{p^\ell} = a_p a_{p^\ell} - p^{k-1} a_{p^{\ell-1}}$ for p prime and $\ell \in \mathbb{Z}^+$.*

Proof. This is a direct consequence of Corollary 3.8 and Corollary 3.16. \square

Exercises

EXERCISE 1. Fill the details in the proof of Lemma 3.1.

EXERCISE 2. Explain $G_n = \mathrm{SL}_2(\mathbb{Z}) G_n = G_n \mathrm{SL}_2(\mathbb{Z})$ and prove $G_n \not\subseteq \Delta_n \mathrm{SL}_2(\mathbb{Z})$ for $n > 1$.

EXERCISE 3. Show that that each class in $\Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})$ is composed by the matrices A such that A or $-A$ have a given lower row.

EXERCISE 4. Show that E_k is obtained putting $m = 0$ in the definition of the Poincaré series.

EXERCISE 5. Learn the definition and properties of the Möbius function μ and deduce Corollary 3.9 from Theorem 3.7.

EXERCISE 6. Prove $\langle f, g \rangle = \int_{\gamma \mathcal{F}} f(z) \overline{g(z)} \Im(z)^k d\mu(z)$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

EXERCISE 7. Prove that $\bigcup_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \gamma \overline{\mathcal{F}}$ gives the band $B = \mathbb{H} \cap \{|\Re(z)| \leq \frac{1}{2}\}$ for some choice of the representatives.

EXERCISE 8. [Warning: This exercise involves large number computations.] According to the theory $\{E_4^3 \Delta, E_6^2 \Delta\}$ is a basis of \mathcal{S}_{24} . Find the matrix of T_2 in this basis, diagonalize it and use the result to find a Hecke basis.

EXERCISE 9. It is known that there exists a basis $\{f_j\}$ of \mathcal{S}_k (called *Miller basis*) such that $f_j(z) = e^{2\pi i j z} + \sum_{n=d+1}^{\infty} a_n^{(j)} e^{2\pi i n z}$ con $a_n^{(j)} \in \mathbb{Z}$. Assuming this, prove that the Fourier coefficients of the Hecke forms are algebraic integers of degree at most $k/12$.

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